Relaxation, homogenization and dimensional reduction in hyperelasticity

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Signed integral functionals with linear growth: Weak* lower semicontinuity in BV without prescribed boundary values

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Outline

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Setting: The functional on $W^{1,1}$

Consider an integral functional of the following form:

$$F(u) := \int_{\Omega} f\big(x, \nabla u(x)\big)\, dx \quad \text{for } u \in W^{1,1}(\Omega; \mathbb{R}^M),$$

with

- $\Omega \subset \mathbb{R}^N$ a bounded domain with C^1 boundary;
- $f : \bar{\Omega} \times \mathbb{R}^{M \times N} \to \mathbb{R}$ a given continuous function;
- linear growth condition: $|f(x, \xi)| \leq C(1 + |\xi|)$.

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"signed": No additional restrictions on the growth of f^-.

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The setting in $W^{1,1}$ is bad for direct methods: **Extend F to BV.**

Setting: The functional on BV

Assume that f has a *recession function*

$$f^{\infty}(x,\xi) = \lim_{s\to+\infty,\eta\to\xi,y\to x} \frac{f(y,s\eta)}{s}, \quad \xi \in \mathbb{R}^{M\times N}\setminus\{0\},\ x\in\bar{\Omega}.$$

In particular, f^{∞} is continuous and 1-homogeneous in ξ.

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In particular, f^∞ is continuous and 1-homogeneous in ξ. Now:

$$F(u) := \int_\Omega df(x, Du) \quad \text{for } u \in BV(\Omega;\mathbb{R}^M), \text{ where}$$

$$df(x,Du) := f(x,\nabla u(x))\,dx + f^\infty\left(x, \frac{dDu^s}{d\,|Du^s|}\right) d\,|Du^s|\,(x).$$

- $Du \in \mathcal{M}(\Omega;\mathbb{R}^{M\times N})$: weak derivative of $u \in BV$;
- Du^s: singular part of Du w.r.t. $\mathcal{L}^N$; $|Du^s|$ is its total variation
- $\nabla u = \frac{dDu^a}{dx}$: density of the abs. cont. part of Du w.r.t. $\mathcal{L}^N$;
- $\frac{dDu^s}{d|Du^s|}$: density of Du^s w.r.t. $|Du^s|$.

W*lsc in BV

Definition (w*lsc)

F is called (sequentially) *weakly* lower semicontinuous* (*w*lsc*) in BV, if

$$Du_n \rightharpoonup^* Du \quad \text{in } \mathcal{M}, u_n \to u \quad \text{in } L^1_{\text{loc}} \quad \Longrightarrow \quad \liminf F(u_n) \geq F(u)$$

Aim

Characterize weak* lower semicontinuity of F in terms of f.

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Quasiconvexity of $f(x, \cdot)$ is clearly necessary, but not sufficient.

- There are many quasiconvex functions with linear growth (Kristensen 1999)
- Nontrivial examples for signed quasiconvex functions?

A simple linear counterexample

Example

The linear functional for $f(x,\xi) = f^\infty(x,\xi) = \xi$,

$$F(u) := \int_0^1 dDu(x), \quad u \in BV((0,1);\mathbb{R})$$

is not w*lsc in BV, in particular along the sequence

$$u_n(x) := \begin{cases} 1 & \text{if } x < \frac{1}{n}, \\ 0 & \text{if } x \geq \frac{1}{n}, \end{cases} \quad \text{whence} \quad Du_n = -\delta_{\frac{1}{n}}.$$

We have

$$F(u_n) = -1 \ \text{ for all } n, \quad u_n \rightharpoonup^* 0, \quad F(0) = 0 > -1.$$

Known results 1: Fonseca&Müller 1993

In our setting, the following is known (see also Ambrosio&Dal Maso 1992):

Theorem (Fonseca&Müller 1993, special case)

If $f \geq 0$, then

$$F \text{ is w}^*\text{lsc} \iff f(x,\cdot) \text{ is quasiconvex for every } x \in \Omega.$$

If f attains negative values, the theorem still holds as long as

$$f^- \text{ has sublinear growth,}$$

i.e.,

$$\forall \varepsilon > 0\ \exists h_\varepsilon \in L^1 : \quad f(x,\xi) \geq -\varepsilon\,|\xi| - |h_\varepsilon(x)|. \qquad (*)$$

At fixed x, if $|h_\varepsilon(x)| < \infty$, this is equivalent to

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The following two conditions are equivalent:

(i) F is w*lsc along sequences with fixed boundary values, i.e.,

$$\liminf F(u_n) \geq F(u) \text{ along all sequences } u_n \rightharpoonup^* u \in BV$$

such that $u_n = u$ on $\partial\Omega$ (in the sense of trace in BV)

(ii) $f(x,\cdot)$ is quasiconvex for every $x \in \bar{\Omega}$

- linear growth of f^- is allowed
- $\partial\Omega$ may be Lipschitz instead of C^1
- f may be Carathéodory instead of continuous (but still continuous in (x,ξ) "as $|\xi| \to \infty$" due to the definition of f^∞)
- if $f = f(\xi)$, a weaker notion of recession function suffices: $f^\infty(\xi) = \limsup_{s\to+\infty,\eta\to\xi} \frac{f(s\eta)}{s}$

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A new characterization: The main theorem

Theorem (Benešová, K. & Kružík 2014)

$F : BV(\Omega;\mathbb{R}^M) \to \mathbb{R}$ is w*lsc if and only if

(i) $f(x_0,\cdot)$ is quasiconvex for every $x_0 \in \bar{\Omega}$, and

(ii) f is qslb at x_0, for every $x_0 \in \partial\Omega$.

- qslb abbreviates "quasi-sublinear from below":
  a sublinear growth condition from below tested by gradients of functions with suitable boundary values
- qslb coincides with p-qscb for $p = 1$, the condition appearing in the corresponding result in $W^{1,p}$ for $p > 1$ (K. 2010; see also K.&Kružík 2013)
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Quasi-sublinear growth from below

Definition (qslb)

We say that f is *quasi-sublinear from below (qslb) at* $x_0 \in \bar{\Omega}$, if

$$\forall \varepsilon > 0 \; \exists \delta > 0, C \geq 0 :$$

$$\int_{\Omega \cap B_\delta(x_0)} f(x, \nabla\varphi(x)) dx \geq -\varepsilon \int_{\Omega \cap B_\delta(x_0)} |\nabla \varphi| \, dx - C$$

for every $\varphi \in W^{1,1}(B_\delta(x_0) \cap \Omega; \mathbb{R}^M)$ with $\varphi = 0$ on $\partial B_\delta(x_0)$.

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Equivalent variant with test functions in BV:

$$\forall \varepsilon > 0 \; \exists \delta > 0, C \geq 0 :$$

$$\int_{\Omega \cap B_\delta(x_0)} df(x, D\varphi) \geq -\varepsilon \, |D\varphi| \, (\Omega \cap B_\delta(x_0)) - C$$

for every $\varphi \in BV(B_\delta(x_0) \cap \Omega; \mathbb{R}^M)$ with $\varphi = 0$ on $\partial B_\delta(x_0)$.

Density argument: F is continous w.r.t. **area-strict convergence**
(generalized Reshetnyak Thm., Kristensen&Rindler'10)

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Equivalent variants for $\mathbf{x_0} \in \partial\Omega$, with outer normal ν_{x_0} and

$$D(x_0) := \{y \in B_1(0) \subset \mathbb{R}^N \mid y \cdot \nu_{x_0} < 0\} :$$

$$\forall \varepsilon > 0 \; \exists C \geq 0 :$$
$$\int_{D(x_0)} f(x_0, \nabla\eta(y))dy \geq -\varepsilon \int_{D(x_0)} |\nabla\eta(y)| \, dy - C \qquad (**)$$
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Remarks on the theorem

- **Continuity of f^∞ in x is crucial:** As in the linear example shown before,

$$F : BV((-1,1);\mathbb{R}) \to \mathbb{R}, \quad F(u) := \int_{-1}^{1} \chi_{(0,1)}(x) dDu(x)$$

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- **Possible generalizations without continuity in x:** The boundary and jump discontinuities of f^∞ in x across a smooth hypersurface are similar; a variant of qslb is then needed also at the jump surfaces inside Ω.

- **Relaxation of non-qslb integrands is not possible:** If f is not qslb at some $x \in \overline{\Omega}$, then the w*lsc envelope of F is $-\infty$.

Remarks on qslb

- **Example:** $f(x,\xi) := a(x)^T \xi t(x)$ is qslb, for continuous functions

$$a : \bar{\Omega} \to \mathbb{R}^M, \quad t : \bar{\Omega} \to \mathbb{R}^N \quad \text{such that } t(x) \cdot \nu_x = 0 \text{ on } \partial\Omega.$$

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- **qslb and quasiconvexity at the boundary:** If f^∞ is qslb at x_0 ($A = \xi = 0$ below), then $f^\infty(x_0, \cdot)$ is qcb at the zero matrix w.r.t. the normal ν_{x_0} (some A, $\xi = 0$), but not vice versa (linear f!).

Definition (qcb: Ball&Marsden 1984; Sprenger 1996)

$f^\infty(x_0, \cdot)$ is **quasiconvex at the boundary (qcb) at the matrix** $\xi \in \mathbb{R}^{M \times N}$ **w.r.t. the normal** ν_{x_0}, if

$$\int_{D(x_0)} f^\infty(x_0, \nabla\eta(y) + \xi)\, dy \geq \int_{D(x_0)} \left(f^\infty(x_0, \xi) + A : \nabla\eta(y)\right) dy$$

for a suitable $A \in \mathbb{R}^{M \times N}$ and every $\eta \in W_0^{1,\infty}(B_1(0); \mathbb{R}^M)$.

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Strategy of the proof

To show: qc and qslb $\implies$ w*lsc along some $u_n \rightharpoonup^* u$:

- Split $u_n - u = v_n + w_n$, with Dv_n "purely concentrating at the boundary" and $w_n = 0$ near $\partial\Omega$, such that Dv_n and Dw_n do not interact in F (**localizing decomposition** in BV).

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- Further split v_n into components living on ε-balls centered on $\partial\Omega$ (**localizing decomposition** again).
- Each piece p_n of v_n is an admissible test function for qslb, and qslb implies that

$$G_\varepsilon(p) := \int_\Omega df(x, Dp) + \varepsilon\,|Dp| - df(x, 0)\,dx$$

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- **Functionals bounded from below are w*lsc along pure concentrations at the boundary**.

Strategy of the proof

Note

The strategy is similar to the one used in $W^{1,p}$ (K. 2010), but we **cannot use the decomposition lemma** to separate concentrations from oscillations!

W*lsc along pure concentrations

Lemma

Suppose that F is bounded from below, $f(x,\cdot)$ is Lipschitz (qc suffices), $u_n \rightharpoonup^* u$ in BV. If

$$\operatorname{supp}|Du_n - Du| \subset (\partial\Omega)_{r_n} := \bigcup_{x\in\partial\Omega} B_{r_n}(x) \quad \text{for some } r_n \searrow 0$$

(purely concentrating at the boundary), then $\liminf F(u_n) \geq F(u)$.

Proof. Let $\varepsilon > 0$.

- Choose $u^* \in BV$ such that $F(u+u^*) \leq \varepsilon + \inf F$.
- Observe that

$$F(u_n) - F(u) - F(u_n + u^*) + F(u + u^*) \to 0,$$

essentially because $|Du^*|(\operatorname{supp}|Du_n - Du|) \leq |Du^*|((\partial\Omega)_{r_n}) \to 0$.

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$$\operatorname{supp}|Du_n - Du| \subset (\partial\Omega)_{r_n} := \bigcup_{x\in\partial\Omega} B_{r_n}(x) \quad \text{for some } r_n \searrow 0$$

(purely concentrating at the boundary), then $\liminf F(u_n) \geq F(u)$.

Proof. Let $\varepsilon > 0$.

- Choose $u^* \in BV$ such that $F(u+u^*) \leq \varepsilon + \inf F$.
- Observe that

$$F(u_n) - F(u) - F(u_n + u^*) + F(u + u^*) \to 0,$$

essentially because $|Du^*|(\operatorname{supp}|Du_n - Du|) \leq |Du^*|((\partial\Omega)_{r_n}) \to 0$.

- Since $F(u_n + u^*) - F(u + u^*) \geq (\inf F) - F(u+u^*) \geq -\varepsilon$, we get that

$$\liminf F(u_k + u) - F(u) \geq -\varepsilon.$$

□

Localizing decomposition

Definition (does not charge)

Let $(u_n) \subset BV$, $K \subset \mathbb{R}^N$ closed. $\mathbf{(Du_n)}$ **does not charge** $\mathbf{K}$, if

$$\sup_{n\in\mathbb{N}} |Du_n| \left((K)_\delta \cap \Omega\right) \underset{\delta\to 0^+}{\longrightarrow} 0, \qquad \text{where } (K)_\delta := \bigcup_{x\in K} B_\delta(x).$$

Lemma (localizing decomposition in BV)

Let

- $K_1, K_2 \subset \bar{\Omega}$ compact with $\bar{\Omega} \subset K_1 \cup K_2$; **(e.g.: $\mathbf{K_1 = \partial\Omega,\ K_2 = \bar{\Omega}}$)**
- $(u_n) \subset BV$ bounded with $u_n \to 0$ in L^1

Then a subsequence decomposes as

$$u_{k(n)} = u_{1,n} + u_{2,n},$$

with $(u_{j,n})_n \subset BV$ bounded s.t. $u_{j,n} \to 0$ in L^1 $(j = 1, 2)$, and:

(i) $\overline{\{\mathbf{u_{1,n} \neq 0}\}} \subset \mathbf{(K_1)_{\frac{1}{n}}}$, $\overline{\{\mathbf{u_{2,n} \neq 0}\}} \subset \mathbf{(K_2)_{\frac{1}{n}} \setminus K_1}$, $\{u_{j,n} \neq 0\} \subset \{u_n \neq 0\}$, $|Du_{j,n}| \leq \left|Du_{k(n)}\right| + \frac{1}{n}\mathcal{L}^N$;

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Let $(u_n) \subset BV$, $K \subset \mathbb{R}^N$ closed. **$(\mathbf{Du_n})$ does not charge $\mathbf{K}$**, if

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The proof is analogous to the corresponding result in $W^{1,p}$ (K. 2010):

- Choose cutoff-functions:
  $\varphi_n \in C^\infty(\mathbb{R}^N; [0,1]), \quad \varphi_n = 1$ on $(K_1)_{\frac{1}{2n}}$, $\varphi_n = 0$ outside $(K_1)_{\frac{1}{n}}$
- Observe: Since $u_k \to 0$ in L^1,

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  strongly as measures.
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Localizing decomposition: non-interaction

Proposition (asymptotical additivity)

Suppose that f is Lipschitz in the second variable, i.e.,

$$|f(x,\xi) - f(x,\eta)| \leq C\,|\xi - \eta|\,,$$

$v \in BV$, $(u_{k(n)}) \subset BV$ and

$$u_{k(n)} = u_{1,n} + u_{2,n}$$

is decomposed into component sequences with the properties listed in the localizing decomposition. Then

$$f(x, Du_{k(n)} + Dv) - f(x, Dv) - \sum_{j=1}^{2} \left[f(x, Du_{j,n} + Dv) - f(x, Dv) \right] \to 0$$

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The end

Thank you for your attention!