Limits of Sobolev Homeomorphisms and Existence of 2D-Traction Free Minimal Deformations

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"You will have to name it," Pierre said to his young wife, in the same tone as if it were a question of choosing a name for little Irène. The one-time Mlle Sklodovska reflected in silence for a moment. Then, her heart turning toward her own country which had been erased from the map of the world, she wondered vaguely if the scientific event would be published in Russia, Germany and Austria- the oppressor countries-and answered timidly:”Could we call it polonium” ? In the Proceedings of the Academy of Science for July 1898 we read: ” If the existence of this new metal is confirmed we propose to call it polonium, from the name of the original country of one of us.”

-from the book MADAME CURIE
I devote my talk to the memory of Józef Marcinkiewicz and all Polish mathematicians whose glorious scientific careers had come to a cruel end during Nazi-Soviet occupation. Józef Marcinkiewicz, Stanisław Saks and Juliusz Paweł Schauder were inspiration to me. I am mindful of them not only as mathematicians.

Józef Marcinkiewicz, along with 22 thousand Polish great patriots who dared to exhibit a love and pride of independent Poland, were executed by NKVD murderers. By the order of J. Stalin, they were shot in the back of the head and buried secretly in mass graves of gloomy forested sites near Starobielsk, Ostaszków and the most documented Katyń (near Smoleńsk).

Recently (April 10, 2010) 96 Polish political elite, flying to Smoleńsk to honor the memory of victims of Katyń perished in a mysterious air crash.
KATYŃ CAROL Someday maybe a great musician will rise up, will transform speechless rows of gravestones into a keyboard, a great Polish song writer will compose a frightening ballad with blood and tears.

[...]
And there will emerge untold stories, strange hearts, bodies bathed in light...
And the Truth again will embody The Spirit with living words-of the sand of Katyń

- Kazimiera Ilłakowiczówna (translated by T.I.)

Heart-felt thanks for listening
Jani Onninen
Syracuse (USA) & Jyväskylä (Finland)

Researcher with Open Hands
Fellows with Ideas and Helping Hands

Jan Cristina  Leonid Kovalev  Jani Onninen  Ngin-Tee Koh
Sobolev Mappings

As we seek greater knowledge about the energy-minimal deformations in Geometric Function Theory and Nonlinear Hyperelasticity, the questions about Sobolev homeomorphisms and their limits become ever more quintessential.
Approximation with Diffeomorphisms


Every homeomorphism $h : \mathbb{X} \to \mathbb{Y}$ between planar open sets that belongs to the Sobolev class $W^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 < p < \infty$, can be approximated in the Sobolev norm by $C^\infty$-smooth diffeomorphisms.

**PROBLEM 1.** The case $p = 1$ still remains open.

Precisely, we have
APPROXIMATION THEOREM

Let $h : X \xrightarrow{\text{onto}} Y$ be a Sobolev homeomorphism between planar open sets

$$h \in \mathcal{W}_{\text{loc}}^{1,p}(X, \mathbb{R}^2), 1 < p < \infty$$

Then there exists a sequence of $C^\infty$-smooth diffeomorphisms $h_j : X \xrightarrow{\text{onto}} Y$ such that
\[ h_j - h \in W^{1,p}_0(X, \mathbb{R}^2), \quad j = 1, 2, \ldots \]

\[ (h_j - h) \rightarrow 0 \text{ uniformly on } X \]

\[ (Dh_j - Dh) \rightarrow 0 \text{ in } \mathcal{L}^p(X, \mathbb{R}^2) \]

\[ \|Dh_j\|_{\mathcal{L}^p(X)} \leq \|Dh\|_{\mathcal{L}^p(X)} \]
Ball-Evans Conjecture (solved)

A straightforward method of triangulation gives an approximation with piecewise affine homeomorphisms. This settles a long-standing Ball-Evans Conjecture. Actually J. Ball attributed this conjecture to L.C. Evans and pointed out its relevance to the regularity of hyperelastic deformations of neo-hookean materials.

Precisely, we have:
COROLLARY

Let $h : \mathbb{X} \to \mathbb{Y}$ be a homeomorphism in the Sobolev space $\mathcal{W}^{1,p}_{\text{loc}}(\mathbb{X}, \mathbb{R}^2)$, $1 < p < \infty$, between planar opens sets. Then there exists a sequence of piecewise affine homeomorphisms $h_j : \mathbb{X} \to \mathbb{Y}$ such that

$h_j - h \in \mathcal{W}^{1,p}_{\text{loc}}(\mathbb{X}, \mathbb{R}^2)$, $j = 1, 2, \ldots$

$(h_j - h) \to 0$ uniformly on $\mathbb{X}$

$(Dh_j - Dh) \to 0$ strongly in $\mathcal{L}^p(\mathbb{X}, \mathbb{R}^2)$

If $h$ arises as piece-wise affine near $\partial\mathbb{X}$, then $h_j \equiv h$ near $\partial\mathbb{X}$. 
Four Steps in the Proof

Step 1. Partition of $X$ into cells.
Partition of $\mathcal{Y}$ into small squares and the induced cellular structure in $\mathcal{X}$
Step 2. We replace $h$ on each cell by $p$-harmonic diffeomorphisms.

The Radó-Kneser-Choquet Theorem for $p$-harmonic mappings comes into play.

G. Alessandrini and M. Sigalotti (2001)
$p$-Harmonic Replacements in Cells

$h : X \rightarrow Y$

$U = h^{-1}(Q)$

$h_{p} = p$-harmonic extension of $h : \partial U \rightarrow \partial Q$
Step 3. On top of the previous cells
Step 4. We then smooth along interfaces of the adjacent cells.
PROBLEM 2. Can Sobolev homeomorphisms in dimensions \( n \geq 3 \) be approximated with diffeomorphisms?

The problem remains widely open even for \( n = 3 \).
It is by no means clear as to whether the celebrated examples by J. Milnor:

*On manifolds homeomorphic to the 7-sphere.* Ann. of Math. (2) **64** (1956), 399–405.

rule out diffeomorphic approximation of Sobolev homeomorphisms in high dimensions.
Bi-Sobolev Homeomorphisms

A bi-Sobolev homeomorphism \( h : X \onto Y \) is a mapping of class \( W^{1,p}(X, Y) \), \( 1 \leq p < \infty \), whose inverse \( h^{-1} : Y \onto X \) belongs to a Sobolev class \( W^{1,q}(Y, X) \), \( 1 \leq q < \infty \).

**PROBLEM 3.** Can \( h \) be approximated with bi-Sobolev diffeomorphisms \( \{h_j\} \) so that \( h_j \to h \) in \( W^{1,p}(X, Y) \) and \( h_j^{-1} \to h^{-1} \) in \( W^{1,q}(Y, X) \) ?
Univalent Extension up to the Boundary of the Approximating Diffeomorphisms \((p = 2)\)


If one is willing to sacrifice the boundary condition \(h_j \in h + W^{1,2}_0(X, \mathbb{R}^2)\) \((traction free approximation)\), then we can approximate with diffeomorphisms \(h_j : X \onto Y\) which extend as homeomorphisms of \(\overline{X} \onto \overline{Y}\).

Precisely, we have
**THEOREM.** Let $\mathbb{X}$ and $\mathbb{Y}$ be Lipschitz domains and $h : \mathbb{X} \rightarrow \mathbb{Y}$ a homeomorphism of Sobolev class $W^{1,2}(\mathbb{X}, \mathbb{R}^2)$. Then,

1) There exist homeomorphisms $h_j : \overline{\mathbb{X}} \rightarrow \overline{\mathbb{Y}}$ converging to $h$ uniformly on $\overline{\mathbb{X}}$ and strongly in $W^{1,2}(\mathbb{X}, \mathbb{R}^2)$.

2) In the interiors the mappings $h_j : \mathbb{X} \rightarrow \mathbb{Y}$ are $C^\infty$-diffeomorphisms.
Weak and Strong Limits are Equal

**THEOREM.** (J. Onninen and T. Iwaniec, 2013)

Let \( X, Y \in \mathbb{R}^2 \) be bounded multiply connected Lipschitz domains and \( h_j : X \onto Y \) homeomorphisms converging weakly in \( W^{1,p}(X, Y) \) to \( h \in W^{1,p}(X, \mathbb{R}^2), \ p \geq 2 \). Then there exists a sequence of \( C^\infty \)-diffeomorphisms \( h_j^* : X \onto Y, \ h_j^* \in h + W^{1,p}_o(X, Y) \) converging strongly in \( W^{1,p}(X, \mathbb{R}^2) \) to \( h \).
Monotone Sobolev Mappings.

Let us point out that homeomorphisms between Lipschitz domains $X \overset{\text{onto}}{\rightarrow} Y$ in $W^{1,p}(X, \mathbb{R}^2)$, $p \geq 2$, and their weak limits extend continuously as monotone mappings of $\overline{X} \overset{\text{onto}}{\rightarrow} \overline{Y}$.

Monotonicity, a concept introduced by C. B. Morrey (1935), means that the preimage $h^{-1}(C)$ of a continuum $C \subset \overline{Y}$ is a continuum in $\overline{X}$.
THEOREM. (J. Onninen and T. Iwaniec, 2013)

Let $h : \overline{X} \onto \overline{Y}$ be a continuous monotone mapping between Lipschitz domains in the Sobolev space $W^{1,p}(X, \mathbb{R}^2), 1 < p < \infty$. Then $h$ can be approximated uniformly and strongly in $W^{1,p}(X, \mathbb{R}^2)$ with continuous mappings $h_j : \overline{X} \onto \overline{Y}$ that are $C^\infty$-diffeomorphic on $X$, $h_j = h : \partial X \onto \partial Y$. 
Uniform approximation of monotone mappings (between compact metric spaces) with homeomorphisms is of great interest in topology. On the other hand: Approximation of monotone Sobolev mappings between manifolds (cellular mappings if \( n \geq 3 \)) with diffeomorphisms is at the very heart of Geometric Function Theory and Nonlinear Hyperelasticity.
Existence of Traction Free Energy-Minimal Monotone Mappings (no Lavrentiev Phenomenon)

Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$ be bounded Lipschitz domains of the same topological type. For a fairly general energy functionals, we conclude:

**THEOREM.** (J. Onninen and T.Iwaniec, 2013) Among all continuous monotone mappings $h : \overline{\mathbb{X}} \onto \overline{\mathbb{Y}}$ of Sobolev class $W^{1,p}(\mathbb{X}, \mathbb{Y})$, $2 \leq p < \infty$, there exists $h_\circ : \overline{\mathbb{X}} \onto \overline{\mathbb{Y}}$ of smallest energy. Moreover, the energy of $h_\circ$ equals exactly the infimum energy among homeomorphisms.
Hopf Laplace Equation

It is legitimate to perform the inner variation which, in case of the Dirichlet energy, gives rise to:

\[ \frac{\partial}{\partial \bar{z}} \left( h_z \bar{h}_{\bar{z}} \right) = 0 \]

**THEOREM** (Iwaniec, Kovalev, Onninen, Duke Math. J. 2013)

Every \( W^{1,2}_{\text{loc}} \)-solution with nonnegative Jacobian is locally Lipschitz continuous, but not necessarily \( C^1 \)-smooth.
The round annulus is too fat to avoid interpenetration of matter. The cuts in $X$ are necessary. They collapse into the corners of $Y$. The energy-minimal deformation is Lipschitz in the entire annulus (but not $C^1$). Outside the cuts it is injective.
Collapsing of the Dirichlet energy-minimal monotone map

\[
\mathcal{h}(z) = \begin{cases} 
\frac{z}{|z|}, & r < |z| < 1 \\
\frac{1}{2} \left( z + \frac{1}{z} \right), & 1 < |z| < R 
\end{cases}
\]

- collapsing into concave boundary
- critical harmonic Nitsche map
In spite of very impressive progress in the field, collapsing phenomenon of the energy-minimal deformations is not fully resolved. The energy-minimal monotone maps that fail to be invertible tell us where to stop the minimizing sequence of homeomorphisms prior to the conditions favorable to the formation of interpenetration of matter. For the Dirichlet energy we have established the following principle: 
(J. Onninen and T. Iwaniec, Calculus of Var. and PDEs, 2014)

The loss of invertibility comes exactly with the loss of the Lagrange equation.
The Nitsche Conjecture

In the early 1960’s German-American mathematician Johannes C.C. Nitsche raised a question of existence of harmonic homeomorphisms between circular annuli

$$h : \mathbb{A}(r, R) \xrightarrow{\text{onto}} \mathbb{A}(r_*, R_*)$$

Nitsche’s conjecture, which is now a theorem, asserts:
A harmonic homeomorphism does exist iff:

\[
\frac{R_*}{r_*} \geq \frac{1}{2} \left( \frac{R}{r} + \frac{r}{R} \right)
\]
No Interpenetration of Matter

(Iwaniec, Koh, Kovalev, Onninen, Inventiones Math. 2011)

**THEOREM.** Among all homeomorphisms $h: X \mapsto Y$ between bounded doubly connected domains such that $\text{Mod } X \leq \text{Mod } Y$

there exists an energy-minimal harmonic diffeomorphism.

(unique up to conformal automorphisms of $X$)
Theoretical prediction of failure of bodies caused by collapsing of matter is a good motivation that should appeal to MATHEMATICAL ANALYSTS and researchers in the ENGINEERING FIELDS.

Thank You for Listening