On the Modeling and Control of the Delamination Problem

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Outline

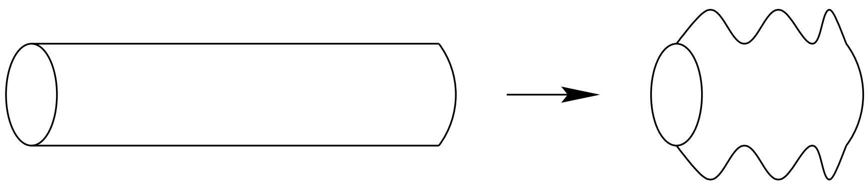
- What is delamination?
- Modeling delamination by HVI
- Energetic approach
- Examples
- Control — first approach

Motivation

Vehicle crash resistance (cars, helicopters)
Design of structural elements with

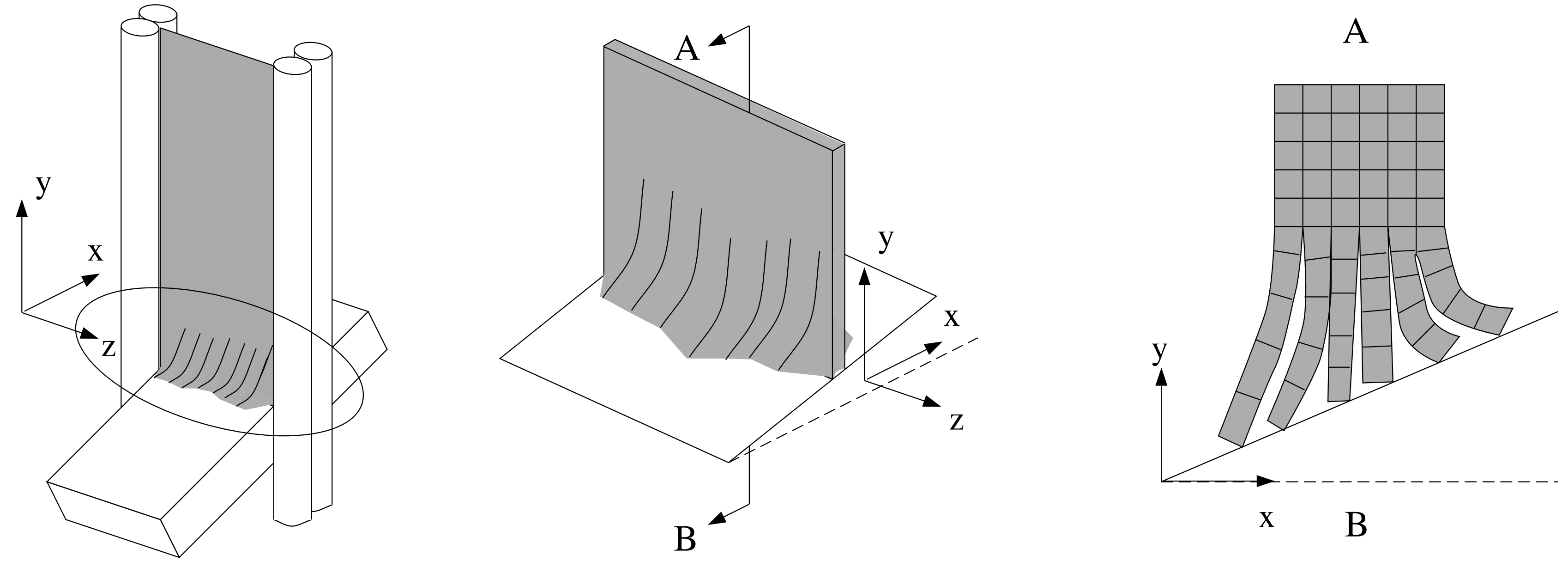
HIGH ENERGY ABSORPTION

Classic design: buckling & plasticity

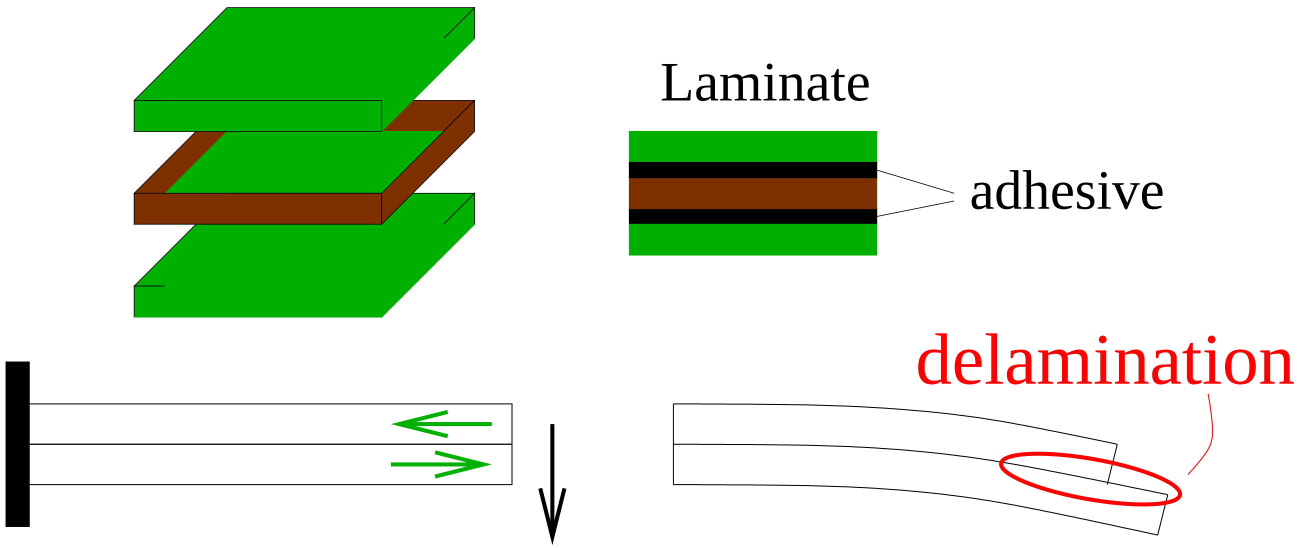


Modern design: use of composite materials
energy absorbed by DELAMINATION

Example

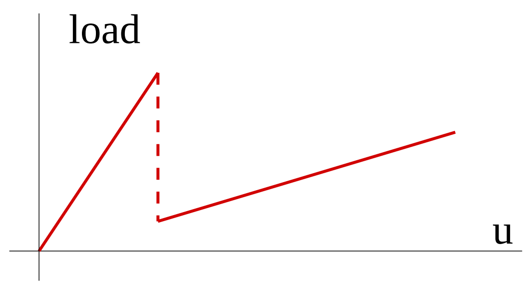


WHAT IS DELAMINATION



Modeling: Unilateral contact conditions + adhesive

Typical load-displacement diagram



VI in MECHANICS (discretized)

Equilibrium: Find $\boldsymbol{u} \in \mathbb{R}^{\boldsymbol{n}}, \boldsymbol{T} \in \mathbb{R}^{\boldsymbol{m}}$, so that

$$\langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{L}, \boldsymbol{v} \rangle + \langle \boldsymbol{T}, \boldsymbol{v} \rangle \qquad \forall \boldsymbol{v} \in \boldsymbol{V}$$

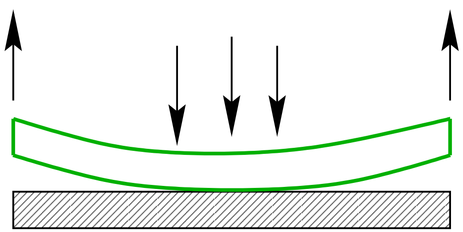
$$-\boldsymbol{T} \in \boldsymbol{\beta}(\boldsymbol{u}) \qquad \text{componentwise}$$

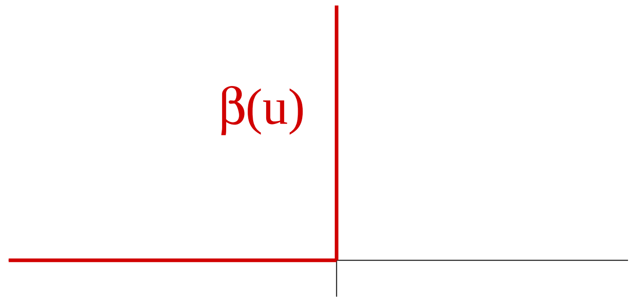
Equilibrium: Find $u \in \mathbb{R}^n, T \in \mathbb{R}^m$, so that

$$\langle Au, v \rangle = \langle L, v \rangle + \langle T, v \rangle \qquad \forall v \in V$$

$$-T \in \beta(u) \qquad \text{componentwise}$$

Example: (unilateral contact b.c.)





if $u_n < 0$ then $T_n = 0$

if $u_n = 0$ then $T_n \geq 0$

$$\beta_n(u) = \begin{cases} 0 & u_n < 0 \\ [0, +\infty] & u_n = 0 \\ \emptyset & u_n > 0 \end{cases}$$

Described by a multifunction $\beta(u)$: $T_n \in \beta_n(u_n)$

VI in MECHANICS (discretized)

Equilibrium: Find $u \in \mathbb{R}^n, T \in \mathbb{R}^m$, so that

$$\langle Au, v\rangle = \langle L, v\rangle + \langle T, v\rangle \qquad \forall v \in V$$

$$-T \in \beta(u) \qquad \text{componentwise}$$

$\beta : \mathbb{R} \longrightarrow \mathbb{R}$ maximal monotone multivalued map

$\beta = \partial j \quad (:= \{z | \langle z, v-u\rangle \leq j(v) - j(u)\}) \quad j$ convex

$\Rightarrow \quad -\langle T, (v-u)\rangle = -\sum T_i(v_i - u_i) \leq \sum(j(v_i) - j(u_i))$

$\Rightarrow$ OUR SYSTEM BECOMES

$$\langle Au, (v-u)\rangle + \sum(j(v_i) - j(u_i)) \geq \langle L, (v-u)\rangle \qquad \forall v \in V$$

equivalent to

$$\langle Au, v\rangle = \langle L, v\rangle + \langle T, v\rangle \qquad \forall v \in V$$

$$-T \in \partial j(u) \qquad \text{componentwise}$$

HEMIVARIATIONAL INEQUALITIES

$$\langle Au, v\rangle = \langle L, u\rangle + \langle T, u\rangle \quad \forall v \in V$$
$$-T \in \partial j(u) \quad \text{componentwise}$$

WHAT IF j IS NOT CONVEX ?

Panagiotopoulos (1985 $\longrightarrow$): take $\overline{\partial} j$ ... Clarke subdifferential

$$\overline{\partial} j = \{w | j^o(u; v) \geq \langle w, v\rangle \; \forall v\}$$

where $j^o(u; v)$ ... Clarke generalized directional derivative

$-T \in \beta(u) = \overline{\partial} j(u)$ (c-w) $\Rightarrow \langle T, v-u\rangle \leq \sum j^o(u; v-u)$

HVI: $\langle Au, v-u\rangle + \sum j^o(u; v-u) \geq \langle L, (v-u)\rangle \quad \forall v \in V$

$$\langle Au, v\rangle = \langle L, u\rangle + \langle T, u\rangle \quad \forall v \in V$$
$$-T \in \overline{\partial} j(u) \quad \text{componentwise}$$

$$\langle Au, v-u\rangle + \sum j(v_i) - j(u_i) \geq \langle L, (v-u)\rangle \quad \forall v \in V \qquad \text{(VI)}$$

$$\sum j^o(u; v-u) \qquad \text{(HVI)}$$

For A symmetric, VI equivalent to

$$\Pi(v) := \frac{1}{2}\langle Av, v\rangle + J(v) - \langle L, v\rangle \to \inf$$

$$\sum j(v_i)$$

VI $\Rightarrow$ $\Pi(v)$ convex

$0 \in \partial\Pi(u)$ 1st order condition, necessary & sufficient

HVI $\Rightarrow$ $J(v)$ (and thus $\Pi(v)$) nonconvex

$0 \in \overline{\partial}\Pi(u)$ only necessary condition

.

Clarke vs. limitting subfifferential

Discretized HVI:

$$\langle \boldsymbol{Au}, \boldsymbol{v} \rangle = \langle \boldsymbol{L}, \boldsymbol{u} \rangle + \langle \boldsymbol{T}, \boldsymbol{u} \rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V},$$
$$-\boldsymbol{T} \in \overline{\partial} \boldsymbol{j}(\boldsymbol{u})$$

Why Clarke ?

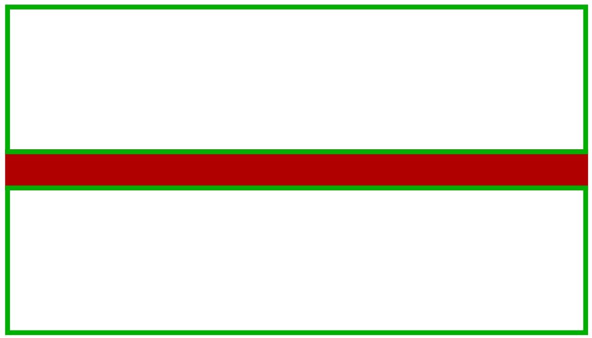
Nature: minimizes potential energy $\mathbf{\Pi}$

HVI just necessary optimality condition

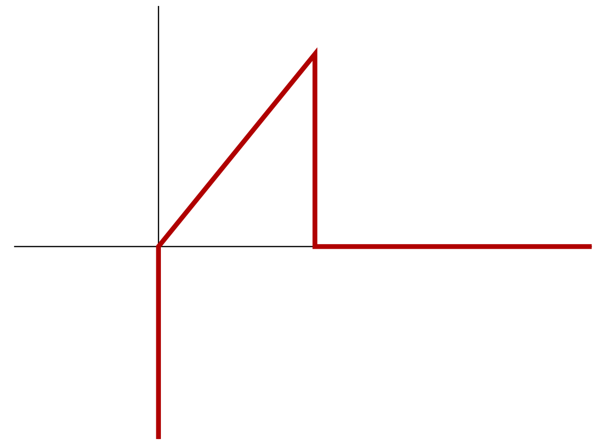
Clarke subdiff. too large

Mordukhovich's limitting subdifferential smaller,

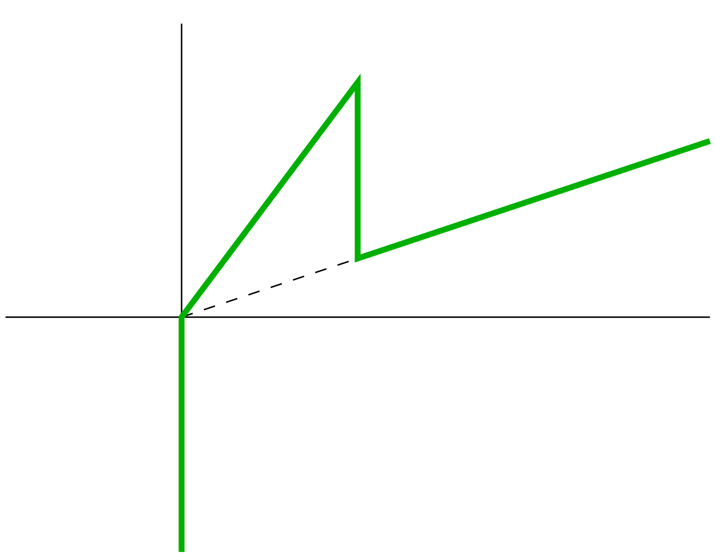
corresponds better to this kind of problems

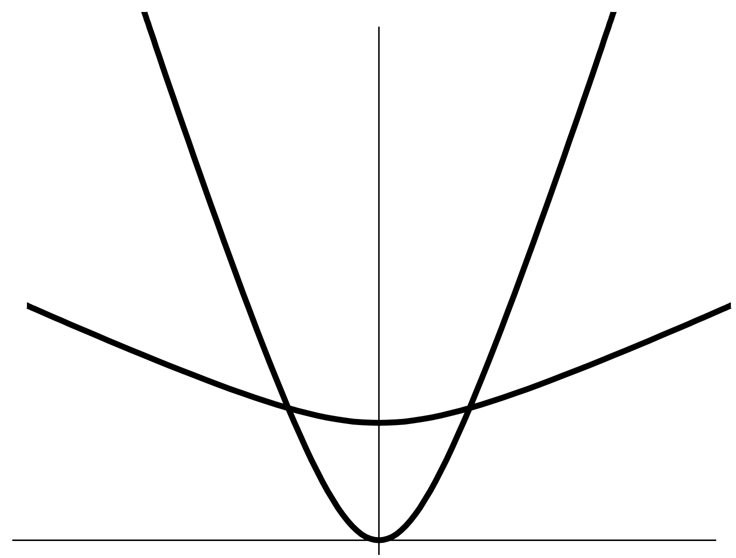


adhesive



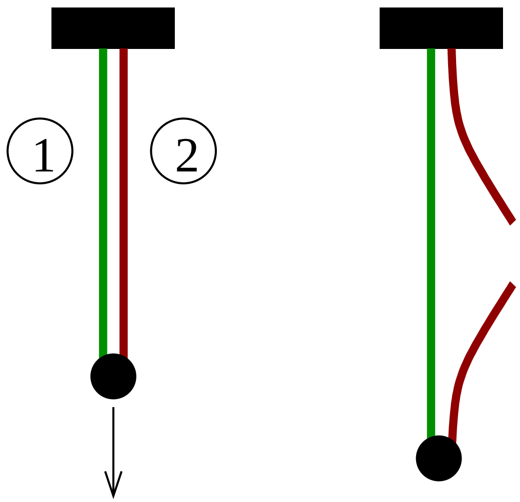
Global model at one point of the contact boundary:

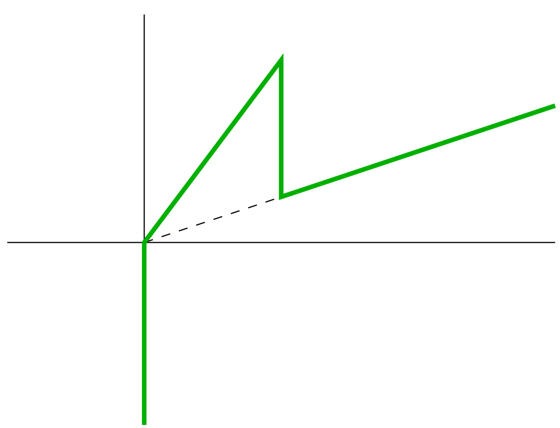


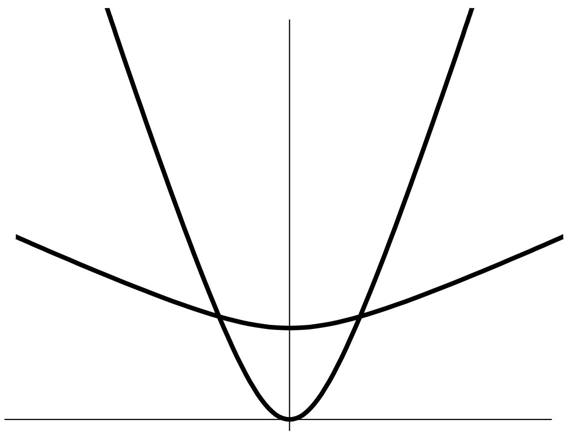


Model example—two strings

Two parallel elastic strings, one breakable



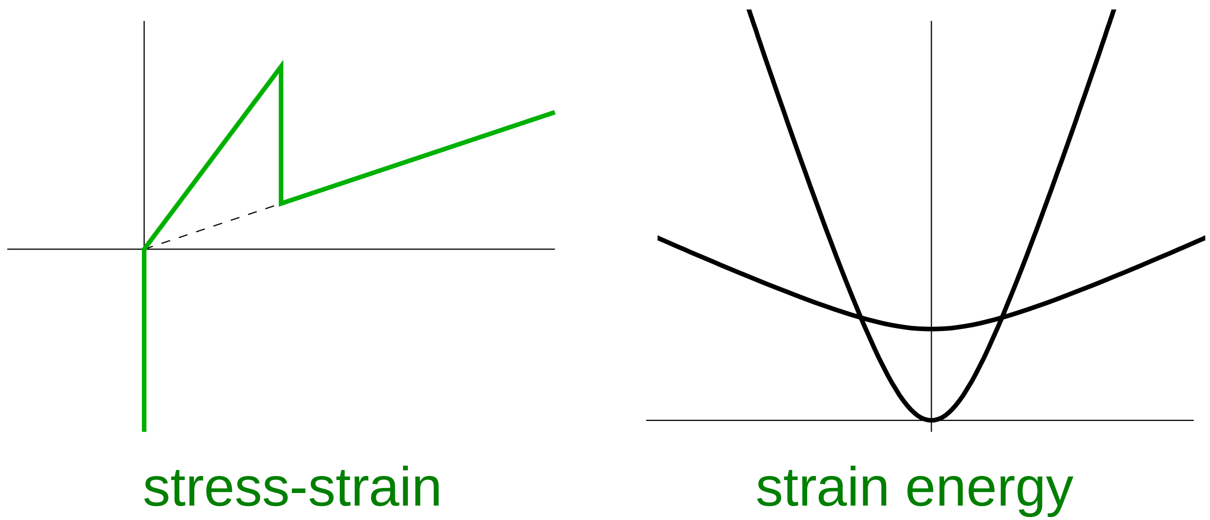




HVI approach (1D system, 1 time step)

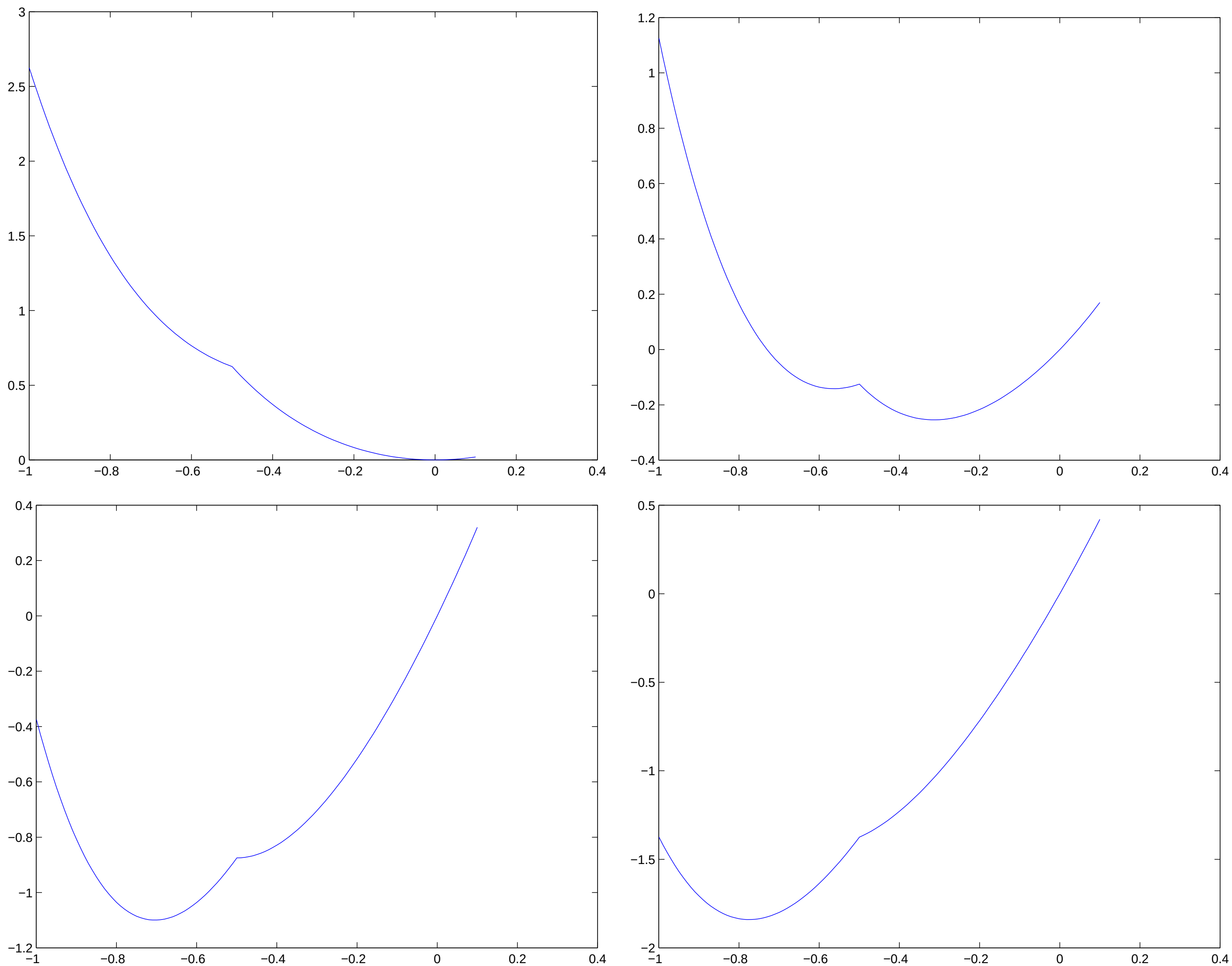
E_1 elasticity of string 1
E_2 elasticity of string 2

Energy: 1st stage (unbroken): $u(E_1 + E_2)u$
2nd stage (broken): uE_1u



Problem:

$$\min_u [\min\{u(E_1 + E_2)u, uE_1u + D\} + fu]$$


3
2.5
2
1.5
1
0.5
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.2
0.5
0
−0.5
−1
−1.5
−2

Energetic approach (1D system, time step k)

E_1 elasticity of string 1
E_2 elasticity of string 2

Energy: 1st stage (unbroken): $u(E_1 + E_2)u$
2nd stage (broken): uE_1u

Stored energy: $\frac{1}{2}u(E_1 + \zeta E_2)u$, $\zeta \in \{0, 1\}$
Free energy: $\frac{1}{2}u(E_1 + \zeta E_2) + f_k u$
Dissipated energy: $(\zeta_{\text{old}} - \zeta)D$

Problem:

$$\min_{u,\zeta} \frac{1}{2}u(E_1 + \zeta E_2)u + f_k u + (\zeta_{\text{old}} - \zeta)D$$

s.t. $\zeta \in \{0, 1\}$

$\zeta_{\text{old}} \geq \zeta$

Theorem: The HVI problem

$$\min_{u}[\min\{u(E_1+E_2)u, uE_1u+\gamma\}+fu]$$

is equivalent to the 'energetic problem'

$$\min_{u,\zeta} \frac{1}{2}u(E_1+\zeta E_2)u + f_k u + (\alpha_{k-1}-\zeta)D$$

$$\text{s.t.} \quad \zeta \in \{0,1\}$$

(also in general multidimensional case).

Relaxation: $\zeta \in \{0,1\} \quad \longrightarrow \quad \zeta \in [0,1]$

Energetic approach, general formulation

Minimize $V(u,\zeta) + R(\zeta - \zeta^{\kappa-1})$

subject to $Lu \geq w^{\kappa}$,

$(u_{\alpha,i} - u_{\beta,j}) \cdot \nu_{ij} \geq 0, \quad (i,j) \in I_{\alpha\beta},\ \alpha, \beta = 1, \ldots, m,$

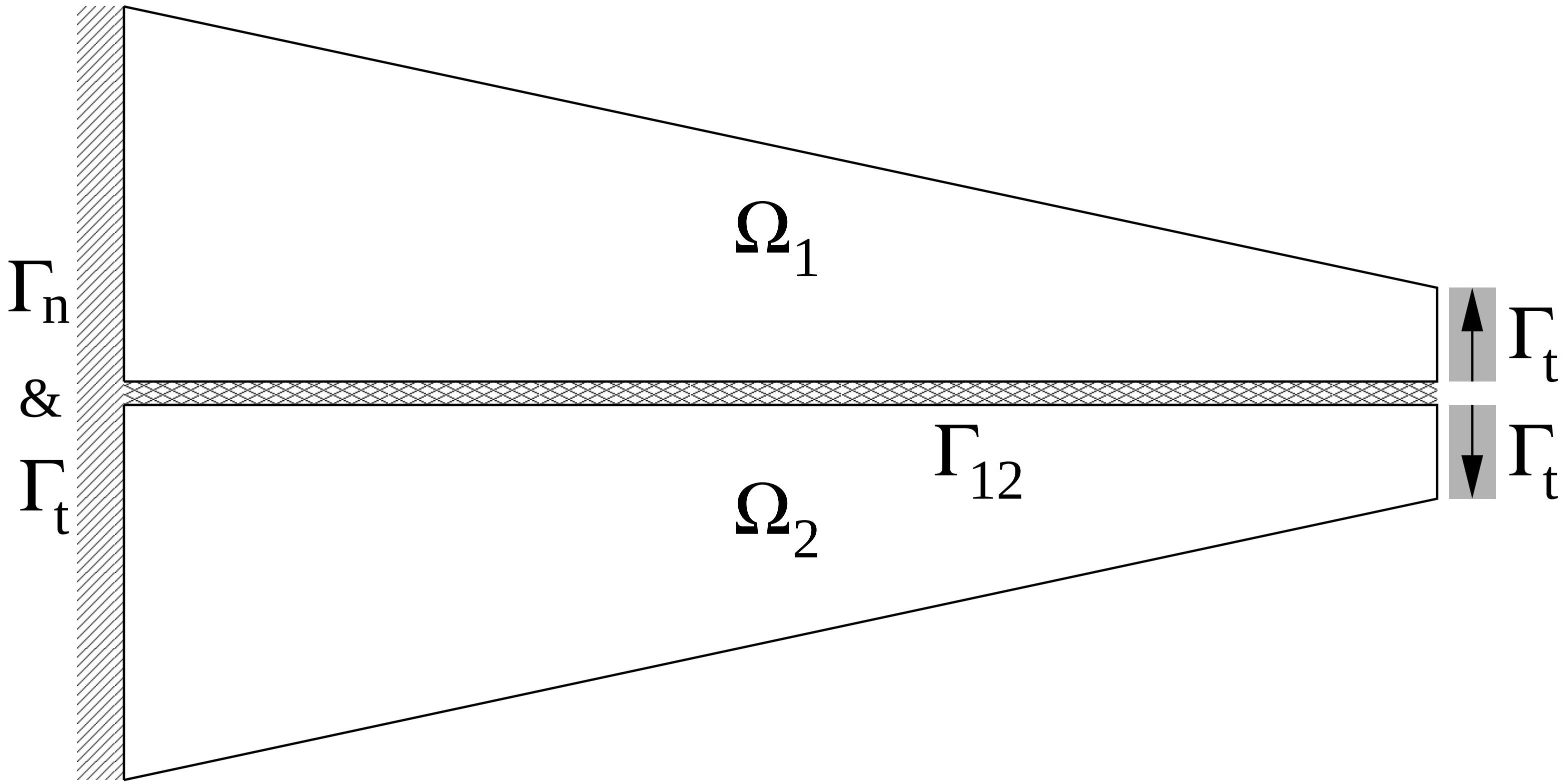
$\zeta^{\kappa-1} \geq \zeta \geq 0$ componentwise.

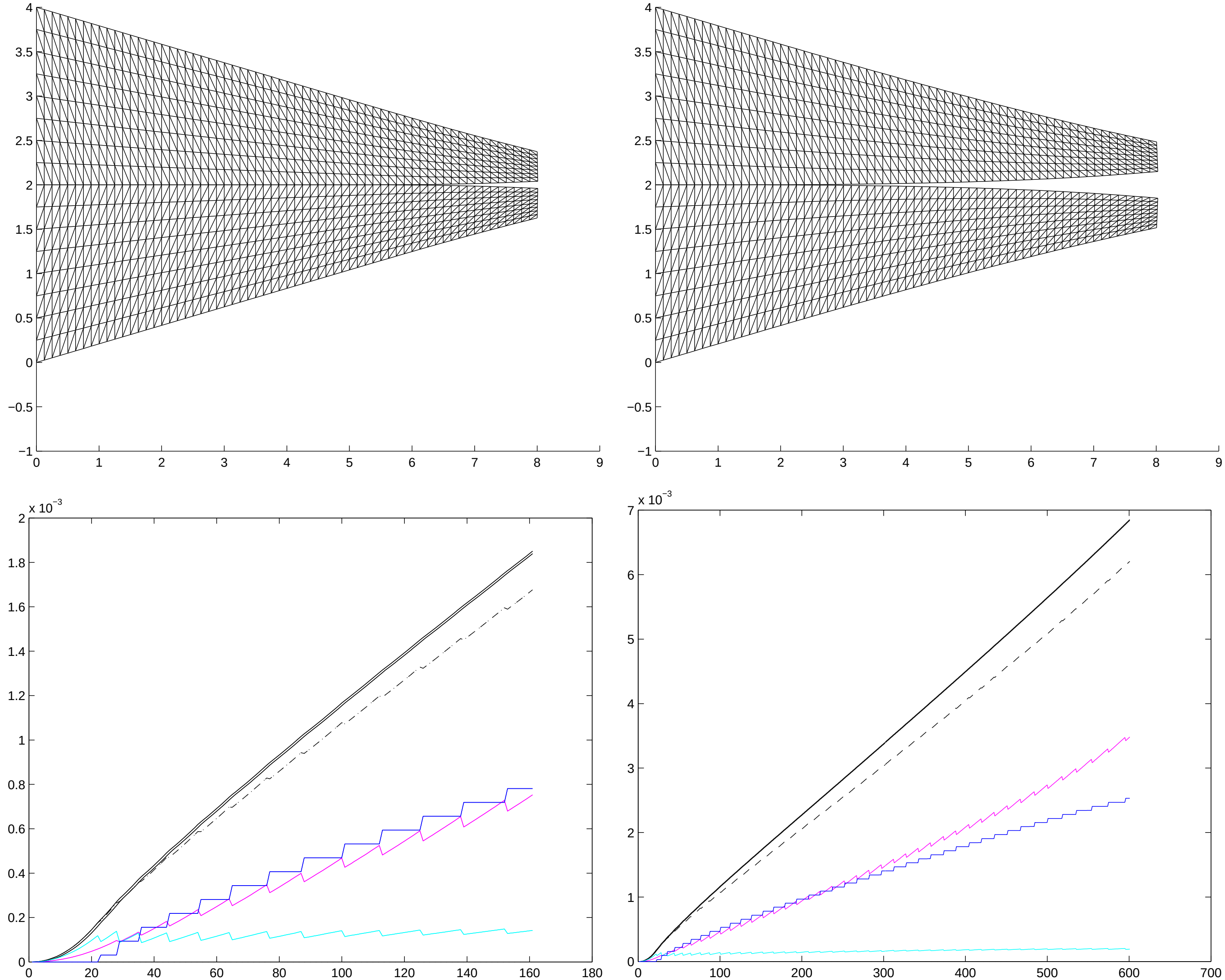
with

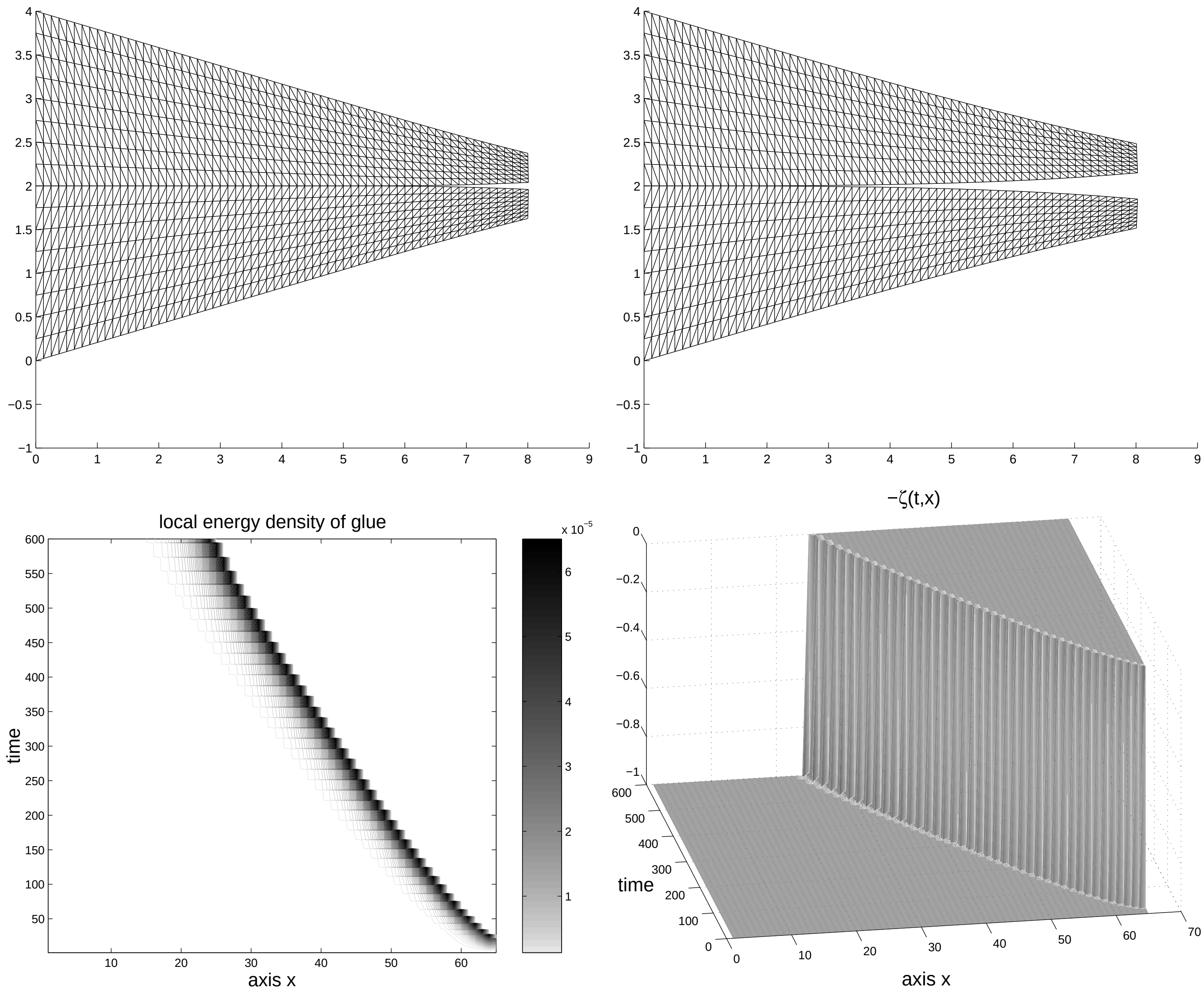
$$V(u,\zeta) = \sum_{\alpha=1}^{2} \left(u_{\alpha}^{T} A_{\alpha} \mathbf{u}_{\alpha} \right) + \sum_{(i,j,k)\in I} \omega_k \zeta_k \left(\mathbf{u}_{1,i} - u_{2,j}\right)^{\top} b(n_i)(u_{1,i} - u_{2,j})$$

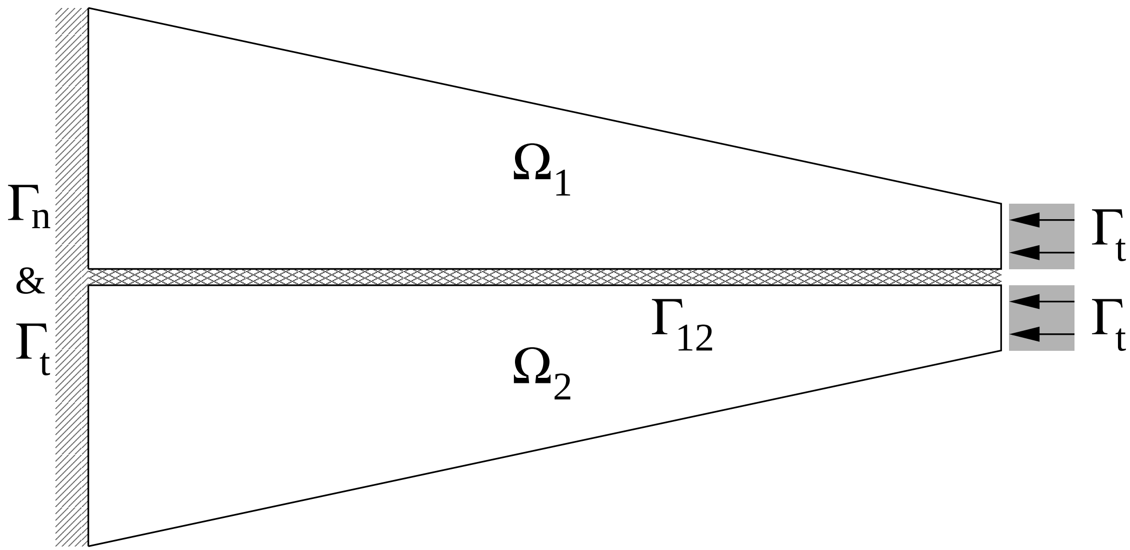
and

$$R(\zeta) = \sum_{(i,j,k)\in I} -\omega_k d(n_i)\zeta_k.$$

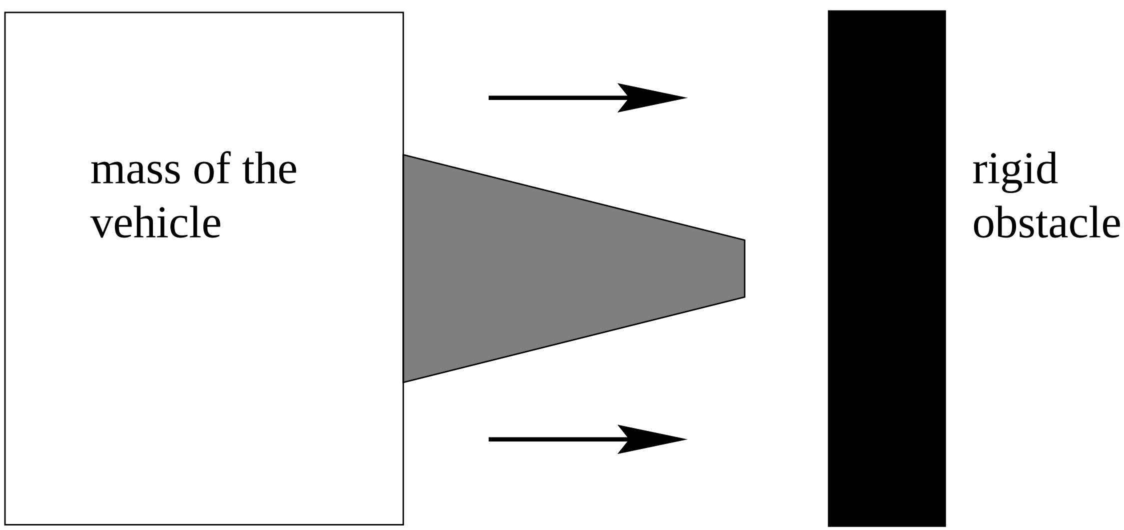

Γn
&
Γt
Ω1
Γt
Γt
Γ12
Ω2


x 10^−3
x 10^−3

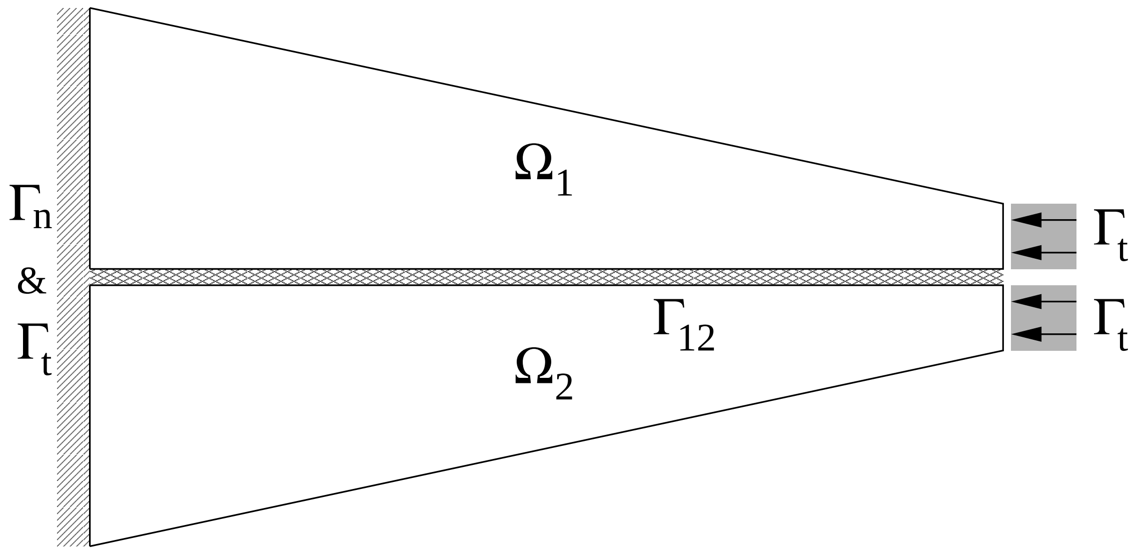

local energy density of glue
time
axis x
x 10^−5
−ζ(t,x)
time
axis x

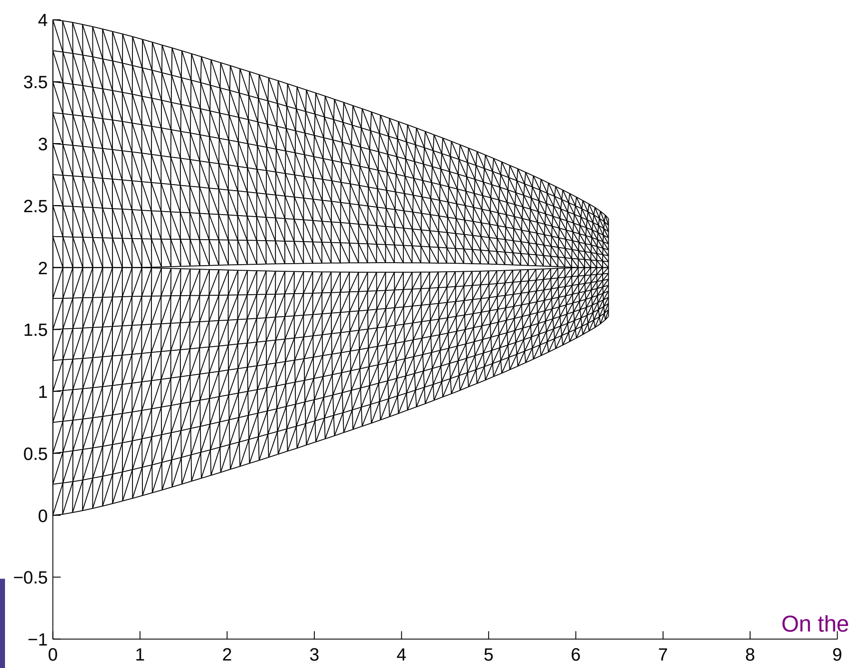


Ω_1
Γ_n
&
Γ_t
Γ_t
Γ_{12}
Γ_t
Ω_2



mass of the vehicle
rigid obstacle


Ω1
Γn
&
Γt
Γt
Γt
Γ12
Ω2


4
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
0
1
2
3
4
5
6
7
8
9

Control of the delamination problem

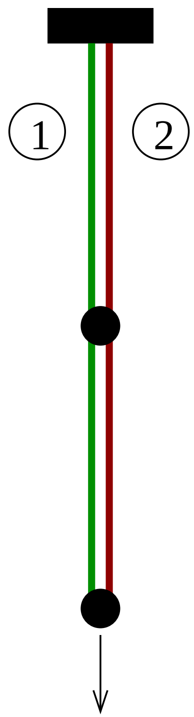
The aim: find such parameters that an objective function depending on the terminal state is minimized.

"Conceptual" MPEC:

$$\begin{array}{rl} \text{minimize} & \varphi(x, y) \\ \text{subject to} & x \in U_{ad}, \\ & y \text{ is the terminal state of the delamination process} \\ & \quad \text{depending on parameter } x, \end{array}$$

Model example—four strings

two + two parallel elastic strings, red breakable



Model example—four strings

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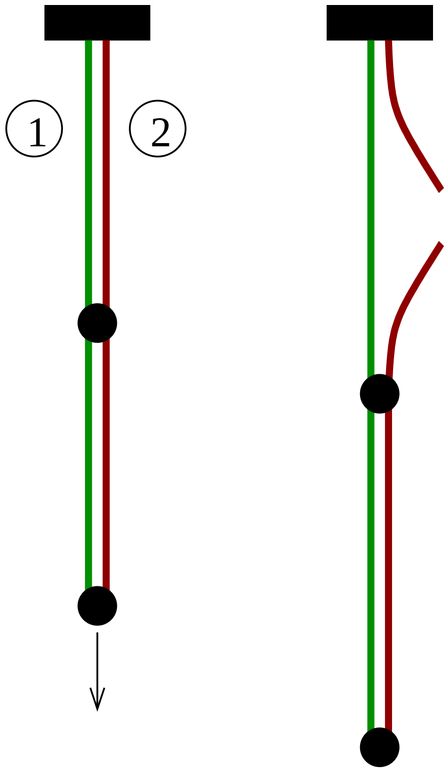

Model example—four strings

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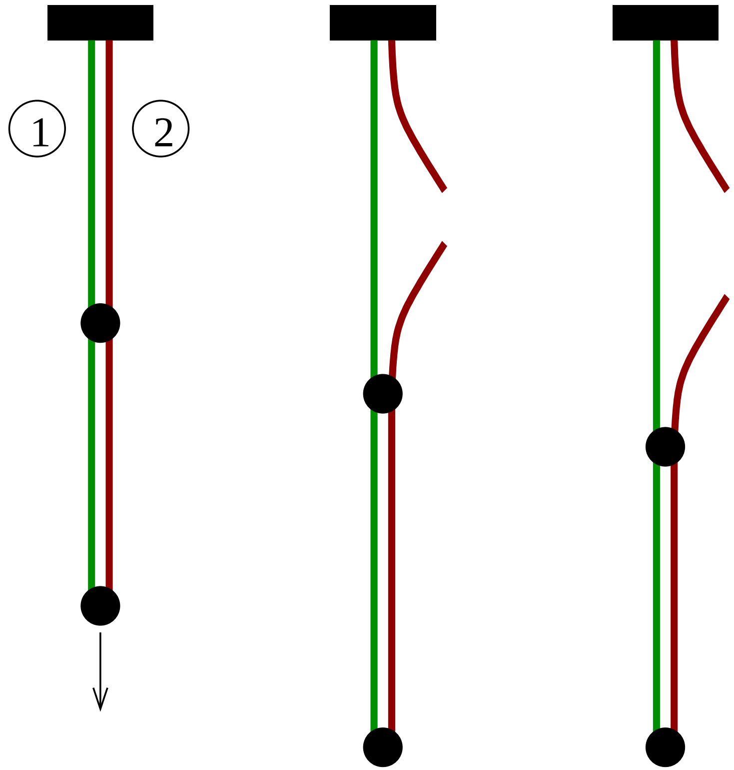
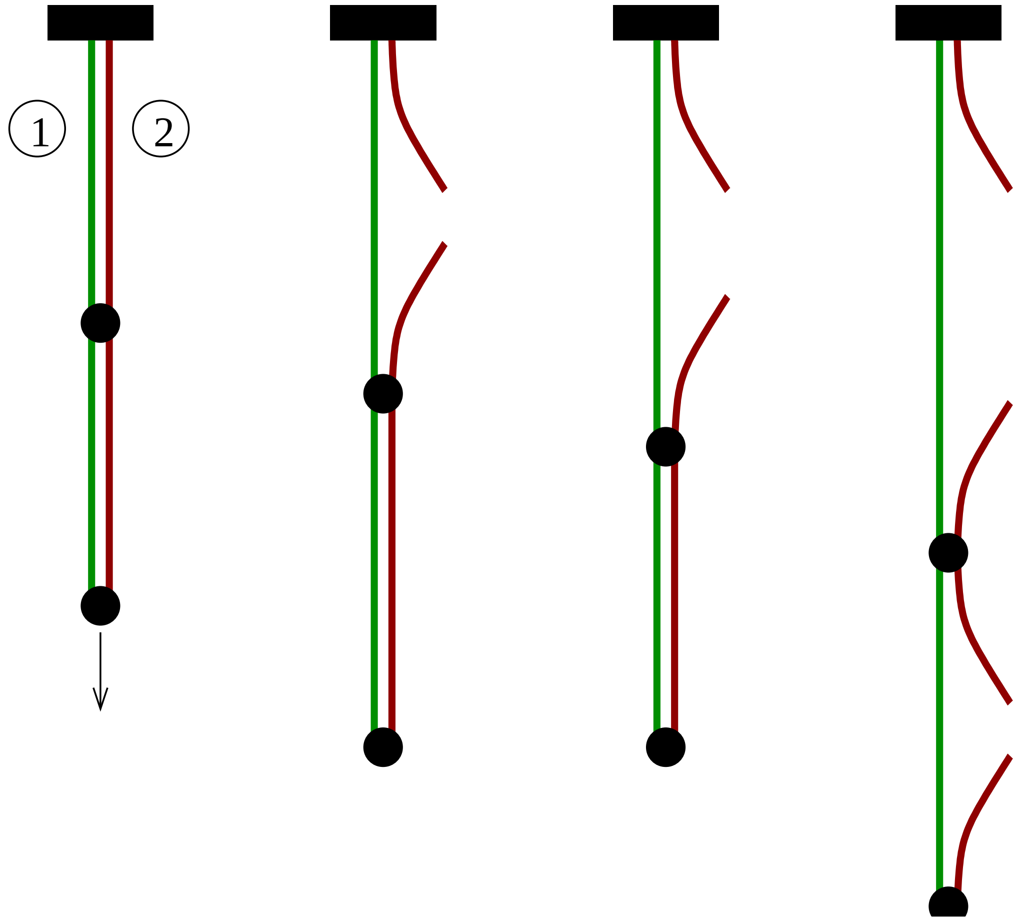


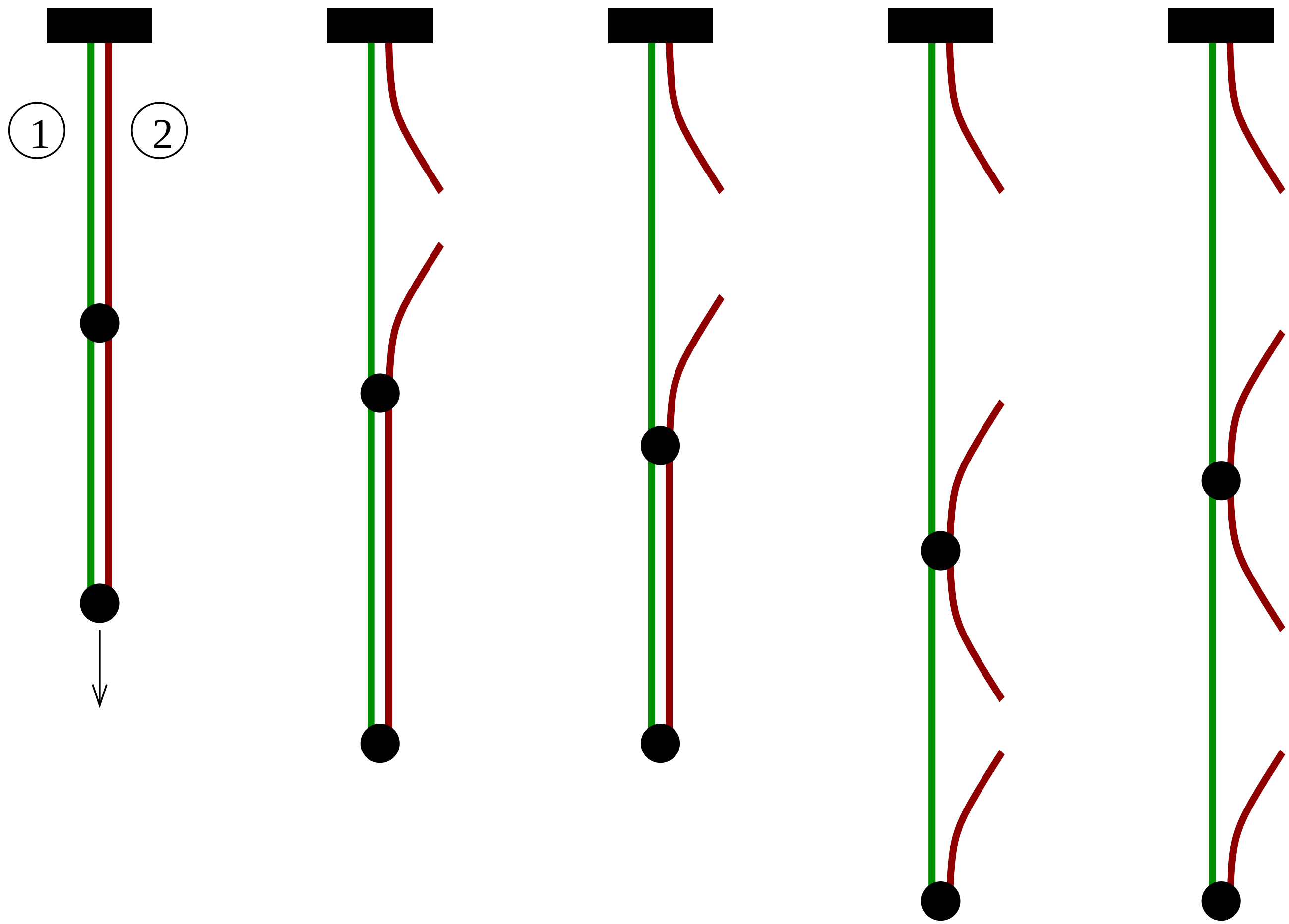
Model example—four strings

two + two parallel elastic strings, red breakable

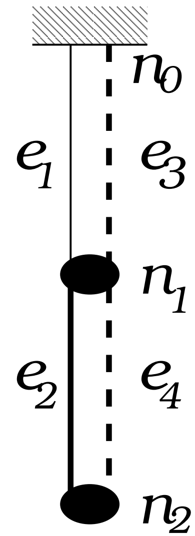


Model example—four strings

two + two parallel elastic strings, red breakable



Goal: find stiffness parameters of the green strings such that, at the terminal time, as much energy is dissipated as possible.


n_0
e_1
e_3
n_1
e_2
e_4
n_2

Model example—four strings

Delamination problem at time i:

$$\min_{u_1^i, u_2^i, \zeta_1^i, \zeta_2^i} \sum_{j=1}^{2} \left(e_j (u_j^i - u_{j-1}^i)^2 + \zeta_j^i e (u_j^i - u_{j-1}^i)^2 + (\zeta_j^{i-1} - \zeta_j^i) ed \right)$$

subject to

$$u_2^i \geq \overline{u}^i$$

$$u_j^i - u_{j-1}^i \geq 0, \quad j = 1, 2$$

$$\zeta^{i-1} \geq \zeta^i \geq 0$$

MPEC:

$$\min_{e_1, e_2, u_1^i, u_2^i, \zeta_1^i, \zeta_2^i} \zeta_1^k + \zeta_2^k$$

subject to

$$e_1 + e_2 = 2$$

$(u_1, u_2, \zeta_1, \zeta_2)$ solves (*) at time k.

Solving the MPECs as NLPs

Idea: replace the equilibrium problem by KKT system.
Convert MPEC:

$$\begin{array}{ll} \text{minimize} & \varphi(x, \tilde{y}) \\ \text{subject to} & \tilde{y} \text{ solves} \left\{ \begin{array}{l} \min\limits_{y} f(x; y) \\ \text{s.t. } g(x; y) \leq 0 \end{array} \right\} \end{array}$$

into NLP

$$\begin{array}{ll} \text{minimize} & \varphi(x, y) \\ \text{subject to} & \nabla_y f(x; y) + \lambda \nabla_y g(x; y) = 0 \\ & g(x; y) \leq 0, \quad \lambda \geq 0 \\ & g(x; y)\lambda \geq 0 \end{array}$$

NLP doesn't satisfy MFCQ but can be solved by (some) current NLP software.

Solving the MPEC

Idea: replace the equilibrium problem (*) by KKT system *for every time step*.

Convert MPEC into NLP:

$$\min_{e_1,e_2,u_1,u_2,\zeta_1,\zeta_2,\text{multipliers}} \zeta_1^k + \zeta_2^k$$

subject to

$$e_1 + e_2 = 2$$

KKT^1

$\ldots$

KKT^N

where N is the number of time steps.

Solving the MPEC

Main trouble: the delamination problem (*) is nonconvex and has several isolated local minima (stationary points).

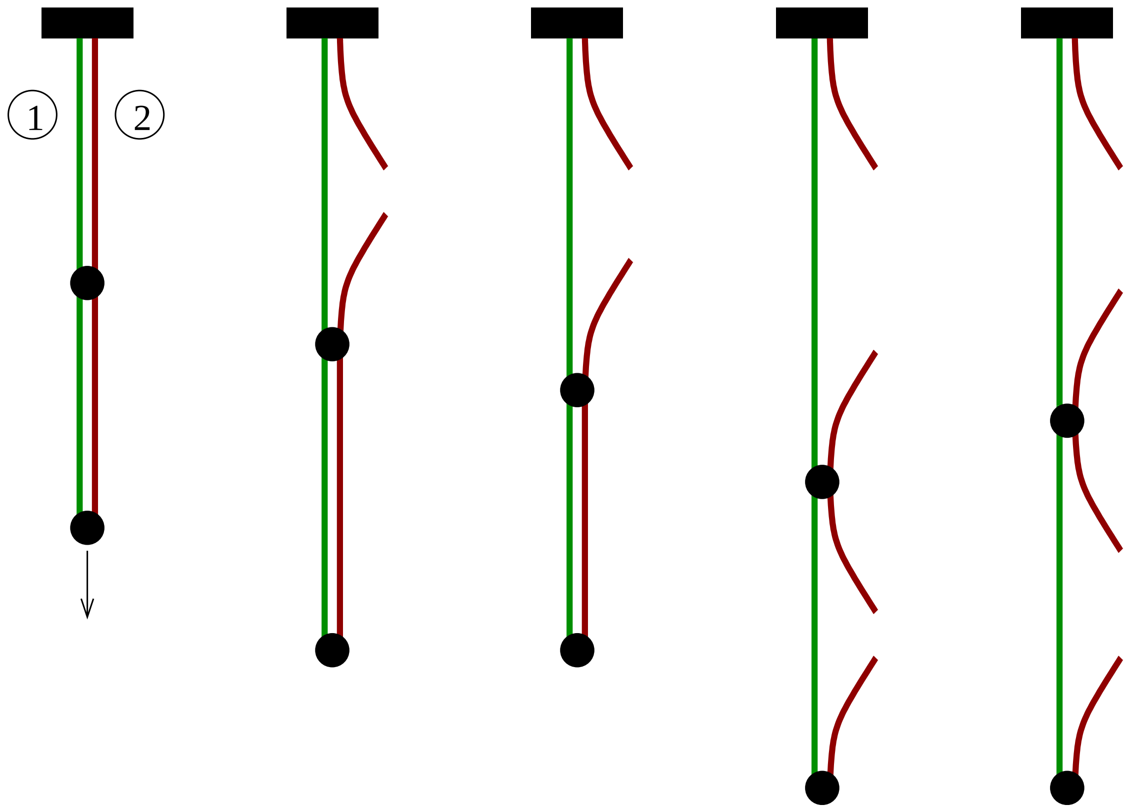
Only one corresponds to a "physical solution" (but not the global minimum).

MPEC's upper level objectives may force the system toward "non-physical" solutions
$\longrightarrow$ (very) fine time discretization may be needed
$\longrightarrow$ (very) large system

Four-string example:
2 "upper-level" variables, 4 "lower-level" variables, 32 time steps
$\longrightarrow$ 354 NLP variables

Model example—four strings

two + two parallel elastic strings, red breakable



“Physical” solution: $\boldsymbol{e_1 = e_2 = 1,}$ $\boldsymbol{\zeta_1^N = \zeta_2^N = 0}$

“Non-physical” solution: $\boldsymbol{e_1 = 1.7, e_2 = 0.3, \zeta_1^N = \zeta_2^N = 0}$