Signed integral functionals with linear growth: Weak* lower semicontinuity in BV without prescribed boundary values

Stefan Krömer

Universität zu Köln

joint work:
Barbora Benešová (Aachen), Martin Kružík (Prague)
Introduction
  Setting
  Aim and problems
  Known results

Weak* lower semicontinuity without prescribed boundary values
  A new characterization
  Quasi-sublinear growth from below

Elements of the proof
  Pure concentrations at the boundary
  Localizing decomposition
Outline

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  Localizing decomposition
Consider an integral functional of the following form:

\[
F(u) := \int_{\Omega} f(x, \nabla u(x)) \, dx \quad \text{for } u \in W^{1,1}(\Omega; \mathbb{R}^M),
\]

with
- \( \Omega \subset \mathbb{R}^N \) a bounded domain with \( C^1 \) boundary;
- \( f : \bar{\Omega} \times \mathbb{R}^{M \times N} \rightarrow \mathbb{R} \) a given continuous function;
- linear growth condition: \( |f(x, \xi)| \leq C(1 + |\xi|) \).
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“signed”: No additional restrictions on the growth of \( f^- \).
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- $f : \bar{\Omega} \times \mathbb{R}^{M \times N} \to \mathbb{R}$ a given continuous function;
- linear growth condition: $|f(x, \xi)| \leq C(1 + |\xi|)$.

**Note**

“signed”: No additional restrictions on the growth of $f^-$. The setting in $W^{1,1}$ is bad for direct methods: **Extend $F$ to $BV$.**
Setting: The functional on $BV$

Assume that $f$ has a recession function

$$f^\infty(x, \xi) = \lim_{s \to +\infty, \eta \to \xi, y \to x} \frac{f(y, s\eta)}{s}, \quad \xi \in \mathbb{R}^{M \times N} \setminus \{0\}, \ x \in \tilde{\Omega}.$$ 

In particular, $f^\infty$ is continuous and 1-homogeneous in $\xi$. 
Setting: The functional on $BV$

Assume that $f$ has a *recession function*

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f^{\infty}(x, \xi) = \lim_{s \to +\infty, \eta \to \xi, y \to x} \frac{f(y, s\eta)}{s}, \quad \xi \in \mathbb{R}^{M \times N} \setminus \{0\}, \ x \in \bar{\Omega}.
\]

In particular, $f^{\infty}$ is continuous and 1-homogeneous in $\xi$. Now:

\[
F(u) := \int_{\Omega} df(x, Du) \text{ for } u \in BV(\Omega; \mathbb{R}^M), \text{ where}
\]

\[
df(x, Du) := f(x, \nabla u(x)) \, dx + f^{\infty}(x, \frac{dDu^s}{d |Du^s|}) \, d |Du^s|(x).
\]

- $Du \in \mathcal{M}(\Omega; \mathbb{R}^{M \times N})$: weak derivative of $u \in BV$;
- $Du^s$: singular part of $Du$ w.r.t. $\mathcal{L}^N$; $|Du^s|$ is its total variation
- $\nabla u = \frac{dDu^a}{dx}$: density of the abs. cont. part of $Du$ w.r.t. $\mathcal{L}^N$;
- $\frac{dDu^s}{d |Du^s|}$: density of $Du^s$ w.r.t. $|Du^s|$. 
W*ls in BV

**Definition (w*ls)**

F is called (sequentially) weakly* lower semicontinuous (w*ls) in BV, if

\[ Du_n \rightharpoonup^* Du \text{ in } M, u_n \rightarrow u \text{ in } L^1_{\text{loc}} \implies \lim \inf F(u_n) \geq F(u) \]

**Aim**

Characterize weak* lower semicontinuity of F in terms of f.

▶ Quasiconvexity of f(x, ·) is clearly necessary, but not sufficient.
▶ There are many quasiconvex functions with linear growth (Kristensen 1999)

▶ Nontrivial examples for signed quasiconvex functions?
W*ls in BV

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Aim

Characterize weak* lower semicontinuity of $F$ in terms of $f$.

Problem

Quasiconvexity of $f(x, \cdot)$ is clearly necessary, but not sufficient.

- There are many quasiconvex functions with linear growth (Kristensen 1999)
- Nontrivial examples for signed quasiconvex functions?
A simple linear counterexample

Example

The linear functional for $f(x, \xi) = f^\infty(x, \xi) = \xi$,

$$F(u) := \int_0^1 dDu(x), \quad u \in BV((0,1); \mathbb{R})$$

is not $w^*\operatorname{lsc}$ in $BV$, in particular along the sequence

$$u_n(x) := \begin{cases} 1 & \text{if } x < \frac{1}{n}, \\ 0 & \text{if } x \geq \frac{1}{n}, \end{cases} \quad \text{whence } Du_n = -\delta_{\frac{1}{n}}.$$

We have

$$F(u_n) = -1 \quad \text{for all } n, \quad u_n \rightharpoonup^* 0, \quad F(0) = 0 > -1.$$
In our setting, the following is known (see also Ambrosio & Dal Maso 1992):

**Theorem (Fonseca & Müller 1993, special case)**

If $f \geq 0$, then

$$F \text{ is } \text{w}^*\text{lsc} \iff f(x, \cdot) \text{ is quasiconvex for every } x \in \Omega.$$  

If $f$ attains negative values, the theorem still holds as long as $f^-$ has sublinear growth,

i.e.,

$$\forall \varepsilon > 0 \exists h_\varepsilon \in L^1 : f(x, \xi) \geq -\varepsilon |\xi| - |h_\varepsilon(x)|. \quad (*)$$

At fixed $x$, if $|h_\varepsilon(x)| < \infty$, this is equivalent to

$$\lim \inf_{|\xi| \to \infty} |\xi|^{-1} f(x, \xi) \geq 0.$$
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Theorem (Kristensen&Rindler 2010)

The following two conditions are equivalent:

(i) $F$ is w*lsclong sequences with fixed boundary values, i.e.,
\[ \liminf F(u_n) \geq F(u) \] along all sequences $u_n \rightharpoonup^* u \in BV$
such that $u_n = u$ on $\partial \Omega$ (in the sense of trace in $BV$)

(ii) $f(x, \cdot)$ is quasiconvex for every $x \in \bar{\Omega}$

- linear growth of $f^-$ is allowed
- $\partial \Omega$ may be Lipschitz instead of $C^1$
- $f$ may be Carathéodory instead of continuous (but still continuous in $(x, \xi)$ “as $|\xi| \to \infty$” due to the definition of $f^\infty$)
- if $f = f(\xi)$, a weaker notion of recession function suffices:
\[ f^\infty(\xi) = \limsup_{s \to +\infty, \eta \to \xi} \frac{f(s\eta)}{s} \]
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Theorem (Kristensen&Rindler 2010)

The following two conditions are equivalent:

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A new characterization: The main theorem

**Theorem (Benešová, K. & Kružík 2014)**

$F : BV(\Omega; \mathbb{R}^M) \rightarrow \mathbb{R}$ is w*lsc if and only if

(i) $f(x_0, \cdot)$ is quasiconvex for every $x_0 \in \bar{\Omega}$, and

(ii) $f$ is qslb at $x_0$, for every $x_0 \in \partial \Omega$.

- qslb abbreviates "quasi-sublinear from below":
  a sublinear growth condition from below tested by gradients of functions with suitable boundary values

- qslb coincides with $p$-qscb for $p = 1$, the condition appearing in the corresponding result in $W^{1,p}$ for $p > 1$ (K. 2010; see also K. & Kružík 2013)

- $f$ can be replaced by $f^\infty$ in (ii)
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Quasi-sublinear growth from below

**Definition (qslb)**

We say that \( f \) is *quasi-sublinear from below (qslb) at \( x_0 \in \bar{\Omega} \)*, if

\[
\forall \varepsilon > 0 \ \exists \delta > 0, \ C \geq 0:\n\int_{\Omega \cap B_\delta(x_0)} f(x, \nabla \varphi(x)) \, dx \geq -\varepsilon \int_{\Omega \cap B_\delta(x_0)} |\nabla \varphi| \, dx - C
\]

for every \( \varphi \in W^{1,1}(B_\delta(x_0) \cap \Omega; \mathbb{R}^M) \) with \( \varphi = 0 \) on \( \partial B_\delta(x_0) \).
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**Equivalent variant with test functions in BV:**

\[
\forall \varepsilon > 0 \ \exists \delta > 0, \ C \geq 0 : \\
\int_{\Omega \cap B_\delta(x_0)} df(x, D\varphi) \geq -\varepsilon |D\varphi| (\Omega \cap B_\delta(x_0)) - C
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for every \( \varphi \in BV(B_\delta(x_0) \cap \Omega; \mathbb{R}^M) \) with \( \varphi = 0 \) on \( \partial B_\delta(x_0) \).

Density argument: \( F \) is continuous w.r.t. **area-strict convergence**

(generalized Reshetnyak Thm., KRISTENSEN&RINDLER’10)
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Equivalent variants for \( x_0 \in \partial \Omega \), with outer normal \( \nu_{x_0} \) and
\[
D(x_0) := \{ y \in B_1(0) \subset \mathbb{R}^N \mid y \cdot \nu_{x_0} < 0 \}:
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\[
\forall \varepsilon > 0 \ \exists C \geq 0 : \\
\int_{D(x_0)} f(x_0, \nabla \eta(y)) \, dy \geq -\varepsilon \int_{D(x_0)} |\nabla \eta(y)| \, dy - C
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for every \( \eta \in W^{1,1}_0(B_1(0); \mathbb{R}^M) \).

\((**\))
Quasi-sublinear growth from below

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We say that $f$ is *quasi-sublinear from below (qslb)* at $x_0 \in \overline{\Omega}$, if

$$\forall \varepsilon > 0 \ \exists \delta > 0, \ C \geq 0 :$$

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for every $\varphi \in W^{1,1}(B_\delta(x_0) \cap \Omega; \mathbb{R}^M)$ with $\varphi = 0$ on $\partial B_\delta(x_0)$.

**Equivalent variants for** $x_0 \in \partial \Omega$, **with outer normal** $\nu_{x_0}$ **and**

$$D(x_0) := \{ y \in B_1(0) \subset \mathbb{R}^N \mid y \cdot \nu_{x_0} < 0 \} :$$

$$\int_{D(x_0)} f^\infty(x_0, \nabla \eta(y)) \, dy \geq 0$$

(***)

for every $\eta \in W^{1,1}_0(B_1(0); \mathbb{R}^M)$. 
Remarks on the theorem

- **Continuity of $f^\infty$ in $x$ is crucial:** As in the linear example shown before,

\[
F : BV((-1,1); \mathbb{R}) \to \mathbb{R}, \quad F(u) := \int_{-1}^{1} \chi_{(0,1)}(x) dDu(x)
\]

is not $w^\ast$lsc.
Remarks on the theorem

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- **Possible generalizations without continuity in $x$:** The boundary and jump discontinuities of $f^\infty$ in $x$ across a smooth hypersurface are similar; a variant of qslb is then needed also at the jump surfaces inside $\Omega$. 
Remarks on the theorem

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- **Possible generalizations without continuity in \( x \):** The boundary and jump discontinuities of \( f^\infty \) in \( x \) across a smooth hypersurface are similar; a variant of qslb is then needed also at the jump surfaces inside \( \Omega \).

- **Relaxation of non-qslb integrands is not possible:** If \( f \) is not qslb at some \( x \in \overline{\Omega} \), then the w* lsc envelope of \( F \) is \(-\infty\).
Remarks on qslb

Example: \( f(x, \xi) := a(x)^T \xi t(x) \) is qslb, for continuous functions
\[
a : \bar{\Omega} \to \mathbb{R}^M, \quad t : \bar{\Omega} \to \mathbb{R}^N
\]
such that \( t(x) \cdot \nu_x = 0 \) on \( \partial \Omega \).
Remarks on qslb

▶ Example: \( f(x, \xi) := a(x)^T \xi t(x) \) is qslb, for continuous functions \( a : \bar{\Omega} \to \mathbb{R}^M, \hspace{1cm} t : \bar{\Omega} \to \mathbb{R}^N \) such that \( t(x) \cdot \nu_x = 0 \) on \( \partial \Omega \).

▶ qslb and quasiconvexity at the boundary: If \( f^\infty \) is qslb at \( x_0 \) \((A = \xi = 0 \) below\), then \( f^\infty(x_0, \cdot) \) is qcb at the zero matrix w.r.t. the normal \( \nu_{x_0} \) \((\text{some } A, \xi = 0)\), but not vice versa (linear \( f \)).

Definition (qcb: Ball&Marsden 1984; Sprenger 1996)

\( f^\infty(x_0, \cdot) \) is quasiconvex at the boundary (qcb) at the matrix \( \xi \in \mathbb{R}^{M \times N} \) w.r.t. the normal \( \nu_{x_0} \), if

\[
\int_{D(x_0)} f^\infty(x_0, \nabla \eta(y) + \xi) \, dy \geq \int_{D(x_0)} \left( f^\infty(x_0, \xi) + A : \nabla \eta(y) \right) \, dy
\]

for a suitable \( A \in \mathbb{R}^{M \times N} \) and every \( \eta \in W^{1,\infty}_0(B_1(0); \mathbb{R}^M) \).
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Strategy of the proof

To show: $qc$ and $qslb$ $\implies$ $w^*lsc$ along some $u_n \rightharpoonup^* u$:

- Split $u_n - u = v_n + w_n$, with $Dv_n$ “purely concentrating at the boundary” and $w_n = 0$ near $\partial \Omega$, such that $Dv_n$ and $Dw_n$ do not interact in $F$ (localizing decomposition in $BV$).
Strategy of the proof

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- Use the theorem of KRISTENSEN & RINDLER along $w_n + u$.
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- Use the theorem of Kristensen & Rindler along $w_n + u$
- Further split $v_n$ into components living on $\varepsilon$-balls centered on $\partial \Omega$ (localizing decomposition again).
Strategy of the proof

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- Each piece $p_n$ of $v_n$ is an admissible test function for $qslb$, and $qslb$ implies that

$$G_\varepsilon(p) := \int_\Omega df(x, Dp) + \varepsilon |Dp| - df(x, 0) \, dx$$

is bounded from below on the admissible test functions.
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To show: $qc$ and $qslb \implies w^\star lsc$ along some $u_n \rightharpoonup^* u$:

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- Use the theorem of Kristensen & Rindler along $w_n + u$
- Further split $v_n$ into components living on $\varepsilon$-balls centered on $\partial \Omega$ (localizing decomposition again).
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- Functionals bounded from below are $w^\star lsc$ along pure concentrations at the boundary.
Strategy of the proof

**Note**

The strategy is similar to the one used in $W^{1,p}$ (K. 2010), but we **cannot use the decomposition lemma** to separate concentrations from oscillations!
Lemma

Suppose that $F$ is bounded from below, $f(x, \cdot)$ is Lipschitz (qc suffices), $u_n \rightharpoonup^* u$ in $BV$. If

$$\text{supp} |Du_n - Du| \subset (\partial \Omega)_{r_n} := \bigcup_{x \in \partial \Omega} B_{r_n}(x) \text{ for some } r_n \downarrow 0$$

(purely concentrating at the boundary), then $\liminf F(u_n) \geq F(u)$.

Proof. Let $\varepsilon > 0$.

- Choose $u^* \in BV$ such that $F(u + u^*) \leq \varepsilon + \inf F$.

- Observe that

$$F(u_n) - F(u) - F(u_n + u^*) + F(u + u^*) \to 0,$$

essentially because $|Du^*| (\text{supp} |Du_n - Du|) \leq |Du^*| ((\partial \Omega)_{r_n}) \to 0$.

- Since $F(u_n + u^*) - F(u + u^*) \geq (\inf F) - F(u + u^*) \geq -\varepsilon$, we get that

$$\liminf F(u_k + u) - F(u) \geq -\varepsilon.$$
Lemma

Suppose that $F$ is bounded from below, $f(x, \cdot)$ is Lipschitz (qc suffices), $u_n \rightharpoonup^* u$ in $BV$. If

$$\text{supp} |Du_n - Du| \subset (\partial \Omega)_{r_n} := \bigcup_{x \in \partial \Omega} B_{r_n}(x)$$

for some $r_n \downarrow 0$ (purely concentrating at the boundary), then $\lim \inf F(u_n) \geq F(u)$. 

Proof. Let $\varepsilon > 0$.

- Choose $u^* \in BV$ such that $F(u + u^*) \leq \varepsilon + \inf F$.

- Observe that

$$F(u_n) - F(u) - F(u_n + u^*) + F(u + u^*) \to 0,$$

essentially because $|Du^*| (\text{supp} |Du_n - Du|) \leq |Du^*| ((\partial \Omega)_{r_n}) \to 0$.

- Since $F(u_n + u^*) - F(u + u^*) \geq (\inf F) - F(u + u^*) \geq -\varepsilon$, we get

$$\lim \inf F(u_k + u) - F(u) \geq -\varepsilon.$$
Lemma

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W*Isco along pure concentrations

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Localizing decomposition

**Definition (does not charge)**

Let $(u_n) \subset BV$, $K \subset \mathbb{R}^N$ closed. $(Du_n)$ does not charge $K$, if

$$\sup_{n \in \mathbb{N}} |Du_n| ((K)_\delta \cap \Omega) \xrightarrow{\delta \to 0^+} 0,$$

where $(K)_\delta := \bigcup_{x \in K} B_\delta(x)$.

**Lemma (localizing decomposition in $BV$)**

Let

- $K_1, K_2 \subset \bar{\Omega}$ compact with $\bar{\Omega} \subset K_1 \cup K_2$;
  (e.g.: $K_1 = \partial \Omega$, $K_2 = \bar{\Omega}$)
- $(u_n) \subset BV$ bounded with $u_n \to 0$ in $L^1$

Then a subsequence decomposes as

$$u_{k(n)} = u_{1,n} + u_{2,n},$$

with $(u_{j,n})_n \subset BV$ bounded s.t. $u_{j,n} \to 0$ in $L^1$ ($j = 1, 2$), and:

(i) $\{u_{1,n} \neq 0\} \subset (K_1)_{\frac{1}{n}}$, $\{u_{2,n} \neq 0\} \subset (K_2)_{\frac{1}{n}} \setminus K_1$,

$\{u_{j,n} \neq 0\} \subset \{u_n \neq 0\}$, $|Du_{j,n}| \leq |Du_{k(n)}| + \frac{1}{n} \mathcal{L}^N$;

(ii) $(Du_{2,n})$ does not charge $K_1$. 
Localizing decomposition

**Definition (does not charge)**

Let \((u_n) \subset BV, K \subset \mathbb{R}^N\) closed. \((Du_n)\) does not charge \(K\), if

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\]

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**Lemma (localizing decomposition in \(BV\))**

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Localizing decomposition: sketch of proof

Given:

- $K_1, K_2 \subset \bar{\Omega}$ compact with $\bar{\Omega} \subset K_1 \cup K_2$;
- $(u_n) \subset BV$ bounded with $u_n \to 0$ in $L^1$

The proof is analogous to the corresponding result in $W^{1,p}$ (K. 2010):

Choose cutoff-functions:

- $\varphi_n \in C^\infty(\mathbb{R}^N; [0, 1])$, $\varphi_n = 1$ on $(K_1)_{\frac{1}{2n}}$, $\varphi_n = 0$ outside $(K_1)_{\frac{1}{n}}$

Observe: Since $u_k \to 0$ in $L^1$,

$$D(\varphi_n u_k) - \varphi_n Du_k = (\nabla \varphi_n) \otimes u_k \to 0 \quad \text{for fixed } n,$$

strongly as measures.

$u_{1,n} := \varphi_n u_{k(n)}$, with $k(n)$ (sufficiently fast) subsequence of $n$ such that

$$\lim_{n} |Du_{k(n)}|((K_1)_{\frac{1}{n}}) = \lim_{n} \lim_{m} |Du_{m}|((K_1)_{\frac{1}{n}})$$

Hence, $(K_1)_{\frac{1}{n}}$ captures everything in $Du_{k(n)}$ charging $K_1$. 
Localizing decomposition: sketch of proof

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Hence, \( (K_1)_{\frac{1}{n}} \) captures everything in \( Du_{k(n)} \) charging \( K_1 \).
Given:

- $K_1, K_2 \subset \tilde{\Omega}$ compact with $\tilde{\Omega} \subset K_1 \cup K_2$;
- $(u_n) \subset BV$ bounded with $u_n \to 0$ in $L^1$

The proof is analogous to the corresponding result in $W^{1,p}$ (K. 2010):

- Choose cutoff-functions:

  $\varphi_n \in C^\infty(\mathbb{R}^N; [0, 1])$, $\varphi_n = 1$ on $(K_1)_{\frac{1}{2n}}$, $\varphi_n = 0$ outside $(K_1)_{\frac{1}{n}}$

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  Hence, $(K_1)_{\frac{1}{n}}$ captures everything in $Du_{k(n)}$ charging $K_1$. 

- $K_1, K_2 \subset \tilde{\Omega}$ compact with $\tilde{\Omega} \subset K_1 \cup K_2$;
- $(u_n) \subset BV$ bounded with $u_n \to 0$ in $L^1$
Localizing decomposition: non-interaction

**Proposition (asymptotical additivity)**

Suppose that \( f \) is Lipschitz in the second variable, i.e.,

\[
|f(x, \xi) - f(x, \eta)| \leq C |\xi - \eta|,
\]

\( \nu \in BV \), \((u_{k(n)}) \subset BV \) and

\[
uk(n) = u_{1,n} + u_{2,n}
\]

is decomposed into component sequences with the properties listed in the localizing decomposition. Then

\[
f(x, Du_{k(n)} + D\nu) - f(x, D\nu) - \sum_{j=1}^{2} [f(x, Du_{j,n} + D\nu) - f(x, D\nu)] \to 0
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Proposition (asymptotical additivity)

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Thank you for your attention!