Semicontinuous energies defined in the space of multiple valued maps

Emanuele Spadaro

Max-Planck-Institut
für Mathematik in den Naturwissenschaften
Leipzig

Multiple valued functions

- SPACE OF UNORDERED Q POINTS.
  Let $M^k \subset \mathbb{R}^n$ be an embedded submanifold ($k \leq n$).

  $$\mathcal{A}_Q(M) := \left\{ \sum_{i=1}^{Q} [\![P_i]\!] : P_i \in M \right\}$$

  $[\![P]\!]$ is the Dirac delta at P.
- METRIC $\mathcal{G}$.
  Let $T_1 = \sum_i [\![P_i]\!]$ and $T_2 = \sum_i [\![S_i]\!]$.

  $$\mathcal{G}(T_1, T_2) := \min_{\sigma \in \mathscr{P}_Q} \sqrt{\sum_i d_M(P_i, S_{\sigma(i)})^2}$$

  where $\mathscr{P}_Q$ is the symmetric group with Q elements.
- MULTIPLE VALUED MAPS.
  Let $\Omega \subset \mathbb{R}^m$ be a regular open set. $u : \Omega \to \mathcal{A}_Q(M)$.

Examples: branched minimal surfaces

Almgren's big regularity paper

- $u : \mathbb{R}^2 \to \mathcal{A}_2(\mathbb{R}^2)$, $\mathbb{R}^2 \sim \mathbb{C}$

$$z \mapsto \{u_1(z), u_2(z)\} \; : \; u_i(z)^2 = z^3.$$

- $\mathrm{graph}(u) := \{(z, w) : w \in \mathrm{supp}\,(u(z))\} \subset \mathbb{R}^2 \times \mathbb{R}^2$

$$\mathrm{graph}(u) = \left\{(z, w) : z^3 = w^2\right\}$$

- $\mathrm{graph}(u)$ is homeomorphic to a disk $\Longrightarrow$ <u>unordered</u> Q points

Model for multi-value microstructures in complex systems

- A material point contains a family of microstructures composed by a finite number of *indistinguishable* individuals, e.g. a finite number of polymers.
- The microstructural descriptor $u(x)$ takes values in a manifold M, e.g. for liquid crystals in nematic order $M = \mathbb{P}^2$ is the projective plane.

The field $u(x)$ is a $Q-$valued map, the multiple values corresponding to a not labeled microstructural family.

Cp. M. Focardi, P.M. Mariano, E. Spadaro, *Multi-value microstructural descriptors for finer state material complexity: analysis of ground states*, preprint (2012).

Lower semicontinuous energies

$$\min \int_{\mathbf{\Omega}} \mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{Du}) \ : \ \mathbf{u} : \mathbf{\Omega} \to \mathcal{A}_{\mathbf{Q}}$$

- Example: ALMGREN‘S DIRICHLET MINIMIZERS.
  First order approximation of the area functional

$$\frac{1}{2}\int_{\Omega} |Du|^2 = \mathcal{H}^m(\mathrm{graph}(u)) - |\Omega| + O(\|Du\|_\infty)$$

- De Lellis & Spadaro (*Memoirs AMS* 2011)

$$|Du| := \sqrt{\sum_{j=1}^{m} \sup_{\varphi} \left| \frac{\partial(\varphi \circ u)}{\partial x_j} \right|^2}$$

  where the supremum is taken among all functions $\varphi : \mathcal{A}_Q \to \mathbb{R}$ with $\mathrm{Lip}(\varphi) \leq 1$.
- Other elliptic integrands – P. Mattila $m = 2$ (*Trans. AMS* 1983)

First order calculus: DIFFERENTIABILITY

Definition

A function $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ is differentiable at $x_0 \in \Omega$ if there exist matrices $L_i \in \mathbb{R}^{n\times m}$ such that

- $$\mathcal{G}\left(u(x), T_{x_0}u(x)\right) = o(|x-x_0|),$$

  with

  $$T_{x_0}u(x) := \sum_{i=1}^{Q} [\![u_i(x_0) + L_i(x-x_0)]\!] \quad \text{and} \quad u(x_0) = \sum_i [\![u_i(x_0)]\!]\,;$$

- $L_i = L_j$ if $u_i(x_0) = u_j(x_0)$.

This way the first-order approximation $T_{x_0}\nu$ is then unambiguously determined:

$$Du_i(x_0) := L_i.$$

Rademacher's theorem and Sobolev functions

Theorem

Let $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ *be a Lipschitz function. Then* u *is differentiable at* $\mathcal{H}^m$*-a.e.* $x \in \Omega$*.*

Definition

A function $u : \Omega \to \mathcal{A}_Q$ belongs to $W^{1,p}$ ($p \in [1, +\infty]$) if u is approximatively differentiable at almost every point of Ω and $|Du| \in L^p(\Omega)$.

Q-integrands

- A measurable map $f : \Omega \times (\mathbb{R}^n)^Q \times \left(\mathbb{R}^{m\times n}\right)^Q \to \mathbb{R}$ is called a *Q-integrand* if, for every permutation π of $\{1,\ldots,Q\}$,

$$f(x, a_1, \ldots, a_Q, A_1, \ldots, A_Q) = f(x, a_{\pi(1)}, \ldots, a_{\pi(Q)}, A_{\pi(1)}, \ldots, A_{\pi(Q)}).$$

- For every Sobolev Q-valued map u, the expression

$$F(u) := \int_\Omega f(x, u(x), Du(x))\, dx$$

where $f(x, u(x), Du(x)) = f(x, u_1(x), \ldots, u_Q(x), Du_1(x), \ldots, Du_Q(x))$ is well defined for almost every in $x \in \Omega$.

Q-quasiconvexity

Definition (Quasiconvexity)

Let $f : (\mathbb{R}^n)^Q \times \left(\mathbb{R}^{m\times n}\right)^Q \to \mathbb{R}$ be a locally bounded Q-integrand. We say that f is *Q-quasiconvex* if

- for every affine Q-valued function

$$u(x) = \sum_{j=1}^{J} q_j \, [\![a_j + L_j \cdot x]\!] \, ,$$

  with $a_i \neq a_j$ for $i \neq j$,
- for every collection of maps $w^j \in W^{1,\infty}(C_1, \mathcal{A}_{q_j})$ with $w^j|_{\partial C_1} = q_j \, [\![a_j + L_j|_{\partial C_1}]\!]$

$$f\big(u(0), Du(0)\big) \leq \int_{C_1} f\big(\underbrace{a_1, \ldots, a_1}_{q_1}, \ldots, \underbrace{a_J, \ldots, a_J}_{q_J}, Dw^1(x), \ldots, Dw^J(x)\big) dx.$$

Semicontinuity result

Theorem (De Lellis, Focardi, Spadaro (*Ann. Acad. Sc. Fenn. Math. 2011*))

Let $p \in [1, \infty[$ *and* $f : \Omega \times (\mathbb{R}^n)^Q \times \left(\mathbb{R}^{m\times n}\right)^Q \to \mathbb{R}$ *be a continuous* Q*-integrand. If* $f(x, \cdot, \cdot)$ *is* Q*-quasiconvex for every* $x \in \Omega$ *and*

$$0 \leq f(x, a, A) \leq C(1 + |a|^q + |A|^p),$$

where

- $q = 0$ *if* $p > m$,
- $q = p^*$ *if* $p < m$
- *and* $q \geq 1$ *is any exponent if* $p = m$,

then the functional F *is weakly lower semicontinuous in* $W^{1,p}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$. *Conversely, if* F *is weakly** *lower semicontinuous in* $W^{1,\infty}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$, *then* $f(x, \cdot, \cdot)$ *is* Q*-quasiconvex for every* $x \in \Omega$.

$$f_k \overset{*}{\rightharpoonup} f \text{ in } W^{1,p}$$

if

- $p \in (1, +\infty]$: $\mathcal{G}(f_k, f) \to 0$ in $L^p(\Omega)$ and $\| |Df_k| \|_{L^p(\Omega)} < +\infty$;
- $p = 1$: $\mathcal{G}(f_k, f) \to 0$ in $L^1(\Omega)$, $\| |Df_k| \|_{L^1(\Omega)} < +\infty$ and $|Df_k|$ is equi-integrable.

Mattila's quadratic elliptic integrands

$$E(u) := \int_\Omega \sum_i \langle A Du_i, Du_i\rangle,$$

where $\mathbb{R}^{n\times m} \ni M \mapsto A\,M \in \mathbb{R}^{n\times m}$ is a linear symmetric map

- *Q-semielliptic* if

$$\int_{\mathbb{R}^m} \sum_i \langle A D f_i, D f_i\rangle \geq 0 \quad \forall\, f \in \mathrm{Lip}(\mathbb{R}^m, \mathcal{A}_Q) \text{ with compact support.} \tag{1}$$

- Q-semiellipticity and Q-quasiconvexity coincide.
  $g(x) = \sum_{i=1}^k [\![f_i(x) + L\cdot x]\!]$ with f compactly supported

$$\begin{aligned} E(g) &= E(f) + k\,\langle A\,L, L\rangle + 2\int_{C_1} \sum_i \langle A\,L, D f_i\rangle \\ &= E(f) + k\,\langle A\,L, L\rangle, \end{aligned}$$

MATTILA'S CONJECTURE: 1-semielliptic implies Q-semielliptic.

Proof: Necessity

F weak*-$W^{1,\infty}$ lower semicontinuous implies $f(x_0,\cdot,\cdot)$ Q-quasiconvex for every $x_0 \in \Omega$.
Set $u(x) = \sum_{j=1}^{J} q_j \, [\![a_j + L_j \cdot x]\!]$

$$z^j(y) := \sum_{i=1}^{q_j} \left[\!\!\left[(w^j(y))_i - a_j - L_j \cdot y\right]\!\!\right]$$

$$v_k^j(y) = \sum_{i=1}^{q_j} \left[\!\!\left[k^{-1}(z^j(ky))_i + a_j + L_j \cdot y\right]\!\!\right]$$

$$u_{k,r}(x) = \sum_{j=1}^{J} \tau_{(1-r)a_j} \left(r \, v_k^j \left(r^{-1} x\right)\right).$$

Note that

- for every r, $u_{k,r} \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(C_r, \mathcal{A}_Q)$ as $k \to +\infty$;
- by the lower semicontinuity of F

$$F\big(u, C_r\big) := \int_{C_r} f\big(x, u, Du\big) \leq \liminf_{k \to +\infty} F\big(u_{k,r}, C_r\big).$$

Sufficienty

$(v_k) \rightharpoonup u \in W^{1,p}(\Omega, \mathcal{A}_Q)$ implies

$$F(u) \leq \liminf_{k\to\infty} F(v_k).$$

Idea: prove that for almost all $x_0 \in \Omega$

$$f(x_0, u(x_0), Du(x_0)) \leq \frac{d\mu}{d\mathcal{L}^m}(x_0),$$

where μ is the weak* limit in the sense of measure of any converging subsequence of $\big(f(x, u_k, Du_k)\mathcal{L}^m \llcorner \Omega\big)$.

Blowup analysis a la Fonseca-Müller (1992).

Step I

Let $u(x_0) = \sum_{j=1}^{J} q_j \llbracket a_j \rrbracket$, with $a_i \neq a_j$ for $i \neq j$. Then, there exist $\rho_k \downarrow 0$ and $(w_k) \subseteq W^{1,\infty}(C_{\rho_k}(x_0), \mathcal{A}_Q)$ such that:

(a) $w_k = \sum_{j=1}^{J} \llbracket w_k^j \rrbracket$ with $w_k^j \in W^{1,\infty}(C_{\rho_k}(x_0), \mathcal{A}_{q_j})$, $\|\mathcal{G}(w_k, u(x_0))\|_{L^\infty(C_{\rho_k}(x_0))} = o(1)$ and $\mathcal{G}(w_k(x), u(x_0))^2 = \sum_{j=1}^{J} \mathcal{G}(w_k^j(x), q_j \llbracket a_j \rrbracket)^2$ for every $x \in C_{\rho_k}(x_0)$;

(b) $⨍_{C_{\rho_k}(x_0)} \mathcal{G}^p(w_k, T_{x_0} u) = o(\rho_k^p)$;

(c) $\lim_{k\uparrow+\infty} ⨍_{C_{\rho_k}(x_0)} f\big(x_0, u(x_0), Dw_k\big) = \frac{d\mu}{d\mathcal{L}^m}(x_0)$.

Step II

For every $\gamma > 0$, there exist $(z_k) \subset W^{1,\infty}(C_1, \mathcal{A}_Q)$ such that $z_k|_{\partial C_1} = T_{x_0} u|_{\partial C_1}$ for every k and

$$\limsup_{k \to +\infty} \int_{C_1} f\big(x_0, u(x_0), Dz_k\big) \leq \frac{d\mu}{d\mathcal{L}^m}(x_0) + \gamma.$$

Q-polyconvexity

Let $N := \min\{m, n\}$, $\tau(n,m) := \sum_{k=1}^{N} \binom{m}{k}\binom{n}{k}$ and define $M : \mathbb{R}^{n\times m} \to \mathbb{R}^{\tau(m,n)}$ as $M(A) := \big(A, \mathrm{adj}_2 A, \ldots, \mathrm{adj}_N A\big)$, where $\mathrm{adj}_k A$ stands for the matrix of all $k \times k$ minors of A.

Definition

A Q-integrand $f : (\mathbb{R}^n)^Q \times \big(\mathbb{R}^{n\times m}\big)^Q \to \mathbb{R}$ is *polyconvex* if there exists a map $g : (\mathbb{R}^n)^Q \times \left(\mathbb{R}^{\tau(m,n)}\right)^Q \to \mathbb{R}$ such that:

(i) the function $g(a_1, \ldots, a_Q, \cdot) : \left(\mathbb{R}^{\tau(m,n)}\right)^Q \to \mathbb{R}$ is convex for every $a_1, \ldots, a_Q \in \mathbb{R}^n$,

(ii) for every $a_1, \ldots, a_Q \in \mathbb{R}n$ and $(L_1, \ldots, L_Q) \in (\mathbb{R}^{n\times m})^Q$ it holds

$$f\big(a_1, \ldots, a_Q, L_1, \ldots, L_Q\big) = g\big(a_1, \ldots, a_Q, M(L_1), \ldots, M(L_Q)\big). \tag{2}$$

Theorem (De Lellis, Focardi, Spadaro (*Ann. Acad. Sc. Fenn. Math. 2011*))

Every locally bounded polyconvex Q-integrand f is Q-quasiconvex.

Explicit examples of polyconvex functions

(a) $f(a_1, \dots, a_Q, L_1, \dots, L_Q) := g\big(\mathcal{G}(L, Q\,[\![0]\!])\big)$ with $g : \mathbb{R} \to \mathbb{R}$ convex and increasing;

(b) $f(a_1, \dots, a_Q, L_1, \dots, L_Q) := \sum_{i,j=1}^{Q} g(L_i - L_j)$ with $g : \mathbb{R}^{n\times m} \to \mathbb{R}$ convex;

(c) $f(a_1, \dots, a_Q, L_1, \dots, L_Q) := \sum_{i=1}^{Q} g(a_i, L_i)$ with $g : \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ measurable and polyconvex.

Remark

*A quadratic integrand is quasiconvex if and only if it is rank-*1 *convex. If* $\min\{m, n\} \leq 2$ *quasiconvexity is equivalent to polyconvexity (F. Terpstra (Math. Ann. 1939).*
This answers to the conjecture by Mattila in this case.

Proof of PC $\Longrightarrow$ QC: Step I

Let f be a Q-integrand, then the following are equivalent:

(i) f is a polyconvex Q-integrand,

(ii) for every choice of vectors $a_1, \ldots, a_Q \in \mathbb{R}^n$ and matrices $A_1, \ldots A_Q \in \mathbb{R}^{n \times m}$, with $A_i = A_j$ if $a_i = a_j$, there exist polyaffine functions $P_j : \mathbb{R}^{n \times m} \to \mathbb{R}$, with $P_i = P_j$ if $a_i = a_j$, such that

$$f\big(a_1, \ldots, a_Q, A_1, \ldots, A_Q\big) = \sum_{j=1}^{Q} P_j(A_j),$$

and

$$f\big(a_1, \ldots, a_Q, L_1, \ldots, L_Q\big) \geq \sum_{j=1}^{Q} P_j(L_j) \quad \text{for every } L_1, \ldots, L_Q \in \mathbb{R}^{n \times m}.$$

Proof of PC $\Longrightarrow$ QC: Step II

$$f\big(\underbrace{a_1,\ldots,a_1}_{q_1},\ldots,\underbrace{a_J,\ldots,a_J}_{q_J},\underbrace{L_1,\ldots,L_1}_{q_1},\ldots\underbrace{L_J,\ldots,L_J}_{q_J}\big)=\sum_{j=1}^{J} q_j\,P_j(L_j)$$

$$\overset{(A)}{=}\int_{C_1}\sum_{j=1}^{J}\sum_{i=1}^{q_j}P_j(Dw_i^j)\le\int_{C_1}f\big(\underbrace{a_1,\ldots,a_1}_{q_1},\ldots,\underbrace{a_J,\ldots,a_J}_{q_J},Dw^1,\ldots,Dw^J\big).$$

Proof of PC $\Longrightarrow$ QC: Step II

$d\omega^j = c_0^j\, dx + \sum_{l=1}^N \sum_{|\alpha|=|\beta|=l} \sigma_\alpha\, c_{\alpha\beta}^{j,l}\, dx_{\bar\alpha} \wedge dy_\beta$, it holds

$$\int_{C_1} \sum_{i=1}^{q_j} P_j(Dw_i^j) = \left\langle T_{w^j,C_1}, d\omega^j \right\rangle,$$

where $P_j(A) = c_0^j + \sum_{l=1}^N \sum_{|\alpha|=|\beta|=l} c_{\alpha\beta}^{j,l}\, M_{\alpha\beta}(A)$.
Since $u|_{\partial C_1} = w|_{\partial C_1}$, it follows that $\partial T_{w,C_1} = \partial T_{u,C_1}$ as rectifiable currents.
Then, setting $u^j(x) = q_j\, [\![a_j + L_j \cdot x]\!]$, by Stokes

$$\sum_{j=1}^J q_j\, P_j(L_j) = \int_{C_1} \sum_{j=1}^J \sum_{i=1}^{q_j} P_j(Du_i^j) = \sum_{j=1}^J \left\langle T_{u^j,C_1}, d\omega^j \right\rangle = \sum_{j=1}^J \left\langle \partial T_{u^j,C_1}, \omega^j \right\rangle$$

$$= \sum_{j=1}^J \left\langle \partial T_{w^j,C_1}, \omega^j \right\rangle = \sum_{j=1}^J \left\langle T_{w^j,C_1}, d\omega^j \right\rangle = \int_{C_1} \sum_{j=1}^J \sum_{i=1}^{q_j} P_j(Dw_i^j).$$

Open questions

- REGULARITY?
  Dirichlet minimizers
  - Hölder continuous (optimal regularity),
  - Analytic outside a set of codimension at least 2,
  - Higher integrability for $|Du|$.
- BRANCHING EFFECTS?

$$u|_{\partial\Omega} = Q\,[\![\varphi]\!] \quad \Longrightarrow \quad u = Q\,[\![\Phi]\!]\ ?$$