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The Intermediate Set and Limiting Superdifferential for Coalition Games: Between the Core and the Weber Set

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Abstract We introduce the intermediate set as an interpolating solution concept between the core and the Weber set of a coalition game. The new solution is defined as the limiting superdifferential of the Lovász extension and thus it completes the hierarchy of variational objects used to represent the core (Fréchet superdifferential) and the Weber set (Clarke superdifferential). From the game-theoretic point of view, the intermediate set is a non-convex solution containing the Pareto optimal payoff vectors, which depend on some ordered partition of the players and the marginal coalitional contributions with respect to the order. A detailed comparison between the intermediate set and other set-valued solutions is provided. We compute the exact form of intermediate set for all games and provide its simplified characterization for the simple games, the clan games and the glove game.

Keywords coalition game · limiting superdifferential · core · Weber set

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1 Introduction

Many important solution concepts for transferable-utility n-person coalition games can be equivalently expressed as formulas involving gradients or generalized gradients of a suitable extension of the given game. This applies to some of the well-known single-valued solutions, such as the Shapley value and the Banzhaf–Coleman index of power. These constructions usually rely on the multilinear extension of coalition games from the discrete cube $\{0,1\}^n$ onto $[0,1]^n$; see [11, Chapter XII], for example. The purpose of such a "differential representation" of the solution is not only computational, but also to provide a new interpretation of the corresponding payoff vectors, which usually revolves around the idea of marginal contributions to a given (possibly virtual) coalition.

The recent progress in variational analysis [10, 15] enables us to construct various kinds of generalized derivatives, the so-called subgradients and supergradients, for a very large family of lower semicontinuous functions. Thus the notion of a unique gradient of a differentiable function is replaced by the concept of a subdifferential (superdifferential) of a possibly nonsmooth function. The elements of a superdifferential—the supergradients—have a close geometric connection with Jacobians of all smooth majorants of the function at the neighborhood of a given point; see A. Among the main superdifferentials count the Fréchet, the limiting and the Clarke superdifferential, respectively.

The representation of some solution concepts by generalized derivatives for selected classes of cooperative games was studied already by Aubin [2]. The authors of [5, 16] use the Lovász extension of a coalition game in order to express the core and the Weber set in terms of its Fréchet and the Clarke superdifferential, respectively.

In this paper we pursue a converse research direction by adopting the idea proposed in [16]: we employ the limiting superdifferential to define directly a new solution concept for coalition games, the so-called intermediate set. Specifically, the intermediate set is the limiting superdifferential of the Lovász extension of the game calculated at the grand coalition. The associated payoff vectors are thus marginal contributions to the grand coalition in the sense conveyed by the limiting superdifferential. However, several questions arise at this point, for instance:

- What are game-theoretical properties of such a solution?
- Is the intermediate set the set of payoff vectors determined by some reasonable principle of profit allocation among players?

The main goal of this paper is to argue that the newly constructed solution is meaningful and interesting from many perspectives. Using the tools of variational analysis only, we will show that the intermediate set

- is always nonempty, subadditive and Pareto optimal solution,
- is a finite union of convex polytopes, and hence it is not generally convex-valued,
- lies in-between the core and the Weber set,
- coincides with the core iff the game is supermodular (convex).

The intermediate set can be viewed as a nonempty interpolant between the core and the Weber set, which is convenient especially whenever the former is empty or small and the latter is huge. Our Theorem 1 provides a clear interpretation of the payoff vectors from the intermediate set: for some ordered partition of the player set, each such vector is a Weber-style marginal vector on the level of blocks of coalitions and, at the same time, no coalition inside each block can improve upon this payoff vector in the sense of marginal coalitional contributions. The intermediate set is thus a solution concept that looks globally like the Weber set, but behaves locally like the core concept.

The article is structured as follows. We fix our notation and terminology in Section 2, where we repeat the basic facts about the core, the Weber set, the Lovász extension and its superdifferentials. Section 3 contains the characterization of the intermediate set based on ordered partitions of the player set (Theorem 1) and the discussion of a distribution process that leads to a payoff vector in the intermediate set. Some motivating examples are also included (Examples 1 and 3). We carry out an in-depth inspection of the properties of the intermediate set and compare it to the various solution concepts in Section 4. The differences among the core, the intermediate set and the Weber set are captured by Table 1. The selected classes of coalition games—the simple games, the clan games and the glove game—are analyzed in Section 5 and the formula from Theorem 1 is refined in order to derive a neat description of the intermediate set. The main part of the paper is concluded with an outlook towards further research in Section 6. Appendix consists of two parts. A brief explanation of the notions from nonsmooth analysis is in Appendix A, with no attempt at a comprehensive discussion of all the results from this area used in the paper. Appendix B contains the proof of the main characterization result, Theorem 1.

2 Core and Weber Set

We use the standard notions and results from cooperative game theory; see [12]. Let $N = \{1, \dots, n\}$ be a finite set of *players*, where n is a positive integer. By 2^N we denote the powerset of N whose elements $A \subseteq N$ are called *coalitions*. A *(transferable utility coalition) game* is a function $v\colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. Any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ is called a *payoff vector*. We introduce the following notation:

$$\mathbf{x}(A) = \sum_{i \in A} x_i, \quad \text{for every } A \subseteq N.$$

We say that a payoff vector $\mathbf{x}$ is *feasible* in a game v whenever $\mathbf{x}(N) \le v(N)$. The set of all feasible payoff vectors in v is denoted by $\mathcal{F}(v)$.

Let $\mathit{\Gamma}(N)$ be the set of all games and $\mathit{\Omega} \subseteq \mathit{\Gamma}(N)$. A *solution* on $\mathit{\Omega}$ is a set-valued mapping $\sigma\colon \mathit{\Omega} \to 2^{\mathbb{R}^n}$ that maps every game $v \in \mathit{\Omega}$ to a set $\sigma(v) \subseteq \mathcal{F}(v)$. We recall the core solution and the Weber set. The *core* of a game v is the convex polytope

$$\mathcal{C}(v) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N),\ \mathbf{x}(A) \ge v(A) \text{ for every } A \subseteq N\}.$$

Let Π_n be the set of all the permutations π of the player set N. Let $v \in \Gamma(N)$ and $\pi \in \Pi_n$. A *marginal vector* of a game v with respect to π is the payoff vector $\mathbf{x}^v(\pi) \in \mathbb{R}^n$ with coordinates

$$x_i^v(\pi) = v\left(\bigcup_{j \le \pi^{-1}(i)} \{\pi(j)\}\right) - v\left(\bigcup_{j < \pi^{-1}(i)} \{\pi(j)\}\right), \quad i \in N. \tag{1}$$

The *Weber set* of v is the convex hull of all the marginal vectors of v,

$$\mathcal{W}(v) = \operatorname{conv}\{\mathbf{x}^v(\pi) \mid \pi \in \Pi_n\}.$$

Since $\mathbf{x}^v(\pi)(N) = v(N)$, the Weber set is a solution on $\Gamma(N)$ in the sense defined above. Moreover, it always contains the core solution; see [23, Theorem 14].

Proposition 1 *$\mathcal{C}(v) \subseteq \mathcal{W}(v)$ for every $v \in \Gamma(N)$.*

The fundamental tool in this paper is the concept of Lovász extension [9]. For every set $A \subseteq N$ let χ_A denote the incidence vector in $\mathbb{R}^n$ whose coordinates are given by

$$(\chi_A)_i = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We write 0 in place of $\chi_\emptyset$. The embedding of 2^N into $\mathbb{R}^n$ by means of the mapping $A \mapsto \chi_A$ makes it possible to interpret a game on 2^N as a real function on $\{0,1\}^n$. Indeed, it suffices to define $\hat{v}(\chi_A) = v(A)$, for every $A \subseteq N$. In the next step we will extend the function $\hat{v}$ onto the whole of $\mathbb{R}^n$. For every $\mathbf{x} \in \mathbb{R}^n$, put

$$\Pi(\mathbf{x}) = \{\pi \in \Pi_n \mid x_{\pi(1)} \ge \cdots \ge x_{\pi(n)}\}.$$

Given $i \in N$ and $\pi \in \Pi(\mathbf{x})$, define

$$V_i^\pi(\mathbf{x}) = \{j \in N \mid x_j \ge x_{\pi(i)}\}.$$

Note that $V_i^\pi(\mathbf{x}) = V_i^\rho(\mathbf{x})$ for every $\pi, \rho \in \Pi(\mathbf{x})$. This implies that any vector $\mathbf{x} \in \mathbb{R}^n$ can be unambiguously written as a linear combination

$$\mathbf{x} = \sum_{i=1}^{n-1} (x_{\pi(i)} - x_{\pi(i+1)}) \cdot \chi_{V_i^\pi(\mathbf{x})} + x_{\pi(n)} \cdot \chi_N. \tag{3}$$

Using the convention $V_0^\pi(\mathbf{x}) = \emptyset$, we can rewrite (3) as

$$\mathbf{x} = \sum_{i=1}^{n} x_{\pi(i)} \cdot \left(\chi_{V_i^\pi(\mathbf{x})} - \chi_{V_{i-1}^\pi(\mathbf{x})}\right). \tag{4}$$

The *Lovász extension* $\hat{v}$ of $v \in \Gamma(N)$ is the function $\mathbb{R}^n \to \mathbb{R}$ defined linearly with respect to the decomposition (4):

$$\hat{v}(\mathbf{x}) = \sum_{i=1}^{n} x_{\pi(i)} \cdot \left(v(V_i^\pi(\mathbf{x})) - v(V_{i-1}^\pi(\mathbf{x}))\right), \quad \text{for any } \mathbf{x} \in \mathbb{R}^n. \tag{5}$$

Observe that the definition of $\hat{v}(\mathbf{x})$ is independent on the choice of $\pi \in \Pi(\mathbf{x})$. Clearly $\hat{v}(\chi_A) = v(A)$ for every coalition $A \subseteq N$. It is easy to see that the Lovász extension $\hat{v}$ of any game v fulfills these properties:

- $\hat{v}$ is continuous and piecewise linear on $\mathbb{R}^n$;
- $\hat{v}$ is positively homogeneous, that is, $\hat{v}(\lambda \cdot \mathbf{x}) = \lambda \cdot \hat{v}(\mathbf{x})$ for every $\lambda \geq 0$ and $\mathbf{x} \in \mathbb{R}^n$;
- the mapping $v \in \Gamma(N) \mapsto \hat{v}$ is linear.

The following easy lemma says that the local behavior of $\hat{v}$ is the same around χ_N as in the neighborhood of 0.

Lemma 1 *For any $\mathbf{x} \in \mathbb{R}^n$ it holds true that*

$$\hat{v}(\mathbf{x} + \chi_N) = \hat{v}(\mathbf{x}) + \hat{v}(\chi_N).$$

Proof This follows directly from the definition (5) together with the identities $\Pi(\mathbf{x}+\chi_N) = \Pi(\mathbf{x})$, $\Pi(\chi_N) = \Pi_n$ and $V_1^\pi(\chi_N) = \ldots = V_n^\pi(\chi_N) = N$ for every $\pi \in \Pi_n$. □

A game $v \in \Gamma(N)$ is called *supermodular* (or *convex*) if the following inequality is satisfied:

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B), \quad \text{for every } A, B \subseteq N.$$

A *submodular* game v is such that $-v$ is supermodular. A game v is called *additive* when $v(A \cup B) = v(A) + v(B)$ for every $A, B \subseteq N$ with $A \cap B = \emptyset$. We will make an ample use of several characterizations of supermodular games appearing in the literature.

Proposition 2 *Let $v \in \Gamma(N)$. Then the following are equivalent:*

1. *v is supermodular;*
2. *$\{\mathbf{x}^v(\pi) \mid \pi \in \Pi_n\} \subseteq \mathcal{C}(v)$;*
3. *$\mathcal{C}(v) = \mathcal{W}(v)$;*
4. *The Lovász extension $\hat{v}$ of v is a concave function.*

Proof Shapley [18] proved 1. ⇒ 2. and Weber [23] showed that 2. ⇒ 3., respectively. The implication 3. ⇒ 1. was shown by Ichiishi [8]. The equivalence between 1. and 4. is the "supermodular" version of the theorem originally proved by Lovász in [9] for submodular games. □

Remark 1 An extensive survey of other conditions equivalent to supermodularity together with (references to) the proofs can be found in [20, Appendix A]. The notion of "convexity" would be somewhat overloaded in this paper since it could refer to both convex games and convex sets of solutions on games. Moreover, convex games have concave Lovász extensions. For those reasons we strictly prefer the term "supermodular game" over "convex game", although the latter is commonly used.

The Lovász extension $\hat{v}$ of a game v can be used to characterize the core solution and the Weber set by using the tools of nonsmooth calculus; the reader is invited to consult Appendix A for all the notions related to superdifferentials of functions. It was shown in [5, Proposition 3] that the core coincides with the Fréchet superdifferential of the Lovász extension at 0, $\mathcal{C}(v) = \hat{\partial}\hat{v}(0)$. Similarly, from [16, Proposition 4.1] we know that the Weber set is the Clarke superdifferential of $\hat{v}$ at 0, $\mathcal{W}(v) = \overline{\partial}\hat{v}(0)$. It may be more natural to use the grand coalition N in place of the empty coalition in those formulas. As a direct consequence of Lemma 1 this is always possible and thus we can shift the computations of the respective superdifferentials to χ_N.

Proposition 3 *For every game $v \in \Gamma(N)$,*

$$\begin{aligned}\mathcal{C}(v) &= \hat{\partial}\hat{v}(\chi_N) = \hat{\partial}\hat{v}(0),\\ \mathcal{W}(v) &= \overline{\partial}\hat{v}(\chi_N) = \overline{\partial}\hat{v}(0).\end{aligned}$$

3 Intermediate Set

This section is composed of two subsections. In the first one we define the intermediate set using the limiting superdifferential. Its characterization based on ordered partitions of the player set is proved in the second subsection.

3.1 Definition and basic properties

As we have already mentioned in the introduction, it may often happen that the core is small or empty and the Weber set is too coarse. For this reason we follow the idea of Boris Mordukhovich, which was mentioned in [16], and by analogy with Proposition 3 we define a new solution concept as $\partial\hat{v}(\chi_N)$, where ∂ is the limiting superdifferential. By its definition—see Appendix A—the limiting superdifferential always lies in-between the Clarke superdifferential and the Fréchet superdifferential. A simple interpretation of the limiting superdifferential is that it coincides with the union of all Fréchet superdifferentials with respect to some sufficiently small neighborhood of the point in question. In Subsection 3.2 we will refine the original (analytic) Definition 1 into a combinatorial description analogous to many solution concepts for coalition games.

Definition 1 Let $v \in \Gamma(N)$. The *intermediate set* $\mathcal{M}(v)$ of v is the set

$$\mathcal{M}(v) := \partial\hat{v}(\chi_N).$$

We will start with a motivating example of the three-player glove game, in which we show the form of the intermediate set. The general formula, which bypasses the computation of Lovász extension and the limiting superdifferential, will be proved later.

Example 1 Consider a game with the player set $N = \{1, 2, 3\}$ in which the first player owns a single left glove, while the remaining two players possess one right glove each. The profit of a coalition is the number of glove pairs the coalition owns:

$$v(A) = \begin{cases} 1 & \text{if } A \in \{\{1,2\}, \{1,3\}, N\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to compute $\mathcal{C}(v), \mathcal{M}(v)$ and $\mathcal{W}(v)$ directly. Since v is both a simple game and a glove game investigated in Subsection 5.1 and 5.3, respectively, we can also use Theorem 2 and 4 to recover $\mathcal{M}(v)$. Thus,

$$\begin{aligned} \mathcal{C}(v) &= \{(1,0,0)\}, \\ \mathcal{M}(v) &= \operatorname{conv}\{(1,0,0),(0,1,0)\} \cup \operatorname{conv}\{(1,0,0),(0,0,1)\}, \\ \mathcal{W}(v) &= \operatorname{conv}\{(1,0,0),(0,1,0),(0,0,1)\}. \end{aligned}$$

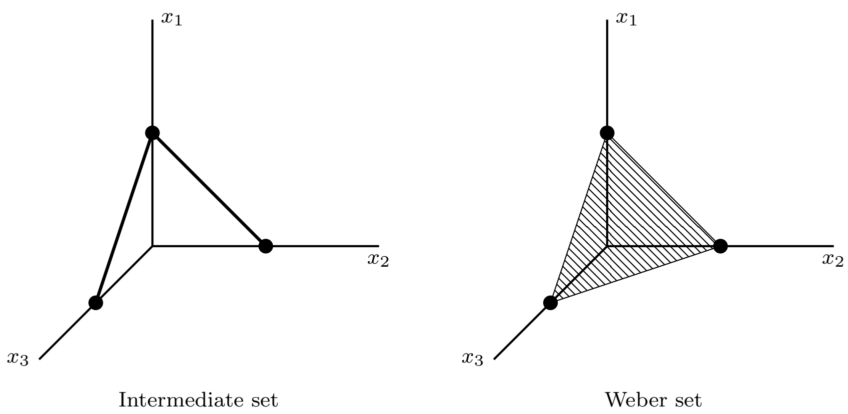


Fig. 1: The intermediate set and the Weber set for the 3-person glove game

We will briefly comment on the shape of the solutions. The core $\mathcal{C}(v)$ is a singleton, reflecting the principle of stability according to which the total payoff goes to the owner of the sparser kind of glove: player 1 has the ability to block effectively the contract among the players. On the other hand, the Weber set $\mathcal{W}(v)$ contains any individually rational and Pareto optimal payoff, which may be difficult to interpret. The intermediate set $\mathcal{M}(v)$ admits two scenaria: player 1 does a deal with either player 2 or player 3, but he does not need both of them in the same time. Once the contract is made, player 1 may decide to share the total profit in an arbitrary ratio. The remaining player (a non-contractor) is thus eliminated from further bargaining.

In the rest of this section, we will show some basic properties of $\mathcal{M}(v)$.

Lemma 2 *For every game $v \in \Gamma(N)$, the intermediate set $\mathcal{M}(v)$ is nonempty and*

$$\mathcal{C}(v) \subseteq \mathcal{M}(v) \subseteq \mathcal{W}(v), \tag{6}$$

where both inclusions may be strict. If v is supermodular, then $\mathcal{C}(v) = \mathcal{M}(v) = \mathcal{W}(v)$. Moreover, we have

$$\mathcal{W}(v) = \operatorname{conv} \mathcal{M}(v).$$

Proof Due to [15, Corollary 8.10, Theorem 9.13] we have that $\mathcal{M}(v)$ is nonempty. Inclusion (6) follows from the relation $\hat{\partial} f(x) \subseteq \partial f(x) \subseteq \overline{\partial} f(x)$ and the equalities from Proposition 3. The last part is a consequence of [15, Theorem 8.49]. □

From the viewpoint of game theory, it makes sense to evaluate superdifferentials of Lovász extensions for core-like solutions at χ_N since it conforms with the idea of marginal contributions to the grand coalition N. On the other hand, from the computational point of view it may be easier to compute the limiting superdifferential at the origin 0 since $\hat{v}$ is a positively homogeneous function.

Lemma 3 *The following identity is satisfied for every game $v \in \Gamma(N)$:*

$$\mathcal{M}(v) = \partial \hat{v}(\chi_N) = \partial \hat{v}(0).$$

Proof It follows from the definitions and from the fact that $\hat{v}$ has the same structure around χ_N and 0 due to Lemma 1. □

Putting together Proposition 3 and Lemma 3, we can now summarize the relations between the discussed solutions and the superdifferentials as follows:

$$\begin{aligned} \mathcal{C}(v) &= \hat{\partial} \hat{v}(\chi_N) = \hat{\partial} \hat{v}(0), \\ \mathcal{M}(v) &= \partial \hat{v}(\chi_N) = \partial \hat{v}(0), \\ \mathcal{W}(v) &= \overline{\partial} \hat{v}(\chi_N) = \overline{\partial} \hat{v}(0). \end{aligned}$$

3.2 Characterization by ordered partitions

In this section we are going to prove the main characterization of the intermediate set, Theorem 1. Its purpose is twofold. First, this result shows that the purely analytic definition of intermediate set can be equivalently stated in terms of the combinatorial and order-theoretic properties of a coalition game. Second, it may be better to use Theorem 1 than the definition based on the limiting superdifferential for the computational reasons. In what follows the main tool is the notion of an ordered partition of the player set, which generalizes the permutations $\pi \in \Pi_n$.

An *ordered partition* of the player set N is a partition of N together with a total order on the coalitions forming the partition. Thus every ordered partition of N is just a K-tuple $P := (B_1, \dots, B_K)$ $(K \geq 1)$ of coalitions $\emptyset \neq B_i \subseteq N$ such that $B_i \cap B_j = \emptyset$ $(i \neq j)$ and $B_1 \cup \dots \cup B_K = N$. Note that there is always a total order on the blocks of the partition and thus the notion of an ordered partition is truly different from that of a coalition structure [3], which is just a partition of the player set. Let

$$\mathcal{P} = \{P \mid P \text{ is an ordered partition of } N\}.$$

The family $\mathcal{P}$ is associated with the following scheme of allocating profits $\mathbf{x}$ among the players in a game v:

1. The players may be split into any ordered partition $P = (B_1, \dots, B_K) \in \mathcal{P}$.
2. Each block of players B_k can distribute the total amount

$$\mathbf{x}(B_k) = v(B_1 \cup \dots \cup B_{k-1} \cup B_k) - v(B_1 \cup \dots \cup B_{k-1})$$

to its members, which can be interpreted as the marginal contribution of coalition B_k to the coalition $B_1 \cup \dots \cup B_{k-1}$ with respect to the ordered partition P.
3. No coalition B in a block B_k may improve upon $\mathbf{x}$ while respecting the given order of coalition blocks, that is,

$$\mathbf{x}(B) \geq v(B_1 \cup \dots \cup B_{k-1} \cup B) - v(B_1 \cup \dots \cup B_{k-1}).$$

Note that the players shares the total of $v(N)$ among them as a consequence of the second principle. The distribution procedure explained above has two extreme cases. Assume that the ordered partition P is the finest possible: $P = (\{\pi(1)\}, \dots, \{\pi(n)\})$ for some permutation $\pi \in \Pi_n$. In this case the allocation scheme in a game v leads to the marginal vectors $\mathbf{x}^v(\pi)$ defined by (1). On the contrary, if the partition contains one block only, $P = (N)$, then all the players (and coalitions) are treated equally, which results in distributing payoffs according to the definition of core. Any ordered partition $P = (B_1, \dots, B_K)$ different from the two extreme cases generates a combination of the principle of marginal distribution on the level of blocks with the core-like stability inside each block of the partition, while respecting the given order of coalitions. Such a distribution process is thus always a mixture of the considerations endogenous to B_i and those which are exogenous to B_i. Also for this reason we have coined the term "intermediate set" for $\mathcal{M}(v)$.

Our main result says that $\mathbf{x} \in \mathcal{M}(v)$ if and only if $\mathbf{x}$ is allocated in accordance with the above distribution principles based on some ordered partition P.

Theorem 1 *For every game $v \in \Gamma(N)$,*

$$\mathcal{M}(v) = \bigcup_{P \in \mathcal{P}} \mathcal{M}_P(v), \tag{7}$$

where $\mathcal{M}_P(v)$ with $P = (B_1, \dots, B_K)$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ such that the folowing two conditions hold for every $k = 1, \dots, K$:

$$\mathbf{x}(B_k) = v(B_1 \cup \dots \cup B_{k-1} \cup B_k) - v(B_1 \cup \dots \cup B_{k-1}), \tag{8a}$$

$$\mathbf{x}(B) \geq v(B_1 \cup \dots \cup B_{k-1} \cup B) - v(B_1 \cup \dots \cup B_{k-1}) \quad \textit{for each } B \subseteq B_k. \tag{8b}$$

Proof See Appendix B. □

The preceding result can serve as an alternative definition of $\mathcal{M}(v)$. Since the union in (7) runs over the family $\mathcal{P}$, computing $\mathcal{M}(v)$ can be quite a complex task. It is known that the number of ordered partitions over an n-element set equals the n-th ordered Bell number. For example, in case $n = 4$ there are already 75 ordered partitions of $\{1, 2, 3, 4\}$.

Remark 2 Note that when we choose $K = 1$ and $B_1 = N$, then $P = (N)$ and the relations (8) are precisely those relations defining the core: $\mathcal{M}_P(v) = \mathcal{C}(v)$. Analogously, setting $K = n$ and each B_k to be equal to a singleton yield a single marginal vector (1): $\mathcal{M}_P(v) = \{\mathbf{x}^v(\pi)\}$ for some $\pi \in \Pi_n$ and $P = \{(\pi(1), \dots, \pi(n))\}$.

We will now present two examples. In the first one, we will make use of Theorem 1 to write the general form of the intermediate set for any 3-player coalition game. In the second one, we further build on the first one and present another game where the three presented solution concepts differ in a significant way.

Example 2 Let $N = \{1, 2, 3\}$. In order to simplify the notation for coalitions we will omit the parentheses and commas so that a coalition $\{i, j\}$ is written as ij. The family $\mathcal{P}$ of all ordered partitions over $\{1, 2, 3\}$ is

$$\begin{aligned}\mathcal{P} =& \{(N), (1, 23), (2, 13), (3, 12), (23, 1), (13, 2), (12, 3)\} \cup \\ & \bigcup_{\pi \in \Pi_n} \{(\pi(1), \pi(2), \pi(3))\}.\end{aligned}$$

Let $v \in \Gamma(N)$. For example, the choice $P = (1, 23)$ gives

$$\begin{aligned}\mathcal{M}_P(v) = \{\mathbf{x} \in \mathbb{R}^3 \,|x_1 = v(1), \mathbf{x}(23) = v(N) - v(1), \\ x_2 \geq v(12) - v(1), x_3 \geq v(13) - v(1)\}.\end{aligned}$$

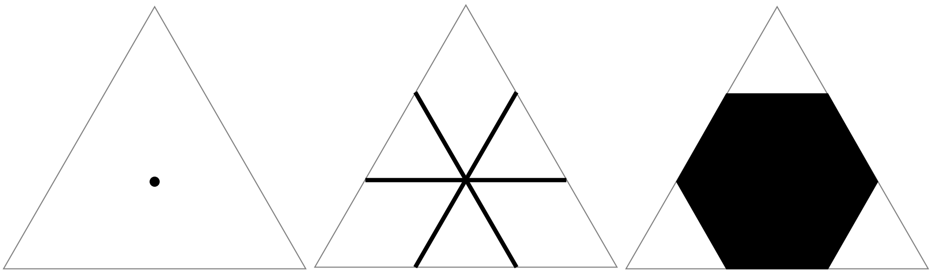
Theorem 1 says that

$$\begin{aligned}\mathcal{M}(v) = \mathcal{C}(v) \cup & \\ & \mathcal{M}_{(1,23)}(v) \cup \mathcal{M}_{(2,13)}(v) \cup \mathcal{M}_{(3,12)}(v) \cup \\ & \mathcal{M}_{(23,1)}(v) \cup \mathcal{M}_{(13,2)}(v) \cup \mathcal{M}_{(12,3)}(v) \cup \\ & \{\mathbf{x}^v(\pi) \mid \pi \in \Pi_n\}.\end{aligned}$$

Example 3 Let $N = \{1, 2, 3\}$ and

$$v(A) = \begin{cases} 0 & \text{if } |A| = 1, \\ 2 & \text{if } |A| = 2, \\ 3 & \text{if } A = N. \end{cases}$$

It is easy to see that v is not supermodular but only superadditive, that is, $v(A \cup B) \geq v(A) + v(B)$ for every $A, B \subseteq N$ with $A \cap B = \emptyset$.



(a) Core (b) Intermediate set (c) Weber set

Fig. 2: The solutions from Example 3 in the barycentric coordinates

The core of this game is single-valued, $\mathcal{C}(v) = \{(1, 1, 1)\}$, while the Weber set $\mathcal{W}(v)$ is the hexagon whose 6 vertices are all the coordinate-wise permutations of the payoff vector $(0, 1, 2)$. The intermediate set is the union of three line segments—see Figure 2. We obtain that $\mathcal{M}_{(i,jk)}(v) = \emptyset$ for every ordered partition (i, jk) of N. On the other hand, $\mathcal{M}_{(ij,k)}(v)$ is the line segment whose endpoints are the two marginal vectors $\mathbf{x}$ with $x_k = 1$. Thus a payoff vector $\mathbf{x}$ is in $\mathcal{M}(v)$ iff it belongs to $\mathcal{M}_{(ij,k)}(v)$ for some ordered partition (ij, k) of N. Note that the example shows that, in general, the intermediate set is not a union of selected faces of the Weber set.

The following lemma presents an additional characterization of supermodularity based on the core solution; cf. Proposition 2.

Lemma 4 *A game v is a supermodular if and only if $\mathcal{C}(v) = \mathcal{M}(v)$. If v is submodular, then $\mathcal{M}(v) = \{\mathbf{x}^v(\pi) \mid \pi \in \Pi_n\}$.*

Proof By Proposition 2 supermodularity is equivalent to $\mathcal{C}(v) = \mathcal{W}(v)$. But this is equivalent to $\mathcal{C}(v) = \mathcal{M}(v)$ since $\mathcal{W}(v) = \operatorname{conv} \mathcal{M}(v)$ and $\mathcal{C}(v)$ is a convex set.

Let v be a submodular game and consider an ordered partition $P = (B_1, \ldots, B_K)$ as in Theorem 1 and let $\mathbf{x} \in \mathcal{M}_P(v)$. If $K = n$, then system (8) generates a marginal vector $\mathbf{x}$. Hence assume that there exists some B_k with $|B_k| \geq 2$. Without loss of generality, we may assume that $k = 2$. For every $i \in B_2$ we obtain

$$\begin{aligned} v(B_1 \cup B_2) - v(B_1) &\overset{\text{(8a)}}{=} \mathbf{x}(B_2) = \mathbf{x}(B_2 \setminus \{i\}) + x_i \\ &\overset{\text{(8b)}}{\geq} v(B_1 \cup B_2 \setminus \{i\}) - v(B_1) + v(B_1 \cup \{i\}) - v(B_1). \end{aligned}$$

By rearranging the previous inequality, we obtain

$$v(B_1 \cup B_2) + v(B_1) \geq v(B_1 \cup B_2 \setminus \{i\}) + v(B_1 \cup \{i\}).$$

Since submodularity provides the converse inequality, we get

$$x_i = v(B_1 \cup \{i\}) - v(B_1).$$

As we obtain this relation for all i, we see that $\mathbf{x}$ is a marginal vector, which finishes the proof. $\square$

Remark 3 The ordered partitions of N are in one-to-one correspondence to strict weak orders on N. Indeed, given $P \in \mathcal{P}$, define a binary relation $\prec_P$ on N as follows: $i \prec_P j$ whenever there are B_k and B_ℓ with $k < \ell$ and $i \in B_k$, $j \in B_\ell$. Otherwise the two elements i and j are incomparable. It is easy to see that $\prec_P$ is a *strict weak order* on N, which means that it satisfies the following conditions:

1. irreflexivity,
2. transitivity,
3. for every $i, j, k \in N$, if i is incomparable with j and j is incomparable with k, then i is incomparable with k.

Conversely, every strict weak order $\prec$ on N gives rise to an ordered partition $\mathcal{P}_\prec$ whose blocks correspond to equivalence classes of incomparability and the order is inherited from $\prec$ in a natural way. The previous results about the representation of the intermediate set can be thus equivalently rephrased in terms of all strict weak orders on the player set. In the light of this interpretation, the core solution corresponds to the unique strict weak order on N in which no pair of players is comparable, while a marginal vector arises from a total order on N.

4 Properties of Intermediate Set

In this section the intermediate set is compared in detail with the core and the Weber set, respectively. We list selected properties and show whether they are satisfied for these solution concepts. Further, we briefly discuss the relation of the intermediate set to other set-valued solutions.

4.1 Comparison with the core and the Weber set

In this subsection some of the properties of the intermediate set are summarized; see Table 1. We follow the approach presented in [12, Section 8.11], where numerous properties and solution concepts are listed together with conditions under which a certain property is satisfied by a given solution concept. For the reader's convenience we repeat the definitions and include the known properties of the core and the Weber set for a direct comparison.

Definition 2 Let $\emptyset \neq \Omega \subseteq \Gamma(N)$. We say that a solution $\sigma \colon \Omega \to 2^{\mathbb{R}^n}$ satisfies

- *nonemptiness (NE)* if $\sigma(v) \neq \emptyset$ for every $v \in \Omega$;
- *convex–valuedness (CON)* if $\sigma(v)$ is convex for every $v \in \Omega$;
- *Pareto optimality (PO)* if $\mathbf{x}(N) = v(N)$ for every $v \in \Omega$ and every $\mathbf{x} \in \sigma(v)$;
- *individual rationality (IR)* if $x_i \geq v(\{i\})$ for every $i \in N$, every $v \in \Omega$ and every $\mathbf{x} \in \sigma(v)$;

- *superadditivity (SUPA)* if $\sigma(v_1) + \sigma(v_2) \subseteq \sigma(v_1 + v_2)$ for every $v_1, v_2 \in \Omega$ such that $v_1 + v_2 \in \Omega$;
- *subadditivity (SUBA)* if $\sigma(v_1) + \sigma(v_2) \supseteq \sigma(v_1 + v_2)$ for every $v_1, v_2 \in \Omega$ such that $v_1 + v_2 \in \Omega$;
- *additivity (ADD)* if σ is both subadditive and superadditive;
- *anonymity (AN)* if $\sigma(\pi v) = \pi(\sigma(v))$ for every $v \in \Omega$ and every $\pi \in \Pi_n$ such that $\pi v \in \Omega$, where πv is defined for every $A \subseteq N$ by $\pi v(\{\pi(i) \mid i \in A\}) = v(A)$, and $\pi(\sigma(v)) = \{(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}) \mid \mathbf{x} \in \sigma(v)\}$;
- *equal treatment property (ETP)* if $x_i = x_j$ for every $\mathbf{x} \in \sigma(v)$, every $v \in \Omega$ and any pair of players $i, j \in N$ that are *substitutes* in v, that is, $v(A \cup \{i\}) = v(A \cup \{j\})$, for each $A \subseteq N \setminus \{i, j\}$;
- *reasonableness (RE)* if for every $v \in \Omega$ and for every $\mathbf{x} \in \sigma(v)$ we have $b_i^{min}(v) \leq x_i \leq b_i^{max}(v)$ for all $i \in N$, where

$$b_i^{min} = \min_{A \subseteq N \setminus \{i\}} (v(A \cup \{i\}) - v(A)),$$
$$b_i^{max} = \max_{A \subseteq N \setminus \{i\}} (v(A \cup \{i\}) - v(A));$$

- *covariant under strategic equivalence (COV)* if for every $v, w \in \Omega$, every $\alpha > 0$ and every additive game z such that $w = \alpha v + z$, we have $\sigma(w) = \alpha\sigma(v) + \{(z(\{1\}), \ldots, z(\{n\}))\}$;
- *null player property (NP)* if for every $v \in \Omega$ and every $\mathbf{x} \in \sigma(v)$, we have $x_i = 0$ whenever player i is a *null player*, that is, $v(A \cup \{i\}) = v(A)$ for all $A \subseteq N$;
- *dummy property (DUM)* if for every $v \in \Omega$ and every $\mathbf{x} \in \sigma(v)$ we have $x_i = v(\{i\})$ whenever player i is a *dummy player*, that is, $v(A \cup \{i\}) = v(A) + v(\{i\})$ for all $A \subseteq N \setminus \{i\}$.

	$\mathcal{C}(v)$	$\mathcal{M}(v)$	$\mathcal{W}(v)$
Nonemptiness	•	✓	✓
Convex–valuedness	✓		✓
Pareto optimality	✓	✓	✓
Individual rationality	✓	•	•
Superadditivity	✓		
Subadditivity		✓	✓
Additivity			
Anonymity	✓	✓	✓
Equal treatment property			
Reasonableness	✓	✓	✓
Covariance	✓	✓	✓
Null player property	✓	✓	✓
Dummy property	✓	✓	✓

Table 1: Fulfillment of selected properties. The mark ✓ means that the property is satisfied on $\Omega = \Gamma(N)$, while • means that only a “significant” subclass of games $\Omega \subsetneq \Gamma(N)$ has the corresponding property. The empty space indicates that the property is not satisfied by every game.

Not all the proofs are presented here. We included only those of them which are nontrivial, important or use the concepts of nonsmooth calculus. In all other cases the reader is referred to an analogous comparison [12, Table 8.11.1]. We do not mention the notoriously known facts about the core, nevertheless they are included in Table 1.

Lemma 5 *Both $\mathcal{M}$ and $\mathcal{W}$ satisfy NE.*

Proof Since the limiting and the Clarke superdifferential of a Lipschitz function are nonempty by [15, Corollary 8.10, Theorem 9.13], both $\mathcal{M}(v)$ and $\mathcal{W}(v)$ are nonempty for any game v. □

It follows directly from the corresponding definitions that both $\mathcal{C}$ and $\mathcal{W}$ satisfy CON. However, the set $\mathcal{M}(v)$ does not have to be convex; see Example 1. Since PO is satisfied by $\mathcal{W}$, it is also satisfied by any smaller solution concept. The example below shows that neither $\mathcal{M}$ nor $\mathcal{W}$ satisfy IR.

Example 4 Let $N = \{1, 2\}$ and v be a game such that $v(\{1\}) = v(\{2\}) = 1$ and $v(N) = 0$. Then it is easy to see that $\mathcal{M}(v) = \{(1, -1), (-1, 1)\}$, which is a non-convex set.

However, in the next lemma we show that IR holds true for $\mathcal{M}$ and $\mathcal{W}$ on a large subclass of games, which includes all superadditive games.

Lemma 6 *Both $\mathcal{M}$ and $\mathcal{W}$ satisfy IR on the following class of games:*

$\Gamma^*(N) = \{v \in \Gamma(N) \mid v(A \cup \{i\}) \geq v(A) + v(\{i\})$ *for all* $A \subseteq N$ *and* $i \in N \setminus A\}$.

Proof Let $v \in \Gamma^*(N)$. It is easy to see from Theorem 1 that any $\mathbf{x} \in \mathcal{M}(v)$ satisfies

$$x_i \geq v(A \cup \{i\}) - v(A) \geq v(\{i\})$$

for every $i \in N$ and every $A \subseteq N \setminus \{i\}$. The proof is analogous for the Weber set. □

Remark 4 The games in $\Gamma^*(N)$ are called *zero-monotonic* or *weakly superadditive*. The class $\Gamma^*(N)$ is investigated in [13], where the authors show that the condition $v \in \Gamma^*(N)$ is equivalent to external stability of $\mathcal{W}(v)$, among others.

Concerning SUPA, SUBA and ADD, the proofs are consequences of the general results about superdifferentials/subdifferentials.

Lemma 7 *$\mathcal{M}$ and $\mathcal{W}$ are subadditive and none of them is additive, in general.*

Proof We can apply the superdifferential sum rule [15, Corollary 10.9, Exercise 10.10] directly to the Lovász extension of a game to obtain this result. □

Anonymity holds true for both $\mathcal{M}$ and $\mathcal{W}$ due to Proposition 3 and Lemma 3, since all the discussed superdifferentials have an analogous property. Since ETP is in general violated by $\mathcal{C}$, it cannot hold for any larger solution concept. Similarly, property RE is true for $\mathcal{W}$ and thus for any solution σ included in $\mathcal{W}$.

Concerning COV, we will first show the following lemma and then a proof that COV holds for all the three solution concepts.

Lemma 8 *If v is an additive game, then $\hat{v}$ is a linear function.*

Proof Consider any vector $\mathbf{x} \in \mathbb{R}^n$ with all the coordinates different. According to (5),

$$\hat{v}(\mathbf{x}) = \sum_{i=1}^{n} x_{\pi(i)} \left(v(V_i^\pi(\mathbf{x})) - v(V_{i-1}^\pi(\mathbf{x})) \right) = \sum_{i=1}^{n} x_{\pi(i)} v(\{\pi(i)\}) = \sum_{i=1}^{n} x_i v(\{i\}).$$

Since $\hat{v}$ is continuous, this formula holds true for any $\mathbf{x} \in \mathbb{R}^n$ so that $\hat{v}$ is indeed linear. □

Lemma 9 *$\mathcal{C}$, $\mathcal{M}$ and $\mathcal{W}$ satisfy COV.*

Proof Let $v, w \in \Omega$ and $\alpha > 0$ be such that $w = \alpha v + z$, where z is an additive game. Since the mapping $v \in \Gamma(N) \mapsto \hat{v}$ is linear, we obtain

$$\hat{w}(\mathbf{x}) = \alpha \hat{v}(\mathbf{x}) + \hat{z}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n.$$

Additivity of z implies linearity of $\hat{z}$ due to Lemma 8. The sought result is a consequence of the superdifferential sum rule [15, Exercise 8.8]. □

As regards the null player property, we have $x_i = 0$ for any marginal vector $\mathbf{x}$ and a null player $i \in N$ in a game v. NP is preserved by passing to the convex hull and thus $x_i = 0$ for every $\mathbf{x} \in \mathcal{W}(v)$, which was to be proved. Since NP and COV implies DUM by [12, Remark 4.1.18], we have completed the whole Table 1.

4.2 Relation to other solution concepts

We will briefly comment on the relation between the intermediate set and selected solution concepts for coalition games. Our sample contains only those candidates that bear a formal resemblance to the intermediate set or those solutions that contain the core. We omit the discussion of the solutions whose position with respect to the intermediate set is clear due to a known result, such as the selectope, which is always at least as large as the Weber set [6]. For the sake of brevity we do not repeat definitions of the discussed solutions, but refer to the literature instead.

Solutions for Coalition Structures A *coalition structure* in an n-person game is an (unordered) partition $\{B_1, \ldots, B_K\}$ of the player set N. Although coalition structures of Aumann and Dreze [3] are used to define various solution concepts such as the core, they differ from the intermediate set in many aspects. Namely the payoff vectors $\mathbf{x}$ associated with games on coalition structures usually satisfy Pareto optimality locally, that is, $\mathbf{x}(B_i) = v(B_i)$ for each block B_i of the partition. This is certainly not the case of a payoff $\mathbf{x} \in \mathcal{M}_P(v)$ since the coalition B_i takes into account its position in an ordered partition $P = (B_1, \ldots, B_K)$ due to the condition (8a). Another point of dissimilarity is that in the core of a game with a coalition structure

$\{B_1, \ldots, B_K\}$, the condition $\mathbf{x}(A) \geq v(A)$ with $A \subseteq N$ goes across all the blocks of partition, while (8b) applies only to the coalitions inside a given block.

Equal Split-Off Set (ESOS) This solution concept is also based on ordered partitions and may attain non-convex values; see [4, Section 4.2]. It follows from Example 1 that $\mathcal{M}(v)$ is not contained in the ESOS of v. Moreover, the additive game from Example 4.2(iv) in [4] shows that ESOS is not a part of $\mathcal{M}$ either.

Equal Division Core (EDC) The solution EDC is another non-convex solution concept, which was introduced by Selten in [17] and consists of "efficient payoff vectors for the grand coalition which cannot be improved upon by the equal division allocation of any subcoalition". Using Example 1 we can show that EDC of v does not contain and is not contained in $\mathcal{M}(v)$: the EDC of this game coincides with the set

$$\{\mathbf{x} \in \mathcal{I}(v) \mid x_1 \geq \tfrac{1}{2} \vee (x_2 \geq \tfrac{1}{2} \wedge x_3 \geq \tfrac{1}{2})\},$$

where $\mathcal{I}(v) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N), x_i \geq v(\{i\}),\ i \in N\}$ is the set of all *imputations* in game v.

Core Cover (CC) This solution was introduced by Tijs and Lipperts [21]. Example 1 yields that CC of the glove game coincides with the core and thus it is strictly smaller than the corresponding intermediate set. The converse strict inclusion is rendered by Example 1 in [21].

Reasonable Set (RS) See [22] for details. Since the intermediate set has the property RE from Definition 2, it holds true that $\mathcal{M}(v)$ is included in $\mathrm{RS}(v)$ whenever $v \in \Gamma^*(N)$.

Dominance Core (DC) The solution DC is the set of all undominated imputations in the game. If $v \in \Gamma^*(N)$ and $\mathrm{DC}(v) \neq \emptyset$, then [4, Theorem 2.13] yields $\mathcal{C}(v) = \mathrm{DC}(v)$ and thus $\mathcal{M}(v)$ contains $\mathrm{DC}(v)$.

In summary, the only remarkable relations are rendered by the last two items: for every game $v \in \Gamma^*(N)$, we have $\mathrm{DC}(v) \subseteq \mathcal{M}(v) \subseteq \mathrm{RS}(v)$.

5 Examples

In this section we analyze three types of games and simplify the formula for the intermediate set from Theorem 1.

5.1 Simple games

In this subsection, we will compute the intermediate set for the class of all simple games. Then we will compare our results to a formula for the core. A game $v \in \Gamma(N)$ is *monotone* if $v(A) \leq v(B)$ whenever $A \subseteq B \subseteq N$ and v is called *simple* if it is monotone with $v(A) \in \{0, 1\}$ and $v(N) = 1$. Every simple game v over the player set N can be identified with the family $\mathcal{V}$ of *winning coalitions* in v as follows:

$$\mathcal{V} = \{A \subseteq N \mid v(A) = 1\}.$$

Conversely, any system of coalitions $\mathcal{V}$ such that $N \in \mathcal{V}$, $\emptyset \notin \mathcal{V}$ and

$$A \subseteq B \subseteq N,\ A \in \mathcal{V} \Rightarrow B \in \mathcal{V},$$

gives rise to a simple game v by putting $v(A) = 1$ if $A \in \mathcal{V}$ and $v(A) = 0$, otherwise. The family of *minimal winning coalitions* in v is

$$\mathcal{V}^m = \{A \in \mathcal{V} \mid B \subsetneq A \Rightarrow B \notin \mathcal{V},\ \text{for every } B \subseteq N\}.$$

Based on the concept of minimal winning coalitions, we are able derive the following formula for $\mathcal{M}(v)$. It states that $\mathcal{M}(v)$ arises as a union of faces of the standard simplex, where each face corresponds to one minimal winning coalition.

Theorem 2 *If $v \in \Gamma(N)$ is a simple game, then*

$$\mathcal{C}(v) = \bigcap_{E \in \mathcal{V}^m} \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x_i = 0 & \text{if } i \in N \setminus E \\ x_i \geq 0 & \text{if } i \in E \\ \sum_{i \in E} x_i = 1 & \end{array} \right\}, \tag{9a}$$

$$\mathcal{M}(v) = \bigcup_{E \in \mathcal{V}^m} \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x_i = 0 & \text{if } i \in N \setminus E \\ x_i \geq 0 & \text{if } i \in E \\ \sum_{i \in E} x_i = 1 & \end{array} \right\}. \tag{9b}$$

Proof The formula for core on simple games (9a) can be derived easily; see [11, Example X.4.6], for instance.

Denote the right–hand side of (9b) by C. Let $\mathbf{x} \in C$, choose any E such that $\mathbf{x}$ lies in the simplex generated by E and define $B_1 := E$, $B_2 := N \setminus E$. We will show that $K = 2$ and B_1, B_2 satisfy relation (8), implying that $\mathbf{x} \in \mathcal{M}(v)$. Due to the construction of C, we have $v(B_1) = v(B_1 \cup B_2) = 1$. This means that from (8) for $k = 2$ we obtain $x_i = 0$ for all $i \in B_2$. Consider thus $k = 1$ and observe that relation (8a) is in this case equivalent to $\mathbf{x}(E) = 1$. Since E is a minimal winning coalition, relation (8b) reads as $\mathbf{x}(B) \geq 0$ for all $B \subsetneq E$, which is satisfied. Thus $x \in \mathcal{M}(v)$.

For showing the converse inclusion, let $\mathbf{x} \in \mathcal{M}(v)$. By Theorem 1 there is an ordered partition $(B_1, \dots, B_K)$ satisfying (8). Denote by l the smallest integer such that $v(B_1 \cup \cdots \cup B_l) = 1$. Find now any $E \in \mathcal{V}^m$ such that $E \subseteq B_1 \cup \cdots \cup B_l$. From (8b) with $k = l$ we see that $\mathbf{x}(E) \geq 1$, which turns into $\mathbf{x}(E) = 1$ due to (8a). Since $\mathbf{x}(N) = 0$ and $x_i \geq 0$ for all $i \in N$, we have finished the proof. □

Remark 5 Formulas (9) are interesting also from the point of variational analysis. While from Definition 4 we see that the limiting superdifferential is a union of the Fréchet ones with respect to a suitable neighborhood, the previous theorem states that in a special case the Fréchet superdifferential can be written as an intersection of the limiting ones. This is a relation which does not hold true in general.

We will compute the intermediate set of the UN Security Council voting scheme; see e.g. [11, Example XI.2.9].

Example 5 The UN Security Council contains 5 permanent members with veto power and 10 non–permanent members. To pass a resolution, all the permanent members and at least 4 non–permanent members have to vote for the proposal. We assume that the players $N = \{1, \dots, 15\}$ are ordered in such a way that the first five are the permanent members and the last ten are the non–permanent members. Then it is easy to show that the corresponding simple game v satisfies

$$\mathcal{C}(v) = \left\{ \mathbf{x} \in \mathbb{R}^{15} \;\middle|\; \mathbf{x} \geq 0,\ \sum_{i=1}^{5} x_i = 1,\ x_i = 0 \text{ for } i = 6, \dots, 15 \right\},$$
$$\mathcal{W}(v) = \left\{ \mathbf{x} \in \mathbb{R}^{15} \;\middle|\; \mathbf{x} \geq 0,\ \sum_{i=1}^{15} x_i = 1 \right\}.$$

As a consequence of stability of core allocations, any payoff $\mathbf{x} \in \mathcal{C}(v)$ is distributed only among the permanent members. On the other hand, the Weber set is the whole 14-dimensional standard simplex in $\mathbb{R}^{15}$, which is too large and contains some payoff vectors whose meaning is problematic. For instance, it is not entirely clear how to interpret a vector

$$\left(0, \dots, 0, \tfrac{1}{10}, \dots, \tfrac{1}{10}\right) \in \mathcal{W}(v).$$

As we will see, this vector is not contained in $\mathcal{M}(v)$.

Given $i \in N$, denote by $\mathbf{e}_i \in \mathbb{R}^{15}$ the vector whose coordinates are $e_j = 1$ if $j = i$ and $e_j = 0$ otherwise. Put

$$\mathcal{D} = \{D \subseteq \{6, \dots, 15\} \mid |D| = 4\}.$$

Theorem 2 yields

$$\mathcal{M}(v) = \bigcup_{D \in \mathcal{D}} \operatorname{conv}\left(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} \cup \{\mathbf{e}_i \mid i \in D\}\right).$$

In other words, $\mathcal{M}(v)$ is a union of $\binom{10}{4}$ 8-dimensional standard simplices, each of which is a convex hull of $\mathbf{e}_i$s corresponding to the five permanent members and four other non–permanent members. These simplices are associated with the ordered partitions having two blocks, $(12345 \cup D, N \setminus (12345 \cup D))$ where $D \in \mathcal{D}$.

5.2 Clan games

We will recover the intermediate set for a class of games which are very close to simple games, the so-called clan games; see [4, Section 5.3].

Definition 3 We say that $v \in \Gamma(N)$ is a *clan game* if there exists a subset $C \subseteq N$ such that $C \notin \{\emptyset, N\}$ and the following properties are satisfied:

$$v(A) \geq 0 \text{ for all } A \subseteq N, \tag{10a}$$
$$v(N) - v(N \setminus \{i\}) \geq 0 \text{ for all } i \in N, \tag{10b}$$
$$v(A) = 0 \text{ for each } A \not\supseteq C, \tag{10c}$$
$$v(N) - v(A) \geq \sum_{i \in N \setminus A} [v(N) - v(N \setminus \{i\})] \text{ whenever } A \supseteq C. \tag{10d}$$

If (10d) is replaced with the following stronger property,

$$v(B) - v(A) \geq \sum_{i \in B \setminus A} [v(B) - v(B \setminus \{i\})] \text{ whenever } B \supseteq A \supseteq C,$$

then we call v a *total clan game*.

The properties (10a)–(10d) are known as nonnegativity, nonnegative marginal contributions to the grand coalition, clan property and union property, respectively. Set C is called a clan. If v is a total clan game, we are able to simplify substantially the formula from Theorem 1. Note that constraints (11b) below are the box constraints and thus system (11) is easily solvable.

Theorem 3 *Let $v \in \Gamma(N)$ be a total clan game and let $P = (B_1, \ldots, B_K)$ be an ordered partition of N. Then $\mathcal{M}_P(v)$ given by (8) is empty whenever $B_1 \not\supseteq C$ and there is $B \subseteq B_1$ with $v(B) > 0$. If $B_1 \supseteq C$, then $\mathcal{M}_P(v)$ is the set of payoff vectors $\mathbf{x}$ such that*

$$\mathbf{x}(B_k) = v(B_1 \cup \cdots \cup B_{k-1} \cup B_k) - v(B_1 \cup \cdots \cup B_{k-1}), \tag{11a}$$
$$v(\{i\}) \leq x_i \leq v(B_1 \cup \cdots \cup B_{k-1} \cup B_k) - v(B_1 \cup \cdots \cup B_{k-1} \cup B_k \setminus \{i\}), \tag{11b}$$

for every $k = 1, \ldots, K$ and all $i \in B_k$.

Proof If $B_1 \not\supseteq C$, then from (10c) we obtain $v(B_1) = 0$ and thus $\mathbf{x}(B_1) = 0$ by (8a). But then (8) cannot have any solution because $x_i \geq v(\{i\}) \geq 0$ for all $i \in B_k$ due to (10a) and $v(B) > 0$ for some $B \subseteq B_1$. For the rest of the proof, assume that $B_1 \supseteq C$.

Consider first any $\mathbf{x}$ which satisfies (8). Then, (11a) is directly (8a). Formula (11b) follows from

$$\begin{aligned} x_i = \mathbf{x}(B_k) - \mathbf{x}(B_k \setminus \{i\}) &\leq v(B_1 \cup \cdots \cup B_{k-1} \cup B_k) - v(B_1 \cup \cdots \cup B_{k-1}) \\ &\quad - v(B_1 \cup \ldots B_{k-1} \cup B_k \setminus \{i\}) + v(B_1 \cup \cdots \cup B_{k-1}), \end{aligned}$$

and the inequality $x_i \geq v(\{i\})$ is obvious.

Conversely, let $\mathbf{x}$ satisfy (11). Then (8a) is of the form (11a) and it suffices to show (8b). If $k = 1$, then the result follows directly from [4, Proposition 5.31]. If $k \geq 2$, fix any $B \subseteq B_k$. Then

$$\begin{aligned}\mathbf{x}(B) = \mathbf{x}(B_k) - \sum_{i \in B_k \setminus B} x_i &\overset{(11b)}{\geq} v(B_1 \cup \dots \cup B_{k-1} \cup B_k) - v(B_1 \cup \dots \cup B_{k-1}) \\ &- \sum_{i \in B_k \setminus B} [v(B_1 \cup \dots \cup B_{k-1} \cup B_k) - v(B_1 \cup \dots \cup B_{k-1} \cup B_k \setminus i)] \\ &\overset{(10)}{\geq} v(B_1 \cup \dots \cup B_{k-1} \cup B) - v(B_1 \cup \dots \cup B_{k-1}),\end{aligned}$$

which was to be proved. Note that we were allowed to use (10) since $B_1 \cup \dots \cup B_{k-1} \supseteq B_1 \supseteq C$ due to the assumption $k \geq 2$. □

As a corollary we directly obtain the following result for the core. In our setting we would be able to prove it only for total clan games. For the proof for clan games we refer to [4, Proposition 5.31].

Corollary 1 *Consider a clan game $v \in \Gamma(v)$. Then we have*

$$\mathcal{C}(v) = \left\{ \mathbf{x} \;\middle|\; \begin{array}{l} \mathbf{x}(N) = v(N) \\ v(\{i\}) \leq x_i \leq v(N) - v(N \setminus \{i\}) \text{ for all } i \in N \end{array} \right\}.$$

5.3 Glove game

In the previous subsections we have managed to compute $\mathcal{M}(v)$ for the classes of simple and total clan games. In this subsection, we will perform the same task for the glove game, which belongs to the class of assignment games [19]. In the glove game, there are $n = p + q$ players and each of them has a glove: either a left one or a right one. When a subset of players forms a coalition, then their joint profit is the number of glove pairs owned together. Specifically, assume that L is the set of all players having the left glove and R is the set of all players having the right glove. Then

$$v(A) = \min\{|A \cap L|, |A \cap R|\}.$$

Without loss of generality, we always assume that $L = \{1, \dots, p\}$, $R = \{p + 1, \dots, p + q\}$ and $p \geq q$.

Although the shape of core for glove game is known, we will reprove the formula for $\mathcal{C}(v)$ and based on it, we will employ Theorem 1 to compute also $\mathcal{M}(v)$.

Lemma 10 *If $p > q$, then $\mathcal{C}(v)$ consists of a single point $\mathbf{x}$ with the following coordinates: $x_l = 0$ for all $l \in L$ and $x_r = 1$ for all $r \in R$.*

Proof Clearly, any $\mathbf{x}$ from the lemma statement satisfies $\mathbf{x} \in \mathcal{C}(v)$. On the other hand, let $\mathbf{x} \in \mathcal{C}(v)$. Then the definition of core yields $\mathbf{x}(L \cup R) = q$. Decompose L into $L = L_1 \cup L_2$ such that $|L_1| = q$ and $|L_2| = p - q \geq 1$. Then $\mathbf{x}(L_1 \cup R) \geq q$, which immediately implies $\mathbf{x}(L_1 \cup R) = q$ and $\mathbf{x}(L_2) = 0$. Since L_2 was chosen in an arbitrary way, we have $\mathbf{x}(L) = 0$. Further, for every $r \in R$ we can deduce $\mathbf{x}(\{1, r\}) \geq v(\{1, r\}) = 1$, which together with $x_1 = 0$ and $\mathbf{x}(R) = q$ implies $x_r = 1$. $\square$

Lemma 11 *If $p = q$, then $\mathcal{C}(v) = \operatorname{conv}\{\chi_L, \chi_R\}$, where χ_L, χ_R are defined by* (2).

Proof Similarly as in the previous lemma, it is not difficult to verify that $\chi_L, \chi_R \in \mathcal{C}(v)$ and thus $\mathcal{C}(v) \supseteq \operatorname{conv}\{\chi_L, \chi_R\}$. Conversely, let $\mathbf{x} \in \mathcal{C}(v)$. Using the definition of core, we obtain $\mathbf{x}(L \cup R) = q$ and $\mathbf{x}(\{l, r\}) \geq 1$ for all $l \in L$ and $r \in R$. But summing q such terms results in $\mathbf{x}(\{l, r\}) = 1$, which further implies $x_{l_1} = x_{l_2}$ for all $l_1, l_2 \in L$ and $x_{r_1} = x_{r_2}$ for all $r_1, r_2 \in R$. This means that $\mathbf{x} = (\lambda, \dots, \lambda, 1 - \lambda, \dots, 1 - \lambda)$ for some $\lambda \in [0, 1]$. $\square$

We will provide a simple way of determining the solution of (8). Note that if $p_{k-1} = q_{k-1}$, then system (8) can be computed directly from Lemmas 10 and 11.

Lemma 12 *Let $(B_1, \dots, B_K)$ be an ordered partition of N. Given $k = 1, \dots, K$, let p_k and q_k be the number of left and right gloves, respectively, owned by $B_1 \cup \dots \cup B_k$.*

- *If $p_{k-1} > q_{k-1}$ and $p_k < q_k$, then system* (8) *does not have a feasible solution.*
- *If $p_{k-1} > q_{k-1}$ and $p_k \geq q_k$, then $\mathbf{x}$ is a solution to system* (8) *if and only if $x_l = 0$ for all $l \in B_k \cap L$ and $x_r = 1$ for all $r \in B_k \cap R$.*
- *If $p_{k-1} < q_{k-1}$ and $p_k > q_k$, then system* (8) *does not have a feasible solution.*
- *If $p_{k-1} < q_{k-1}$ and $p_k \leq q_k$, then $\mathbf{x}$ is a solution to system* (8) *if and only if $x_l = 1$ for all $l \in B_k \cap L$ and $x_r = 0$ for all $r \in B_k \cap R$.*

Proof We will prove only the first two statements since the proof of the last two assertions is completely analogous. Assume that $p_{k-1} > q_{k-1}$ and consider any solution $\mathbf{x}$ of (8). Then we have

$$\mathbf{x}(B_k) = \min\{p_k, q_k\} - \min\{p_{k-1}, q_{k-1}\} = \min\{p_k, q_k\} - q_{k-1}. \tag{12}$$

Taking $B = \{r\}$ for any $r \in B_k \cap R$ results in $x_r \geq 1$. Similarly, by taking $B = \{l\}$ for $l \in B_k \cap L$ we get $x_l \geq 0$. This results in

$$\mathbf{x}(B_k) = \mathbf{x}(B_k \cap L) + \mathbf{x}(B_k \cap R) \geq 0 + (q_k - q_{k-1}) = q_k - q_{k-1}. \tag{13}$$

Combining formulas (12) and (13) leads to

$$q_k \leq \min\{p_k, q_k\}. \tag{14}$$

If $p_k < q_k$, then formula (14) cannot be satisfied and thus, system (8) does not have any feasible solutions. On the other hand, if $p_k \geq q_k$, then from (12) we see that $\mathbf{x}(B_k) = q_k - q_{k-1}$, and (13) further implies that $\mathbf{x}(B_k \cap L) = 0$ and $\mathbf{x}(B_k \cap R) = q_k - q_{k-1}$. But this means that $x_r = 1$ for all $r \in B_k \cap R$ and one inclusion has been proved.

To finish the proof, we must show that for $p_k \geq q_k$ and for $\mathbf{x}$ with $x_l = 0$ for all $l \in B_k \cap L$ and $x_r = 1$ for all $r \in B_k \cap R$, the payoff vector $\mathbf{x}$ solves (8). Then

$$\begin{aligned}\mathbf{x}(B_k) &= q_k - q_{k-1} = \min\{p_k, q_k\} - \min\{p_{k-1}, q_{k-1}\} \\ &= v(B_1 \cup \cdots \cup B_{k-1} \cup B_k) - v(B_1 \cup \cdots \cup B_{k-1}).\end{aligned}$$

Consider any $B \subseteq B_k$ and assume that B contains a players with left gloves and b players with right gloves. Then

$$\begin{aligned}\mathbf{x}(B) = b &\geq \min\{a + p_{k-1} - q_{k-1}, b\} = \min\{a + p_{k-1}, b + q_{k-1}\} - q_{k-1} \\ &= \min\{a + p_{k-1}, b + q_{k-1}\} - \min\{p_{k-1}, q_{k-1}\} \\ &= v(B_1 \cup \cdots \cup B_{k-1} \cup B) - v(B_1 \cup \cdots \cup B_{k-1}),\end{aligned}$$

which concludes the proof. □

We will prove the main theorem of this section. It says that every $\mathbf{x} \in \mathcal{M}(v)$ can be generated via Theorem 1 by choosing coalitions $B_1, \ldots, B_{q+1}$ such that: (i) $B_1, \ldots, B_q$ are 2-player coalitions containing a pair of players each of which owns one right and one left glove, respectively, (ii) the coalition B_{q+1} contains only the players possessing left gloves or $B_{q+1} = \emptyset$ if $p = q$, that is, $B_{q+1} \subseteq L$.

Theorem 4 *Let $\mathbf{x} \in \mathbb{R}^n$. Then $\mathbf{x} \in \mathcal{M}(v)$ if and only if there exists $\tilde{L} \subseteq L$ with $|\tilde{L}| = q$ and a bijection $\rho\colon \tilde{L} \to R$ such that the following conditions are satisfied:*

$$x_l + x_{\rho(l)} = 1 \text{ for all } l \in \tilde{L}, \tag{15a}$$

$$x_l \geq 0 \text{ for all } l \in \tilde{L}, \tag{15b}$$

$$x_l = 0 \text{ for all } l \in L \setminus \tilde{L}. \tag{15c}$$

Proof Let $\mathbf{x}$ satisfy (15). We can enumerate the elements of $\tilde{L}$ as $l_1, \ldots, l_q$ and define the coalitions

$$B_1 = \{l_1, \rho(l_1)\}, \ldots, B_q = \{l_q, \rho(l_q)\}, B_{q+1} = L \setminus \tilde{L}.$$

Then it is easy to verify that $\mathbf{x} \in \mathcal{M}(v)$ due to Theorem 1 and using the partition above.

For the proof of the second inclusion, denote by p_k the number of left gloves owned by players $B_1 \cup \cdots \cup B_k$ and by q_k the number of right gloves owned by the same players. Put $p_0 = q_0 = 0$. To prove the statement, we will construct ρ by a variant of finite induction. There are three possibilities: $p_1 = q_1$, $p_1 < q_1$ or $p_1 > q_1$.

If $p_1 = q_1$, then define two sets $L_1 := B_1 \cap L$ and $R_1 := B_1 \cap R$. Lemma 11 states that $\mathbf{x}$ is a solution to system (8) if and only if there exists $\lambda \in [0, 1]$

such that $x_l = \lambda$ for all $l \in L_1$ and $x_r = 1-\lambda$ for all $r \in R_1$. Since $|L_1| = |R_1|$, we can define a bijection $\rho\colon L_1 \to R_1$. Now observe that

$$x_l + x_{\rho(l)} = \lambda + (1-\lambda) = 1$$

for every $l \in L_1$. Hence, (15a)–(15b) holds true for L_1.

If $p_1 > q_1$, then we deduce from Lemma 12 that there are two possibilities: either there exists $k > 0$ such that $p_1 > q_1, \dots, p_{k-1} > q_{k-1}$ with $p_k = q_k$ or $p_1 > q_1, \dots, p_K > q_K$. We will consider only the first possibility and return to the second one at the end of the proof. Define $L_1 := (B_1 \cup \dots \cup B_k) \cap L$ and $R_1 := (B_1 \cup \dots \cup B_k) \cap R$. Due to Lemmas 10 and 12 this implies that $x_l = 0$ for all $l \in L_1$ and $x_r = 1$ for all $r \in R_1$. But since $p_k - p_0 = q_k - q_0$, there is a bijection ρ between L_1 and R_1 and, similarly as in the case $p_1 = q_1$, we observe that $x_l + x_{\rho(l)} = 1$ and $x_l \geq 0$ for all $l \in L_1$.

If $p_1 < q_1$, we will proceed as in the case $p_1 > q_1$. Note that due to our assumption that there are more left gloves than right gloves ($p_K > q_K$), it cannot happen that $p_1 < q_1, \dots, p_K < q_K$.

Applying this procedure multiple times, we have managed to find an index k, sets $\hat{L}$ and $\hat{R}$ and a bijection $\rho\colon \hat{L} \to \hat{R}$ such that the following properties are satisfied:

1. $p_k = q_k$ and $p_{k+1} > q_{k+1}, \dots, p_K > q_K$,
2. $x_l + x_{\rho(l)} = 1$ and $x_l \geq 0$ for all $l \in \hat{L}$,
3. $\hat{L} \cup \hat{R} = B_1 \cup \dots \cup B_k$ and $|\hat{L}| = |\hat{R}| = p_k$.

The rest of the proof is straightforward. From Lemma 10 we obtain that $x_l = 0$ for all $l \in L \setminus \hat{L}$ and $x_r = 1$ for all $r \in R \setminus \hat{R}$. Find any $L' \subseteq L \setminus \hat{L}$ such that $|L'| = |R \setminus \hat{R}|$, define $\tilde{L} := \hat{L} \cup L'$ and extend bijection $\rho\colon \hat{L} \to \hat{R}$ to a bijection $\rho\colon \tilde{L} \to R$. Then any such $\tilde{L}$ and ρ satisfy (15), which completes the proof. $\square$

The intermediate set $\mathcal{M}(v)$ is a finite union of convex polytopes. We will now compute these polytopes for the case $p > q$. Denote by $\hat{\mathbf{x}}$ the unique vector in $\mathcal{C}(v)$, thus $\hat{x}_l = 0$ for all $l \in L$ and $\hat{x}_r = 1$ for all $r \in R$, and define the following set of bijections:

$$\Psi := \{\rho\colon \tilde{L}^\rho \to R \mid \tilde{L}^\rho \subseteq L,\ \rho \text{ is a bijection}\}.$$

For every $\rho \in \Psi$ introduce q vectors as follows: given $l \in \tilde{L}^\rho$, put $\mathbf{x}^{\rho,l}$ to be equal to $\hat{\mathbf{x}}$ except two coordinates, specifically $x_l^{\rho,l} = 1$ and $x_{\rho(l)}^{\rho,l} = 0$. Finally, set

$$B^\rho := \operatorname{conv}\left\{\hat{\mathbf{x}}, \bigcup_{l \in \tilde{L}^\rho} \mathbf{x}^{\rho,l}\right\}.$$

Corollary 2 *Using the notation above, assume that $p > q$. Then*

$$\mathcal{M}(v) = \bigcup_{\rho \in \Psi} B^\rho.$$

Proof Fix $\rho \in \Psi$ and observe that $\mathbf{x} \in B^\rho$ is equivalent to the existence of $\lambda_l \geq 0$ for $l \in \tilde{L}^\rho$ with $\sum_{l\in\tilde{L}^\rho} \lambda_l \leq 1$ such that

$$\mathbf{x} = \left(1 - \sum_{l\in\tilde{L}^\rho} \lambda_l\right)\hat{\mathbf{x}} + \sum_{l\in\tilde{L}^\rho} \lambda_l \mathbf{x}^{\rho,l}.$$

But this is equivalent to our claim by Theorem 4. □

Example 6 Consider the glove game with $N = \{1, \dots, 7\}$, $L = \{1, \dots, 4\}$ and $R = N \setminus L$. Let $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$ and put

$$\begin{aligned}
\mathbf{x} &:= (0, \lambda_1, \lambda_2, \lambda_3, 1 - \lambda_1, 1 - \lambda_2, 1 - \lambda_3),\\
\hat{\mathbf{x}} &:= (0, 0, 0, 0, 1, 1, 1),\\
\mathbf{x}_1 &:= (0, 1, 0, 0, 0, 1, 1),\\
\mathbf{x}_2 &:= (0, 0, 1, 0, 1, 0, 1),\\
\mathbf{x}_3 &:= (0, 0, 0, 1, 1, 1, 0).
\end{aligned}$$

Lemma 10 gives $\mathcal{C}(v) = \{\hat{\mathbf{x}}\}$. Using Theorem 4 we can show that $\mathbf{x} \in \mathcal{M}(v)$. Moreover, Corollary 2 implies that

$$\mathbf{x} = (1 - \lambda_1 - \lambda_2 - \lambda_3)\hat{\mathbf{x}} + \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3.$$

We have shown that $\operatorname{conv}\{\hat{\mathbf{x}}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is one of the polyhedral components of $\mathcal{M}(v)$.

6 Conclusions

We have inserted a new solution concept—the intermediate set—in-between the core and the Weber set. While computing the limiting superdifferential may be a daunting task in general, we were able to arrive at the formula in Theorem 1, which is the main computational tool in this paper. The achieved characterization by ordered partitions of the player set makes it possible to interpret the payoffs in the intermediate set as marginal coalitional contributions determined by some order of coalition blocks and satisfying the conditions (8b).

We will outline some ideas for the future research on this topic.

1. The family $\mathcal{P}$ of all ordered partitions P of the player set N (or, equivalently, the family of all the strict weak orders on N) is in one-to-one correspondence with the set of all nonempty faces of the permutohedron of order n; see [24]. The algebraic structure of the corresponding face lattice determines the geometric composition of the convex components $\mathcal{M}_P(v)$ of the intermediate set $\mathcal{M}(v)$. Namely $\mathcal{M}(v)$ can be viewed as a polyhedral complex whose cells are all $\mathcal{M}_P(v)$ with $P \in \mathcal{P}$. This observation could be vital for studying the following problem, which is motivated by the examples and results presented in the paper, cf. Example 1, Example 3 and Theorem 2: When the core of a coalition game is an intersection of (selected) components $\mathcal{M}_P(v)$?

2. Many solution concepts (the core, the Shapley value etc.) can be axiomatized on various classes of games. Is there an axiomatization of the intermediate set on a suitable class of coalition games?
3. The coincidence of the core with the Weber set is essential for the characterization of extreme rays of the cone of supermodular games presented in [20]. There can be a large gap between the core and the Weber set outside the family of supermodular games. Thus the intermediate set may be a useful tool for describing the properties of games in cones including the supermodular cone such as the cone of exact games or the cone of superadditive games.

Appendix

A Superdifferentials

In this section we will define the selected concepts of variational (nonsmooth) analysis, mainly various superdifferentials which generalize the superdifferential of convex functions. Since these superdifferentials will be computed only for the Lovász extension, which is piecewise linear, we will confine to defining superdifferentials only for such functions. Even though the computation of these objects may be rather a challenging task, see e.g. [1,7], the presented framework allows for a significant simplification. For the general approach based on upper semicontinuous functions, we refer the reader to [15], where a normal cone to a set is constructed and a superdifferential is defined based on it.

The standard monographs on variational analysis [10,14,15] follow the approach usual in convex analysis by dealing with subdifferentials instead of superdifferentials. However, most of the results can be easily transformed to the setting of superdifferentials, usually by reversing inequalities only.

Definition 4 Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a piecewise affine function and $\bar{\mathbf{x}} \in \mathbb{R}^n$. We say that $\mathbf{x}^* \in \mathbb{R}^n$ is a

- *Fréchet supergradient* of f at $\bar{\mathbf{x}}$ if there exists neighborhood $\mathcal{X}$ of $\bar{\mathbf{x}}$ such that for all $\mathbf{x} \in \mathcal{X}$ we have
$$f(\mathbf{x}) - f(\bar{\mathbf{x}}) \leq \langle \mathbf{x}^*, \mathbf{x} - \bar{\mathbf{x}} \rangle;$$
- *limiting supergradient* of f at $\bar{\mathbf{x}}$ if for every neighborhood $\mathcal{X}$ of $\bar{\mathbf{x}}$ there exists $\mathbf{x} \in \mathcal{X}$ such that $\mathbf{x}^*$ is a Fréchet supergradient of f at $\mathbf{x}$;
- *Clarke supergradient* of f at $\bar{\mathbf{x}}$ if
$$\mathbf{x}^* \in \operatorname{conv}\{\mathbf{y} \mid \forall \text{neighborhood } \mathcal{X} \text{ of } \bar{\mathbf{x}} \ \exists \mathbf{x} \in \mathcal{X} \cap D \text{ with } \mathbf{y} = \nabla f(\mathbf{x})\},$$
where
$$D := \{\mathbf{x} \in \mathbb{R}^n \mid f \text{ is differentiable at } \mathbf{x}\}.$$

The collection of all (Fréchet, limiting, Clarke) supergradients of f at $\bar{\mathbf{x}}$ is called *(Fréchet, limiting, Clarke) superdifferential* and it is denoted by $\hat{\partial} f(\bar{\mathbf{x}})$, $\partial f(\bar{\mathbf{x}})$ and $\overline{\partial} f(\bar{\mathbf{x}})$, respectively.

Remark 6 The previous definition can be found e.g. in [15, Definition 8.3]. Note that in the original definition term $o(\|\mathbf{x} - \bar{\mathbf{x}}\|)$ is added. Because we work with piecewise affine functions, this term is superfluous. This also means that the Fréchet superdifferential coincides with the standard superdifferential for convex functions. Similarly, the limiting procedure is simplified for the case of limiting superdifferential.

From the definition it can be seen that

$$\hat{\partial} f(\bar{\mathbf{x}}) \subseteq \partial f(\bar{\mathbf{x}}) \subseteq \overline{\partial} f(\bar{\mathbf{x}}), \quad \bar{\mathbf{x}} \in \mathbb{R}^n,$$

where all the inequalities may be strict. According to [15, Theorem 8.49] we have the following relation between the limiting and the Clarke superdifferential for every piecewise affine function f:

$$\overline{\partial} f(\bar{\mathbf{x}}) = \operatorname{conv} \partial f(\bar{\mathbf{x}}).$$

We will show the differences among the three discussed superdifferentials.

Example 7 Let $f \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in (-\infty, 0], \\ 0 & \text{if } x \in [0, 1], \\ x - 1 & \text{if } x \in [1, \infty). \end{cases}$$

This function is depicted in Figure 3. Consider points $\bar{x} = 0$ and $\bar{y} = 1$. The locally supporting

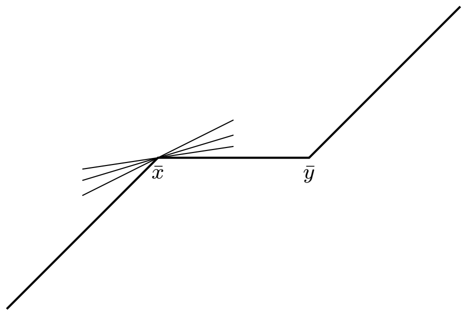


Fig. 3: Supergradients for a piecewise affine function f

hyperplanes from the definition of Fréchet superdifferential at $\bar{x}$ are depicted in the figure. Note that there are no affine majorants for f at $\bar{y}$ and thus the Fréchet superdifferential is empty at this point. Altogether, we obtain

$$\begin{aligned} \hat{\partial} f(\bar{x}) &= [0, 1], & \hat{\partial} f(\bar{y}) &= \emptyset, \\ \partial f(\bar{x}) &= [0, 1], & \partial f(\bar{y}) &= \{0, 1\}, \\ \overline{\partial} f(\bar{x}) &= [0, 1], & \overline{\partial} f(\bar{y}) &= [0, 1]. \end{aligned}$$

Thus all the superdifferentials coincide at $\bar{x}$, but they differ to a great extent at $\bar{y}$.

B Proof of Theorem 1

To prove Theorem 1, consider first a game $v \in \Gamma(N)$, fix $\bar{\mathbf{x}} \in \mathbb{R}^n$ and choose any $\pi \in \Pi(\bar{\mathbf{x}})$. Then there are necessarily unique integers

$$0 = L_0 < L_1 < \cdots < L_K = n$$

such that $L_k - L_{k-1}$ is the number of coordinates of $\bar{\mathbf{x}}$ which have the k–th greatest distinct value in the order given by π:

$$\bar{\mathbf{x}}_{\pi(1)} = \cdots = \bar{\mathbf{x}}_{\pi(L_1)} > \bar{\mathbf{x}}_{\pi(L_1+1)} = \cdots = \bar{\mathbf{x}}_{\pi(L_2)} > \cdots > \bar{\mathbf{x}}_{\pi(L_{K-1}+1)} = \cdots = \bar{\mathbf{x}}_{\pi(L_K)}.$$

Define

$$B_k := \{\pi(L_{k-1}+1), \dots, \pi(L_k)\}$$

and observe that B_k is independent of the choice of $\pi \in \Pi(\bar{\mathbf{x}})$. Take any $\mathbf{x}$ sufficiently close to $\bar{\mathbf{x}}$ and select some $\rho \in \Pi(\mathbf{x})$. Then $\rho \in \Pi(\bar{\mathbf{x}})$ and

$$\begin{aligned} &V_i^\rho(\mathbf{x}) \subseteq V_i^\rho(\bar{\mathbf{x}}), && i = 1, \dots, n, \\ &V_{L_k}^\rho(\mathbf{x}) = V_{L_k}^\rho(\bar{\mathbf{x}}) = B_1 \cup \dots \cup B_k, && k = 1, \dots, K. \end{aligned}$$

This allows us to write $\hat{v}$ in a separable structure

$$\hat{v}(\mathbf{x}) = \sum_{k=1}^{K} \hat{v}_k(\mathbf{x}_{B_k}), \tag{16}$$

where $\mathbf{x}_{B_k}$ is the restriction of $\mathbf{x}$ to components B_k and $\hat{v}_k : \mathbb{R}^{|B_k|} \to \mathbb{R}$ is defined as

$$\hat{v}_k(\mathbf{y}) = \sum_{i=1}^{|B_k|} y_{\varphi(i)} \left[v(B_1 \cup \dots \cup B_{k-1} \cup V_i^\varphi(\mathbf{y})) - v(B_1 \cup \dots \cup B_{k-1} \cup V_{i-1}^\varphi(\mathbf{y})) \right],$$

where $\varphi \in \Pi(\mathbf{y})$. Then we can employ a slightly modified version of [5, Proposition 3] to obtain the following result.

Lemma 13 *For any $k \in \{1, \dots, K\}$:*

$$\hat{\partial}\hat{v}_k(\bar{\mathbf{x}}_{B_k}) = \left\{ \mathbf{x}^* \;\middle|\; \begin{array}{l} \mathbf{x}^*(B_k) = v(B_1 \cup \dots \cup B_{k-1} \cup B_k) - v(B_1 \cup \dots \cup B_{k-1}), \\ \mathbf{x}^*(B) \geq v(B_1 \cup \dots \cup B_{k-1} \cup B) - v(B_1 \cup \dots \cup B_{k-1}) \textit{ for all } B \subseteq B_k \end{array} \right\}.$$

Proof The definition of Fréchet superdifferential and the piecewise affinity of $\hat{v}_k$ give

$$\hat{\partial}\hat{v}_k(\bar{\mathbf{x}}_{B_k}) = \{\mathbf{x}^* |\ \hat{v}_k(\mathbf{y}) - \hat{v}_k(\bar{\mathbf{x}}_{B_k}) \leq \langle \mathbf{x}^*, \mathbf{y} - \bar{\mathbf{x}}_{B_k} \rangle \text{ for all } \mathbf{y} \text{ close to } \bar{\mathbf{x}}_{B_k}\}.$$

Consider now any $\mathbf{x}^* \in \hat{\partial}\hat{v}_k(\bar{\mathbf{x}}_{B_k})$, any $B \subseteq B_k$ and put $\mathbf{y} = \bar{\mathbf{x}} + c\chi_B$, where $c > 0$ is sufficiently small. Denoting a to be the common value of $\bar{\mathbf{x}}$ on B_k, we obtain

$$\begin{aligned} \hat{v}_k(\bar{\mathbf{x}}_{B_k}) &= a\left[(v(B_1 \cup \dots \cup B_{k-1} \cup B_k) - v(B_1 \cup \dots \cup B_{k-1})\right], \\ \hat{v}_k(\mathbf{y}_{B_k}) &= a\left[(v(B_1 \cup \dots \cup B_{k-1} \cup B_k) - v(B_1 \cup \dots \cup B_{k-1} \cup B)\right] \\ &\quad + (a+c)\left[(v(B_1 \cup \dots \cup B_{k-1} \cup B) - v(B_1 \cup \dots \cup B_{k-1})\right], \end{aligned}$$

so that

$$\hat{v}_k(\mathbf{y}_{B_k}) - \hat{v}_k(\bar{\mathbf{x}}_{B_k}) = c\left[(v(B_1 \cup \dots \cup B_{k-1} \cup B) - v(B_1 \cup \dots \cup B_{k-1})\right],$$

By realizing that $\langle \mathbf{x}^*, \mathbf{y}_{B_k} - \bar{\mathbf{x}}_{B_k} \rangle = c\mathbf{x}^*(B)$ and combining all the previous relations, it follows that

$$\mathbf{x}^*(B) \geq v(B_1 \cup \dots \cup B_{k-1} \cup B) - v(B_1 \cup \dots \cup B_{k-1}).$$

Performing the similar procedure for $\mathbf{y} = \bar{\mathbf{x}} - c\chi_{B_k}$, we obtain the first inclusion.

Consider now any $\mathbf{x}^*$ from the right–hand side of the formula in Lemma 13. First, we realize that since $\hat{v}$ is piecewise linear, we need only show that

$$\hat{v}_k(\mathbf{y}) - \hat{v}_k(\bar{\mathbf{x}}_{B_k}) \leq \langle \mathbf{x}^*, \mathbf{y} - \bar{\mathbf{x}}_{B_k} \rangle \tag{17}$$

for those $\mathbf{y} \geq \bar{\mathbf{x}}_{B_k}$ sufficiently close to $\bar{\mathbf{x}}_{B_k}$. Moreover, from the previous paragraph we know that we have already shown this formula for all $\mathbf{y} = (\bar{\mathbf{x}} + c\chi_B)_{B_k}$, where $c > 0$ is small. Fix now any $\mathbf{y} \geq \bar{\mathbf{x}}_{B_k}$ sufficiently close to $\bar{\mathbf{x}}_{B_k}$ and take any $\varphi \in \Pi(\mathbf{y})$. Then

$$\mathbf{y} \in C := \operatorname{conv}\left\{\bar{\mathbf{x}}_{B_k}, \bar{\mathbf{x}}_{B_k} + c\chi_{\{\varphi(1)\}}, \dots, \bar{\mathbf{x}}_{B_k} + c\chi_{\{\varphi(1), \dots, \varphi(|B_k|)\}}\right\}.$$

Since $\hat{v}_k$ is linear on C and since formula (17) holds for all the extreme points of C, it must be true for $\mathbf{y}$ as well. This finishes the proof. □

The decomposition (16) together with Lemma 13 imply immediately that Theorem 1 holds true.

References

1. L. Adam, M. Červinka, and M. Pištěk. Normally admissible partitions and calculation of normal cones to a finite union of polyhedral sets. *Set-Valued and Variational Analysis*, in press. http://dx.doi.org/10.1007/s11228-015-0325-8, 2015.
2. J.-P. Aubin. Coeur et valeur des jeux flous à paiements latéraux. *Comptes Rendus de l'Académie des Sciences. Série A*, 279:891–894, 1974.
3. R. Aumann and J. Dreze. Cooperative games with coalition structures. *International Journal of Game Theory*, 3(4):217–237, 1974.
4. R. Branzei, D. Dimitrov, and S. Tijs. *Models in cooperative game theory*, volume 556 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 2005.
5. V. I. Danilov and G. A. Koshevoy. Cores of cooperative games, superdifferentials of functions, and the Minkowski difference of sets. *Journal of Mathematical Analysis and Application*, 247:1–14, 2000.
6. J. Derks, H. Haller, and H. Peters. The selectope for cooperative games. *International Journal of Game Theory*, 29:23–38, 2000.
7. R. Henrion and J. Outrata. On calculating the normal cone to a finite union of convex polyhedra. *Optimization*, 57(1):57–78, 2008.
8. T. Ichiishi. Super-modularity: Applications to convex games and to the greedy algorithm for LP. *Journal of Economic Theory*, 25(2):283–286, 1981.
9. L. Lovász. Submodular functions and convexity. In A. Bachem, B. Korte, and M. Grötschel, editors, *Mathematical Programming: The State of the Art*, pages 235–257. Springer-Verlag, Berlin, 1983.
10. B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation I*. Springer, 2006.
11. G. Owen. *Game theory*. Academic Press Inc., San Diego, CA, third edition, 1995.
12. B. Peleg and P. Sudhölter. *Introduction to the theory of cooperative games*, volume 34 of *Theory and Decision Library. Series C: Game Theory, Mathematical Programming and Operations Research*. Springer, Berlin, second edition, 2007.
13. C. Rafels and S. Tijs. On the cores of cooperative games and the stability of the Weber set. *International Journal of Game Theory*, 26(4):491–499, 1997.
14. R. T. Rockafellar. *Convex Analysis*. Princeton Univ. Press, 1970.
15. R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis*. Springer, 1998.
16. N. Sagara. Cores and Weber sets for fuzzy extensions of cooperative games. *Fuzzy Sets and Systems*, in press. http://dx.doi.org/10.1016/j.fss.2014.11.017, 2014.
17. R. Selten. Equal share analysis of characteristic function experiments. In H. Sauermann, editor, *Contributions to Experimentation in Economics*, volume 3, pages 130–165. J.C.B. Mohr (Paul Siebeck), 1972.
18. L. S. Shapley. Cores of convex games. *International Journal of Game Theory*, 1:11–26, 1972.
19. L. Shapley and M. Shubik. The assignment game I: The core. *International Journal of Game Theory*, 1(1):111–130, 1971.
20. M. Studený and T. Kroupa. Core-based criterion for extreme supermodular games. Submitted to *Discrete Applied Mathematics*. http://arxiv.org/abs/1410.8395.
21. S. Tijs and F. A. S. Lipperts. The hypercube and the core cover of n-person cooperative games. *Cahiers du Centre d'Études de Researche Opérationelle*, 24:27–37, 1982.
22. L. Gerard-Varet and S. Zamir. Remarks on the reasonable set of outcomes in a general coalition function form game. *International Journal of Game Theory*, 16(2):123–143, 1987.
23. R. J. Weber. Probabilistic values for games. In A. E. Roth, editor, *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, pages 101–120. Cambridge University Press, 1988.
24. G. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.