Quasiconvexity conditions when minimizing over homeomorphisms in the plane

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Relaxation, homogenization and dimensional reduction in hyperelasticity
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Balance of momentum

- Take a deformation $y : \Omega \to \mathbb{R}^d$, that it has to satisfy the balance of momentum

$$\varrho \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \mathrm{div}\mathcal{P} = f$$

  with f the volume force and $\mathcal{P}$ the Piola-Kirchhoff stress tensor.
- In the static case this reduces to

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Hyperelasticity

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- For a wide-class of materials – hyperelastic materials – we may instead prescribe just one *scalar quantity*, the stored energy W and set

$$\mathcal{P} = \frac{\mathrm{d}W}{dA}\Big|_{A=\nabla y}$$

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Minimization of the energy

- Stable states in elasticity are found through (the Euler-Lagrange equation of the momentum balance)

$$\left.\begin{array}{ll} \text{Minimize} & \int_\Omega W(\nabla y)\mathrm{d}x \\ \text{subject to} & y \in \mathcal{A} = \text{set of deformations.} \end{array}\right\} \qquad (1)$$

- Ω a regular domain.

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So, the energy W and the set of deformation $\mathcal{A}$ form together the model of the elastic behaviour.

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What we want to do is the following:

- Characterize precisely the set of energies for which stable states exist
- This is motivated by providing a "safe set" in constitutive modelling

The set of deformations

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A deformation is a vectorial mapping $\bar{\Omega} \mapsto \mathbb{R}^3$ that is smooth enough, injective except perhaps on the boundary, and orientation preserving.

[Ciarlet; 1988]

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$\rightsquigarrow$ these conditions are sometimes written as "$\det(\nabla y) > 0$"

Elastic deformations

Elasticity means that the specimen returns to its original state when releasing all loads

No defects in the specimen

- How smooth is smooth enough?
  1. Since the *deformation gradient* is the crucial quantity it is natural to work with Sobolev spaces $W^{1,p}(\Omega;\mathbb{R}^d)$
  2. y is continuous
  3. In our modelling, we shall require that *the inverse of the deformation* y^{-1} *is in the same class as the deformation itself*

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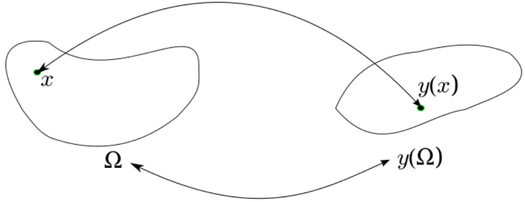
Motivation: The body is deformed from Ω to $y(\Omega)$. Then we change the reference configuration to $y(\Omega)$ and want that the mapping that moves each material point to its original position is an admissible deformation, too.



$\rightsquigarrow$ In a way, we may relate it to *reversibility* of elastic processes (by the same path).

[Giaquinta, Modica, Souček; 1998], [Fonseca, Gangbo; 1995], [Ball; 1981], [Ciarlet, Nečas; 1985], [Šverák; 1988], [Ball, 2002], [Iwaniec, Kovalev, Onninen; 2011]

Deformations in elasticity-Summary

Taking also the preserving of the orientation into account, we take the set of the deformations as the bi-Sobolev maps

$$W^{1,p,-p}(\Omega;\mathbb{R}^3) = \{y; y \text{ homeomorphism, } y \in W^{1,p}(\Omega;\mathbb{R}^3), \\ y^{-1} \in W^{1,p}(y(\Omega);\Omega) \text{ and } \det(\nabla y) > 0 \text{ a.e. on } \Omega\}$$

- Let us note that the constrain $\det(\nabla y) \geq 0$ is "included" when demanding a deformation to be a Sobolev homeomorphism since such cannot change the sign of the determinant in dimension 2,3

[Hencl, Malý; 2010]

Group structure of the set of deformations

The set of the deformations may have a group structure. This models that

- A composition of two deformations is again a deformation.
- Take two deformations $y : \Omega \mapsto \mathbb{R}^3$ and $z : y(\Omega) \mapsto \mathbb{R}^3$. Then the composition of these two deformations is

$$z(y(x)), \qquad x \in \Omega$$

- But the relevant variable is actually the *deformation gradient*

$$(\nabla z)(y(x))(\nabla y)(x)$$

$\rightsquigarrow$ "multiplication should is allowed"

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$\rightsquigarrow$ this is of course possible only on particular bi-Sobolev classes as *bi-Lipschitz maps, quasiconformal maps*

The stored energy

On the stored energy we have only two key requirements:

1. W is continuous on its domain
2. W has a growth that prevents shrinking of volume of positive measure to zero

$$W(A) \to +\infty \qquad \text{whenever } \det(A) \to 0_+ \tag{2}$$

$\rightsquigarrow$ in some situations, we may prescribe some growth etc.

Posing the problem

$$\left.\begin{array}{ll}\text{Minimize} & \int_\Omega W(\nabla y)\mathrm{d}x \\ \text{subject to} & y \in \mathcal{A}.\end{array}\right\} \qquad (1)$$

with

$$\mathcal{A} = W^{1,p,-p}(\Omega;\mathbb{R}^3)$$

an W satisfying

$$W(A) \to +\infty \qquad \text{whenever } \det(A) \to 0_+$$

Under which (minimal) additional conditions on the stored energy W does (1) admit a solution?

Posing the problem

we shall concentrate only on the case when $p = \infty$, i.e.

$$\mathcal{A} = W^{1,\infty,-\infty}(\Omega;\mathbb{R}^3) = \{y; y \text{ homeomorphism, } y \in W^{1,\infty}(\Omega;\mathbb{R}^3),$$
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Notice that

- In this case the set of deformations *has* a group structure
- There actually ex. $\gamma > 0$ s.t. $\det(\nabla y) \geq \gamma$ a.e. on Ω $\rightsquigarrow$ this implies that condition (2) *does not pose any restriction*.

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with

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Under which (minimal) conditions on the stored energy W does (1) admit a solution?

Refining the problem

- A usual approach is to employ the *direct method*
- Crucial ingredients:
  1. closedness of $\mathcal{A}$ under appropriate weak convergence ($\rightsquigarrow$ this is OK in our case under the convergence below)
  2. coercivity of W that enforces this weak convergence of the minimization sequence
  3. A corresponding lower semicontinuity of W

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If $\{y_k\}_{k>0}$ is a sequence of bi-Lipschitz, orientation preserving homeomorphisms such that

$$y_k \overset{*}{\rightharpoonup} y \quad \text{and} \quad y_k^{-1} \overset{*}{\rightharpoonup} y^{-1}$$

under which minimal conditions on W does it hold that

$$\int_\Omega W(\nabla y)\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega W(\nabla y_k)\mathrm{d}x?$$

"Little" Cheating

When studying existence of minimizers, the Lipschitz case is usually the easiest. But it has an intrinsic difficulty in combining the coercivity and the wlsc conditions in the direct method:

- To get the coercivity, we have to set $W(A) = +\infty$ whenever $|A| > K$, $\det(A) < \varepsilon$, for some $K, \varepsilon > 0$.
- But to prove the wlsc, we will construct an aid sequence which might have a norm outside these bounds.

Intermezzo: Quasiconvexity

- Let us "ignore" for a moment all the information about inverses, determinant and just take

$$\{y_k\}_{k>0} \text{ Lipschitz mappings such that } y_k \overset{*}{\rightharpoonup} y$$

[Morrey; 1952], [Ball, Murat; 1984], [Dacorgona; 1989]

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Quasiconvexity

Suppose W is continuous on $\mathbb{R}^3$. W is weakly* lower semicontinuous on $W^{1,\infty}(\Omega;\mathbb{R}^3)$ if and only if it is quasiconvex, i.e.

$$W(Y) \leq \frac{1}{|\Omega|}\int_\Omega W(\nabla\varphi)\,\mathrm{d}x.$$

for all $Y \in \mathbb{R}^{3\times 3}$ and all $\varphi \in W^{1,\infty}(\Omega,\mathbb{R}^3); \varphi = Yx$ on $\partial\Omega$.

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- In mechanics, one often interprets quasiconvexity as the *principle of virtual displacements*

[Morrey; 1952], [Ball, Murat; 1984], [Dacorgona; 1989]

Mechanical Intuition

In words the quasiconvexity condition says

Among all deformations with affine boundary data, the affine one has the lowest energy.

$\rightsquigarrow$ This tells us that we can simplify the situation: If we want to know about existence of minimizers, we just have to study affine boundary data.

Sufficient conditions

- Since $\mathcal{A} \subset W^{1,\infty}(\Omega;\mathbb{R}^3)$, W is sure weakly* lower semicontinuous if it is quasiconvex
- Yet, the condition

$$W(Y) \leq \frac{1}{|\Omega|}\int_\Omega W(\nabla\varphi)\,\mathrm{d}x.$$

for all $Y \in \mathbb{R}^{3\times 3}$ and all $\varphi \in W^{1,\infty}(\Omega,\mathbb{R}^3); \varphi = Yx$ on $\partial\Omega$. is not natural.

- *Why should we test also with non-deformations* $\rightsquigarrow$ particularly when looking at the mechanical intuition?

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Ball's open problems

- The problem posed here also touches the following fundamental problem of mathematical elasticity

  Prove the existence of energy minimizers for elastostatics for quasiconvex stored-energy functions satisfying

  $$W(A) \to +\infty \qquad \text{whenever } \det(A) \to 0_+$$

- $\rightsquigarrow$ this formulation as well as the approach here are motivated by including injectivity and non-penetration as requirements for deformations

[Ball; 2002]

Necessary and sufficient conditions

- Remeber, the condition

$$W(Y) \leq \inf_{\varphi \in W^{1,\infty}(\Omega, \mathbb{R}^d); \varphi = Yx \text{ on } \partial\Omega} \frac{1}{|\Omega|} \int_\Omega W(\nabla \varphi)\, \mathrm{d}x.$$

  is not natural.
- *Why should we test also with non-deformations* $\rightsquigarrow$ particularly when looking at the principle of virtual displacements?

Conjecture

If $\{y_k\}_{k>0}$ is a sequence of bi-Lipschitz, orientation preserving homeomorphisms such that $y_k \overset{*}{\rightharpoonup} y$ and $y_k^{-1} \overset{*}{\rightharpoonup} y^{-1}$ then

$$\int_\Omega W(\nabla y)\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega W(\nabla y_k)\mathrm{d}x$$

if and only if *it is bi-quasiconvex*, i.e.

$$W(Y) \leq \frac{1}{|\Omega|} \int_\Omega W(\nabla \varphi)\, \mathrm{d}x.$$

for all $Y \in \mathbb{R}^{3\times 3}$ with $\det(Y) > 0$ and all $\varphi \in \mathcal{A}; \varphi = Yx$ on $\partial\Omega$.

Necessary and sufficient conditions

- this is still an open problem . . . but . . .

Necessary and sufficient conditions

Proposition [B.B& M.Kr., 2013]

If $\{y_k\}_{k>0}$ is a sequence of bi-Lipschitz, orientation preserving homeomorphisms in the plane (i.e. $\Omega \subset \mathbb{R}^2, y : \Omega \mapsto \mathbb{R}^2$) such that $y_k \overset{*}{\rightharpoonup} y$ and $y_k^{-1} \overset{*}{\rightharpoonup} y^{-1}$ then

$$\int_\Omega W(\nabla y)\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega W(\nabla y_k)\mathrm{d}x$$

if and only if

$$W(Y) \leq \frac{1}{|\Omega|}\int_\Omega W(\nabla\varphi)\,\mathrm{d}x.$$

for all $Y \in \mathbb{R}^{3\times 3}$ with $\det(Y) > 0$ and all $\varphi \in \mathcal{A}; \varphi = Yx$

Constructing a cut-off as key ingredient

- The key ingredient in the proof of this proposition is the construction of some kind of cut-off
- Indeed, take $\{y_k\}_{k>0}$ is a sequence of bi-Lipschitz, orientation preserving homeomorphisms s.t. $y_k \overset{*}{\rightharpoonup} Yx$ (for simplicity)
- If $\forall k$ we had $y_k = Yx$ on $\partial\Omega$, then (from def.)

$$W(Y) \leq \liminf_{k\to\infty} \frac{1}{|\Omega|} \int_\Omega W(\nabla y_k)\,\mathrm{d}x.$$

$\Rightarrow$ weak* lower semicontinuity

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Constructing a cut-off as key ingredient

Reformulating once again

Suppose $\{y_k\}_{k>0}$ is a sequence of bi-Lipschitz, orientation preserving homeomorphisms s.t. $y_k \overset{*}{\rightharpoonup} Yx$, $y_k^{-1} \overset{*}{\rightharpoonup} Y^{-1}x$. Then find another sequence $\{w_k\}_{k>0}$ of bi-Lipschitz, orientation preserving homeomorphisms such that

- $w_k = Yx$ on $\partial\Omega$,
- $\left|\{w_k \neq y_k\}\right| \to 0$.

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- this is a consequence of using *Young measures* $\rightsquigarrow$ a useful tool in such situations
- Notice: $det(Y) > 0$

[Kinderlehrer, Pedregal; 1991, 1992, 1994]

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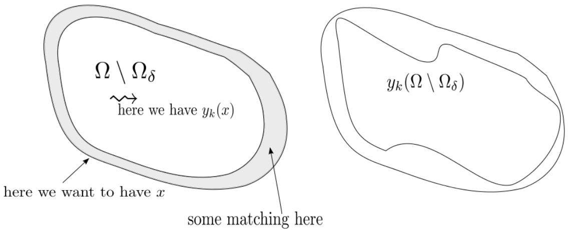
We can imagine a cut-off by the following picture ($Y = \mathrm{Id}$ here):



Cut-off and surjectivity of the trace operator

Notice: the cut-off technique is very much related to characterizing the trace operator.

- What we need to do is to find some $w_k \in W^{1,p,-p}(\Omega)$ on Ω_δ with *prescribed* boundary data, such that the norm of w_k is controlled by a "suitable" norm at the boundary
- This is is equivalent to constructing an extension operator; in other words

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We seek a characterization of the set $\mathcal{X}^p$ such that

$$\mathrm{Tr} : W^{1,p,-p}(\Omega;\mathbb{R}^2) \xrightarrow{onto} \mathcal{X}^p$$

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- What we need to do is to find some $w_k \in W^{1,p,-p}(\Omega)$ on Ω_δ with *prescribed* boundary data, such that the norm of w_k is controlled by a "suitable" norm at the boundary
- This is is equivalent to constructing an extension operator; in other words

We seek a characterization of the set $\mathcal{X}^p$ such that

$$\mathrm{Tr} : W^{1,p,-p}(\Omega;\mathbb{R}^2) \xrightarrow{onto} \mathcal{X}^p$$

$\rightsquigarrow$ this is completely open unless $p = \infty$.

Sidenote: Characterizing the trace operator is of independent interest

Suppose we want to solve

$$\left.\begin{array}{ll} \text{Minimize} & \int_\Omega \tilde{W}(\nabla y)\mathrm{d}x \\ \text{subject to} & y \in W^{1,2}(\Omega);\ y = g \text{ on } \partial\Omega. \end{array}\right\}$$

with

$$W(A) = \begin{cases} |A|^2 + \frac{|\mathrm{cof}(A)|^2}{\det(A)} & \det(A) > 0 \\ +\infty & \text{else} \end{cases}$$

- The energy is polyconvex
- But existence of minimizers depends also on the boundary data $g \rightsquigarrow$ In which class should they be?

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Constructing a cut-off - difficulties

- How is the cut-off constructed usually?
- Take η_ℓ smooth cut-off function and take the convex combination

$$\eta_\ell y_k + (1 - \eta_\ell) Yx$$

- But our constrains $\det(\cdot) > 0$ as well as the invertibility *are not convex*

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Sidenote: Constructing a cut-off under the $\det \neq 0$ constraint

- ... in some situations we can find remedy in *convex integration*
- Take, for examle, the case when $\{y_k\}_{k>0}$ is a sequence of Lipschitz mappings such that ∇y_k is invertible a.e. (*as a matrix*!)
- This is again a *non-convex* constraint but convex integration results help here

[Dacorogna, Marcellini; 1999], [B.B., M.Kr., G. Path; 2012], [Rindler, Koumatos, Wiedemann; 2013]

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Cut-off in the plane

[B.B. & M.Kr; 2013]

Suppose $\{y_k\}_{k>0}$ is a sequence of bi-Lipschitz, orientation preserving homeomorphisms s.t. $y_k \overset{*}{\rightharpoonup} Yx$, $y_k^{-1} \overset{*}{\rightharpoonup} Y^{-1}x$. Then there exists another sequence $\{w_k\}_{k>0}$ of bi-Lipschitz, orientation preserving homeomorphisms such that

- $w_k = Yx$ on $\partial\Omega$,
- $\big|\{w_k \neq y_k\}\big| \to 0$.

Working in the plane

- Also working in the plane makes the situation simpler
- Here, we rely on two crucial things:
  1. The boundary of domains in the plane is *one-dimensional* (e.g. the boundary of the some square)
  2. In the plane, we have bi-Lipschitz *extension* theorems at our disposal

Bi-Lipschitz extension in the plane

[Daneri, Pratelli; 2011]

There exists a geometric constant C such that every L bi-Lipschitz map u defined on the boundary of the unit square admits a CL^4 bi-Lipschitz extension into the square that coincides with u on the boundary.

- What does this mean?

[Tukia; 1980], [Huuskonen,Partanen,Väisälä; 1995], [Tukia,Väisälä;1981,1984]

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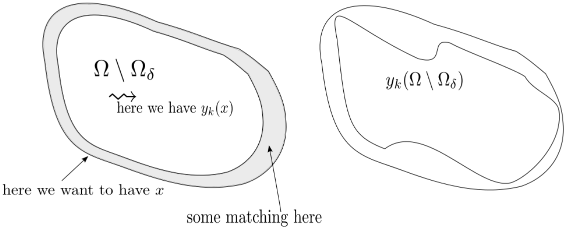
Why is this useful in our case?

- Remember, we wanted to solve a boundary value problem
  $\rightsquigarrow$ *now we can, but on the boundary of the square....*
- So we introduce squares in the grey area



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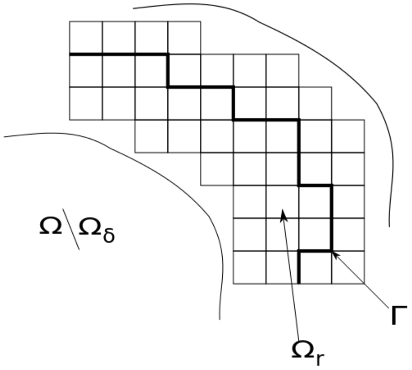
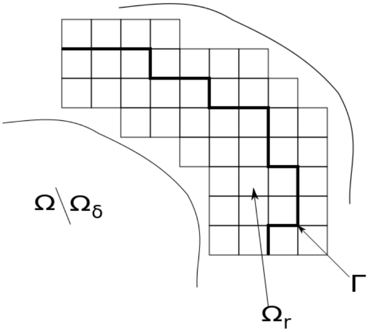


Defininge the cut-off function on the grid



- On "inner" edges we take $y_k(x)$
- On "outer" edges we take Yx
- On "crosses" we to the matching

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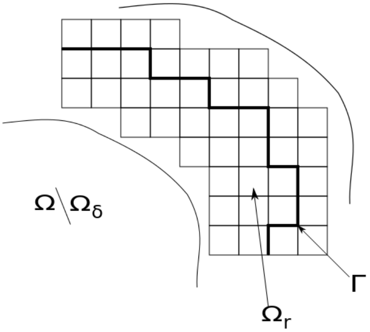

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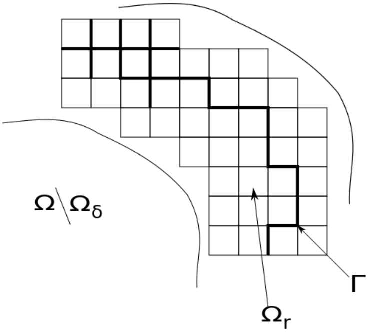

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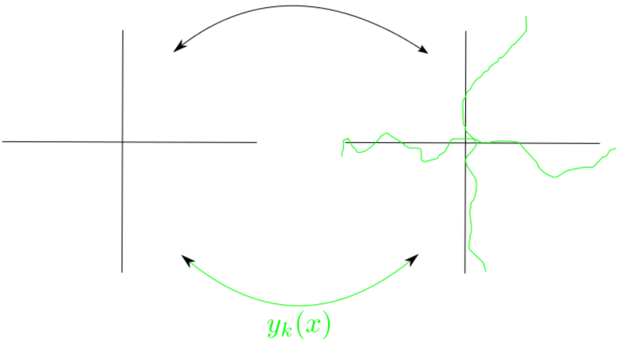

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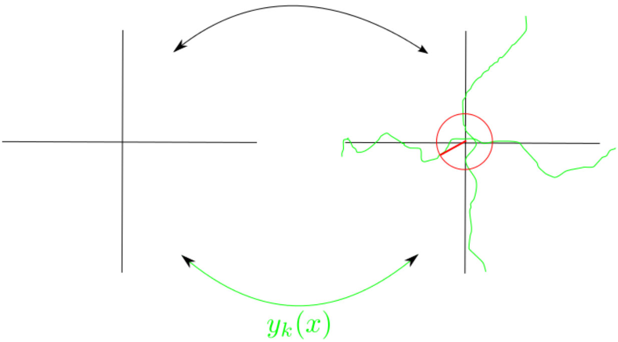

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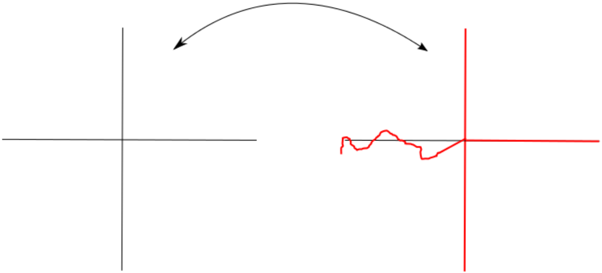
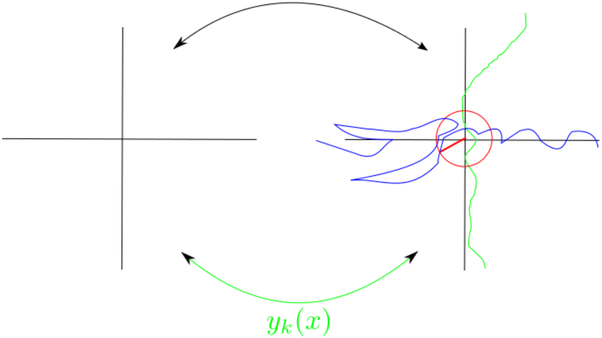


Proof-Summary

- We reduced the problem of constructing a cut-off to constructing it just on the crosses
- $\rightsquigarrow$ thus, we can define a "matching function" that is still bi-Lipschitz on the grid of the squares
- The bi-Lipschitz extension theorem then allows to
- Recall: this allows us to have the quasiconvexity condition (*principle of virtual displacements*) tested just by deformations (in order to anyway obtain weak* lower semicontinuity)

Affine matching - best we can do? (B.B. & M. Ka.-*work in progress*

The affine matching is very much dependent on having Lipschitz, i.e. "affine like" functions $\rightsquigarrow$ we want to go beyond that to bi-Sobolev



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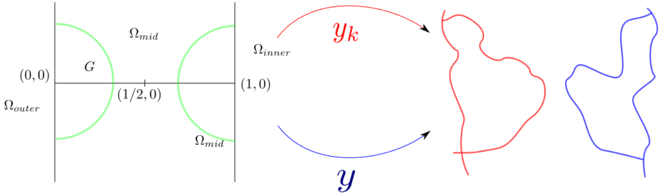

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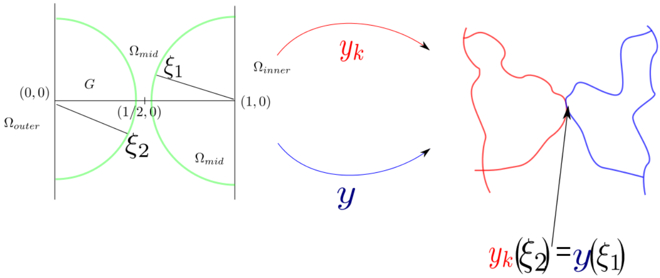

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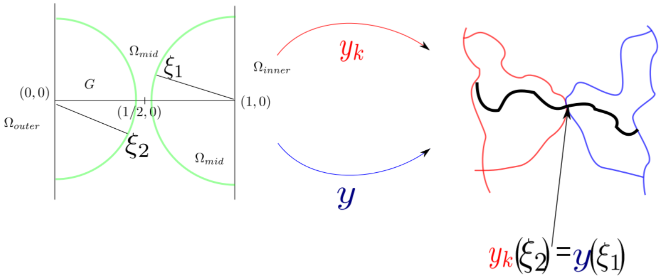

Beyond affine matching - quasiconformal case (B.B. & M. Ka.).-*work in progress*

- We seek to take quasiconformal maps i.e. homeomorphisms in $W^{1,2}(\Omega;\mathbb{R}^3)$ such that

$$|\nabla y(x)| \leq K \det(\nabla y(x)) \qquad \text{a.e. in } \Omega$$

- Such mappings take *infinitesimal circles to infinitesimal ellipses of a bounded eccentricity*
- Qualify as deformations: Inverse also quasiconformal, composition also quasiconformal
- A characterization of traces is available.

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Summary

- It is relevant in elasticity to solve minimization problems on subsets of Sobolev functions, where *non-linear, non-convex* restrictions are posed
- Although sufficiency conditions for existence of minima are generally known, if and only if conditions are still a challange
- Here we extended the quasiconvexity condition also to the case when minimizing over bi-Lipschitz, orientation preserving functions
- A larger class of cost functions can be admitted now

Thank you for your attention!