Lower semicontinuity of integral functionals

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based on joint works with
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September 7, 2016

Extreme-Value Theorem - B. Bolzano ($\approx$ 1830)

Let $f : [a, b] \to \mathbb{R}$ *be continuous for some* $-\infty < a < b < +\infty$. *Then there is* $x_0 \in [a, b]$ *such that* $f(x) \geq f(x_0)$ *for all* $x \in [a, b]$. *In other words,* $f(x_0) = \min_{[a,b]} f$.

Proof is simple (for us, now)

and is based on the Bolzano-Weierstrass theorem: *A bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

- Find $\{x_k\} \subset [a, b]$ such that $\lim_{k\to\infty} f(x_k) = \inf_{[a,b]} f$. (i.e. find a "minimizing" sequence).
- $x_{k_m} \to x_0$ for $m \to \infty$ (Bolzano-Weierstrass)
- Use continuity to infer $\inf_{[a,b]} f = \lim_{m\to\infty} f(x_{k_m}) = f(x_0)$.

It is enough if f is only lower semicontinuous, i.e., if

$$\liminf_{m\to\infty} f(x_{k_m}) \geq f(x_0) \ .$$

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Pros and Cons

- ☺ f can be quite general (nonsmooth, nonconvex,...).
- ☺ The proof allows for generalizations to infinite dimensional spaces, various topologies,...
- ☹ The proof does not tell us how to find x_0, i.e. it is non-constructive.

Why is minimization important?

“Nothing takes place in the world whose meaning is not that of some maximum or minimum.”

Leonhard Euler

Applications

- Mathematical Programming/Optimization/Variational Analysis
- Calculus of Variations (Mechanics, Physics,. . .)
- PDEs, Variational Methods
- Optimal Control (Engineering,. . .)

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The direct method

is a far more advanced setting of the proof of the Extreme-Value Theorem. It was designed by D. Hilbert and S. Zaremba around 1900.

If we replace $\mathbb{R}$ by some Hilbert space H (for instance by L^2) then bounded sequences in H contain subsequences converging **only** in the weak topology. This is "convergence of weighted averages". We say that $u_k \to u$ weakly if

$$\forall h \in H : \langle u_k, h\rangle \to \langle u, h\rangle \text{ if } k \to \infty \ .$$

- For example, if $H := L^2(0,1)$ and $u_k(x) := \sin kx$ then $u_k \to 0$ weakly as $k \to \infty$.
- norm (strong) convergence implies the weak one.

The direct method (of the calculus of variations)

- H is a Hilbert space (Think again about L^2)
- $I : H \to \mathbb{R}$ is coercive (i.e. $I(u) \to \infty$ if $\|u\| \to \infty$)
- I is weakly lower semicontinuous (i.e. $\liminf_{k\to\infty} I(u_k) \geq I(u)$ whenever $u_k \to u$ weakly).

Then there is $u \in H$ such that $I(u) = \inf_H I$.

- Coercivity means that $I(u) \geq I(0)$ if $\|u\| > C$ for some $C > 0$, so

$$\inf_H I = \inf_{\|u\| \leq C} I .$$

- Weak lower semicontinuity implies lower semicontinuity.
- Later on, this program was extended to a "nice" Banach space B (e.g. L^p, $1 < p \leq +\infty$).

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Convexity of I

If $I : B \to \mathbb{R}$ is continuous and **convex** *then I is weakly lower semicontinuous (wlsc).*

- Convex analysis arguments (Hahn-Banach theorem)
- In particular, Mazur's lemma (it "upgrades" weak convergence to strong one).
  If $L = \lim_{k\to\infty} I(u_k)$, then for every $\varepsilon > 0$ there is k_0 such that for all $k \geq k_0$ we have $I(u_k) \leq L + \varepsilon$. By Mazur's lemma, there are $n(\varepsilon) \in \mathbb{N}$, $0 \leq \lambda_k(\varepsilon)$, $\sum_{k=k_0}^{n} \lambda_k = 1$ such that

$$\|u - \sum_{k=k_0}^{n} \lambda_k u_k\| \leq \varepsilon \ .$$

Hence,

$$I(\sum_{k=k_0}^{n} \lambda_k u_k) \leq \sum_{k=k_0}^{n} \lambda_k I(u_k) \leq L + \varepsilon \ .$$

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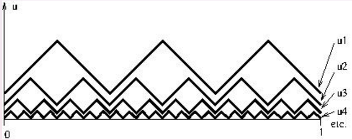
Nonconvexity could be fatal....(at least in 1D)

$$I(u) := \int_0^1 (1 - |u'|)^2 + u^2 \, \mathrm{d}x \ .$$

Consider $\{u_k\}$ a sequence of zig-zag functions driving I to its infimum.



$$u_k \to 0 \text{ in } L^2(0,1)$$

$$u_k' \to 0 \text{ weakly in } L^2(0,1)$$

$$0 = \inf I = \lim_{k\to\infty} I(u_k) < I(0) = 1$$

No weak lower semicontinuity and no minimizer because $I(u) > 0$.

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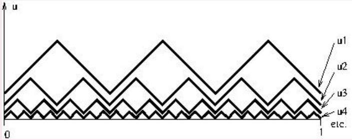

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Weakly continuous I (Reshetnyak, 1968)

Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary, $u : \Omega \to \mathbb{R}^n$ belongs for $p > n$ to the Sobolev space

$$W^{1,p}(\Omega; \mathbb{R}^n) := \{y \in L^p(\Omega; \mathbb{R}^n);\ \nabla y \in L^p(\Omega; \mathbb{R}^{n\times n})\} \ .$$

Set

$$I(u) := \int_\Omega \det \nabla u \, \mathrm{d}x \ .$$

If

$$u_k \to u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^n) \ .$$

Then

$$\lim_{k\to\infty} I(u_k) = I(u) \ .$$

Why is it so?

....because determinant is the divergence.

If $\varphi \in C_0^\infty(\Omega)$ the **strong** convergence of $u_k \to u$ and the **weak** convergence of partial derivatives of u_k allows us to write ($n = 2$):

$$\int_\Omega \varphi \det \nabla u_k \,\mathrm{d}x = \int_\Omega \frac{\partial}{\partial x_1}\left(u_k^1 \frac{\partial u_k^2}{\partial x_2}\right)\varphi + \frac{\partial}{\partial x_2}\left(-u_k^1 \frac{\partial u_k^2}{\partial x_1}\right)\varphi \,\mathrm{d}x$$

$$= -\int_\Omega \left(u_k^1 \frac{\partial u_k^2}{\partial x_2}\right)\frac{\partial \varphi}{\partial x_1} + \left(u_k^1 \frac{\partial u_k^2}{\partial x_1}\right)\frac{\partial \varphi}{\partial x_2} \,\mathrm{d}x$$

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Density of $C_0^\infty(\Omega)$ in $L^{p/(p-n)}(\Omega)$ finishes the argument.

We can replicate the above calculation for all other subdeterminants/minors of the gradient matrix.

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Convexity and weak convergence of minors together

J.M. Ball's notion of **polyconvexity** (1977) with important applications to mathematical elasticity

$$W(F) := \begin{cases} h(F, \operatorname{cof} F, \det F) & \text{if } \det F > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

$$\operatorname{cof} F := (\det F) F^{-\top}$$

$$h : \mathbb{R}^{19} \to \mathbb{R} \text{ is } \textbf{convex}$$

Example (for $\det F > 0$):

$$W(F) := |F|^p + |\operatorname{cof} F|^{p/2} + \frac{1}{\det F} .$$

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Weak lower semicontinuity

$$I(u) := \int_\Omega W(\nabla u(x))\,\mathrm{d}x$$

is weakly lower semicontinuous in $W^{1,p}(\Omega;\mathbb{R}^3)$, if $p > 3$.

The proof is based on convexity of h and special properties of determinants and cofactors, namely if $u_k \rightharpoonup u$ in $W^{1,p}$ for $p > 3$ then

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Generalization to arbitrary dimensions possible (compensated compactness).

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Polyconvexity

- It is relatively easy to construct polyconvex functions.
- Examples for various crystallographic structures.
- It allows us to ensure injectivity of deformations and their orientation preservation ($\det > 0$).
- Included in commercial numerical (FE) software.

Existence of minimizers in elasticity

(i) W polyconvex, $W(F) = +\infty$ if $\det F \leq 0$

(ii) $W(F) = W(RF)$ for all $R \in \mathrm{SO}(3)$ and all $F \in \mathbb{R}^{3\times3}$

(iii) $W(F) \to +\infty$ if $\det F \to 0_+$

(iv) $|F|^p \leq W(F)$

Minimizers of I exist in $W^{1,p}(\Omega; \mathbb{R}^3)$.

Necessary conditions

- Polyconvexity shows that convexity does not have to be necessary for wlsc if I depends on gradients.
- Indeed, the reason is that gradients are not arbitrary functions.

Quasiconvexity (Morrey, 1952)

$W : \mathbb{R}^{m\times n} \to \mathbb{R}$ is (Morrey's) **quasiconvex** if for all $F \in \mathbb{R}^{m\times n}$ and all $\varphi \in C_0^\infty((0,1)^n; \mathbb{R}^m)$ (or $\varphi \in C^\infty_{(0,1)^n-\mathrm{per}}(\mathbb{R}^n; \mathbb{R}^m)$)

$$W(F) \le \int_{(0,1)^n} W(F + \nabla\varphi(x))\,\mathrm{d}x \ .$$

If $1 < p < +\infty$ and $0 \le W(F) \le C(1 + |F|^p)$ then $I(u) := \int_\Omega W(\nabla u(x))\,\mathrm{d}x$ is wlsc on $W^{1,p}(\Omega; \mathbb{R}^m)$ if and only if W is quasiconvex.

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Do we have a necessary condition?

....yes, we do.

- Quasiconvex functions are **rank-one convex**, i.e.,

$$W(\lambda A + (1 - \lambda B)) \leq \lambda W(A) + (1 - \lambda) W(B)$$

  whenever $0 \leq \lambda \leq 1$ and $\text{rank}(A - B) = 1$.

- $g(t) := W(A + t(a \otimes b))$ is convex for all $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, and all $A \in \mathbb{R}^{m \times n}$, i.e., if smooth then $g'' \geq 0$.
- Quasiconvex functions are continuous and locally Lipschitz.
- If $m = 1$ or $n = 1$ rank-one convex means convex as well as quasiconvex.

Example (Alibert & Dacorogna, 1992)

Let $m = n = 2$, $|F|^2 := \sum_{ij} F_{ij}^2$ and

$$W_\gamma(F) := |F|^4 - 2\gamma|F|^2 \det F \ .$$

Then

- W_γ is convex iff $|\gamma| \leq 2\sqrt{2}/3$
- W_γ is polyconvex iff $|\gamma| \leq 1$
- there is $\varepsilon > 0$ such that W_γ is quasiconvex iff $|\gamma| \leq 1 + \varepsilon$
- W_γ is rank-one convex iff $|\gamma| \leq 2/\sqrt{3}$

It is an open problem if $\varepsilon = 2/\sqrt{3} - 1$, i.e., if rank-one convexity and quasiconvexity of W_γ coincide.

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Ordering of convexity notions

In general dimensions,

convexity $\implies$ polyconvexity $\implies$ quasiconvexity $\implies$ rank-1 convexity

and no implication can be reversed.

Quasiconvexity vs. rank-one convexity

- It has been open for about 40 years if rank-one convexity implies quasiconvexity.
- The answer is **no** if $m \geq 3$ and $n \geq 2$ due to V. Šverák's counterexample from 1992.
- The case $m = 2$ and $n \geq 2$ is still open, in particular, we do not know what happens on 2×2 matrices; cf. the Alibert & Dacorogna example.

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Quasiconvexity vs. rank-one convexity

- It has been open for about 40 years if rank-one convexity implies quasiconvexity.
- The answer is **no** if $m \geq 3$ and $n \geq 2$ due to V. Šverák's counterexample from 1992.
- The case $m = 2$ and $n \geq 2$ is still open, in particular, we do not know what happens on 2×2 matrices; cf. the Alibert & Dacorogna example.

Counterexample

Then take $m = 3$ and $n = 2$

$$\varphi(x) := (2\pi)^{-1}(\sin 2\pi x_1, \sin 2\pi x_2, \sin 2\pi(x_1 + x_2)) \ . \tag{1}$$

We see that $\nabla\varphi \subset L$ where

$$L := \left\{ \begin{pmatrix} r & 0 \\ 0 & s \\ t & t \end{pmatrix} \ ; \ \ r, s, t \in \mathbb{R} \right\} \ .$$

Moreover, the only rank-one matrices in L are multiples of the following three ones:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \ , \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \ .$$

Define a rank-one convex $f : L \to \mathbb{R}$,

$$f\left(\begin{pmatrix} r & 0 \\ 0 & s \\ t & t \end{pmatrix}\right) = -rst \ .$$

Counterexample

By simple calculation

$$\int_{(0,1)^2} f(\nabla\varphi(x))\,\mathrm{d}x < 0\ .$$

Then f can be slightly modified and extended from L to the whole space to a rank-one convex function which fails to be quasiconvex.

Other counterexamples?

- Essentially **no**! There exist (to the BOMK) only variants of the previous example.
- Different examples desperately needed to understand the problem better.
- Does rank-one convexity and frame invariance ($W(F) = W(RF)$ for any $R \in \mathrm{SO}(n)$) imply quasiconvexity? This would be important for elasticity.
- If $m = n = 2$ is it true that $\int_{(0,1)^2} W(F + \nabla\varphi^\top)\,\mathrm{d}x \geq W(F)$? for all $\varphi \in C_0^\infty((0,1)^2; \mathbb{R}^2)$ and all $F \in \mathbb{R}^{2\times 2}$? The negative answer means we have a counterexample for $m = n = 2$ (and, I guess, interesting job offers ☺), the affirmative answer shows a special property for 2×2.

Non-positive integrands and quasiconvexity

Quasiconvexity is always necessary and still sufficient for wlsc of I on $W^{1,p}$ $(1 < p < +\infty)$ if

$$-|F|^q \leq W(F) \leq C(1 + |F|^p)$$

and $q < p$.

However, if $p = q$ then it is not the case.

Counterexample

Take $B(0,1)$ a unit ball in $\mathbb{R}^n$, centered at zero, $u \in C_0^\infty(B(0,1);\mathbb{R}^n)$ and extend u by zero to the whole $\mathbb{R}^n$. Then $\int_{B(0,1)} \det \nabla u(x)\,\mathrm{d}x = 0$.
Take $\varrho \in \mathbb{R}^n$ a unit vector ρ such that $D_\varrho := \{x \in B(0,1);\ x \cdot \varrho < 0\}$ and

$$\int_{D_\varrho} \det \nabla u(x)\,\mathrm{d}x < 0 \ .$$

Denote $u_k(x) := u(kx)$ for all $k \in \mathbb{N}$; then, $u_k \to 0$ weakly in $W^{1,n}(B(0,1);\mathbb{R}^n)$ (even in measure) but for all $k \in \mathbb{N}$

$$\int_{D_\varrho} \det \nabla u_k(x)\,\mathrm{d}x \to \int_{D_\varrho} \det \nabla u(x)\,\mathrm{d}x < \int_{D_\varrho} \det \nabla 0\,\mathrm{d}x = 0$$

by our construction ☹.
On the other hand, for the other half-ball ☺,

$$\int_{D_{-\varrho}} \det \nabla u_k(x)\,\mathrm{d}x \to \int_{D_{-\varrho}} \det \nabla u(x)\,\mathrm{d}x > \int_{D_{-\varrho}} \det \nabla 0\,\mathrm{d}x = 0 \ ,$$

What has happened?

$\{u_k\}$ is such that $\|\nabla u_k\|_{L^n} = \|\nabla u\|_{L^n}$ but support of u_k shrinks to zero as $k \to \infty$.
In other words, gradients concentrate at the origin (boundary of D_ϱ).
This can destroy/help wlsc.

Conditions at the boundary?

- This problem addressed by N. Meyers in 1965 who gave a necessary and sufficient condition in terms of sequences, not in terms of integrands.
- If W is positively p-homogeneous then it is (besides quasiconvexity) sufficient and necessary if

$$\int_{D_\varrho} W(\nabla\varphi(x))\,\mathrm{d}x \geq 0$$

for all $\varphi \in C_0^\infty(B(0,1);\mathbb{R}^n)$ and any ϱ which coincides with the outer unit normal to $\partial\Omega$.
- This shows that the domain enters the game.
- More general situations can be treated S. Krömer (2010), S. Krömer & MK (2013) but many questions remain (higher-order gradients)

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Other kinds of problems – nonreflexivity

$$\text{minimize } J(u) := \int_0^1 ((x-1)^2+\varepsilon)|u'(x)|\,\mathrm{d}x + (u(1)-1)^2 \ ,$$
$$u \in W^{1,1}(0,1) \ \&\ u(0) = 0 \ .$$

$$J(u) \geq \varepsilon \int_0^1 |u'(x)|\,\mathrm{d}x + (u(1)-1)^2 = \varepsilon(u(1)-u(0)) + (u(1)-1)^2$$
$$= \varepsilon u(1) + (u(1)-1)^2 \geq (4\varepsilon-\varepsilon^2)/4 \ .$$

On the other hand, taking

$$u_k(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq 1-1/k \\ k\frac{2-\varepsilon}{2}x - (k-1)\frac{2-\varepsilon}{2} & \text{otherwise} \end{cases}$$

we see that $J(u_k) \to (4\varepsilon-\varepsilon^2)/4$ as $k \to \infty$, so it is a minimizing sequence. Notice that if $0 \leq \varepsilon < 2$ $u_k \to 0$ in $L^1(0,1)$ and (u'_k) concentrates at $x = 1$. Consequently, no minimizer exists in the admissible class of competitors. However, if $\varepsilon \geq 2$ then $u = 0$ is the minimizer.

Other functionals

$$I(u) := \int_\Omega W(u(x))\,\mathrm{d}x \ ,$$

if $\mathcal{A}u = 0$ where $\mathcal{A}$ is a first-order partial differential operator. (The case $\mathcal{A}$ =curl is included.)
You can think about $\mathcal{A} =$ div, i.e., solenoidal fields.

- A notion of $\mathcal{A}$-quasiconvexity. If $W \geq 0$ necessary and sufficient conditions given by Fonseca & Müller (1999).
- Negative integrands S. Krömer & MK & G. (2014).
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Conclusions

- Lower semicontinuity is an important issue in many areas of mathematics.
- Many open problems (characterization of quasiconvexity, relation to rank-one convexity), New ideas are missing.
- Algebraic constraints on the fields ($\det \nabla u > 0$) (talk by B. Benešová).
- Growth conditions suitable for mechanics......
- Description of limit behavior by means of parametrized measures (Young measures). Need to characterize them!
- Perhaps look at B. Benešová, MK: *Weak lower semicontinuity of integral functionals and applications.* Preprint arXiv:1601.00390.
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