The constraint of positive Jacobian in Sobolev spaces with exponent below the space dimension

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(joint work with K. Koumatos, E. Wiedemann)

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Paris, 26 March 2014

Positive Jacobian constraints in Calculus of Variations

- Let $\Omega \subset \mathbb{R}^d$.
- *Given:* **Minimization problem** (e.g. in nonlinear elasticity):

$$\int_\Omega f(x, \nabla u(x))\, \mathrm{d}x \quad \to \quad \text{min} \qquad \text{over } u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^d).$$

with **orientation-preserving constraints**:

$$\det \nabla u > 0 \quad \text{a.e.} \qquad \text{or}$$
$$\det \nabla u \geq r > 0 \quad \text{a.e.} \qquad \text{or}$$
$$\det \nabla u = r > 0 \quad \text{a.e.}$$

- Motivation: Elasticity theory, fluid dynamics, . . . (but there one should look at the *distributional* determinant)

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- Motivation: Elasticity theory, fluid dynamics, ... (but there one should look at the *distributional* determinant)
- *Main difficulty:* Constraint is heavily nonlinear and non-convex!

Differential inclusions

- Consider the constraint

$$\det \nabla u > 0 \quad \text{a.e.}$$

- Assume we satisfy the constraint "approximately", i.e.

$$(u_j) \subset \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d) \quad \text{with} \quad u_j|_{\partial\Omega} = u|_{\partial\Omega}$$

and

$$\int_{\{\det \nabla u_j < 0\}} \underbrace{\left|\det \nabla u_j(x)\right|^{p/d}}_{\in \mathrm{L}^1} \,\mathrm{d}x \;\to\; 0.$$

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- **Question 1:** Can we find new sequence $(v_j) \subset \mathrm{W}^{1,p}_u(\Omega;\mathbb{R}^d)$ such that

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- **Question 2:** Given $g \in \mathrm{W}^{1-1/p,p}(\partial\Omega;\mathbb{R}^d)$ (trace space to $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$), does there exist

$$u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d) \quad \text{with} \quad u|_{\partial\Omega} = g \quad \text{and} \quad \det \nabla u > 0 \qquad ?$$

  (Note: No "compatibility" assumption on g!)

The case $p \geq d$

- Say $u_j = u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$ with

$$\det \nabla u = 0 \quad \text{a.e.},$$

  this satisfies the constraint $\int_{\{\det \nabla u < 0\}} \left|\det \nabla u(x)\right|^{p/d} \, \mathrm{d}x = 0$.

- Can find $v_j \to u$ in $\mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$ such that $v_j|_{\partial\Omega} = u|_{\partial\Omega}$ and

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- **Observation:** Question 1 is unsolvable if $p \geq d$:
In this case, $\det \nabla v_j \in \mathrm{L}^1(\Omega)$ and by Stokes' Theorem,

$$\int_\Omega \det \nabla v_j \,\mathrm{d}x = \int_\Omega dv_j^1 \wedge \cdots \wedge dv_j^2 = \int_{\partial\Omega} v_j^1 \wedge dv_j^2 \wedge \cdots \wedge dv_j^d$$
$$= \int_{\partial\Omega} u^1 \wedge du^2 \wedge \cdots \wedge du^d = \cdots = \int_\Omega \det \nabla u \,\mathrm{d}x = 0.$$

So $\det \nabla v_j > 0$ a.e. is impossible. □

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- Similar argument shows: Also *Question 2* unsolvable if $p \geq d$.

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So $\det \nabla v_j > 0$ a.e. is impossible. □
- Similar argument shows: Also *Question 2* unsolvable if $p \geq d$.
- We will henceforth consider the "soft" case $1 < p < d$.

Young measures

Let

- $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be continuous,
- $|f(x, A)| \leq C(1 + |A|^p)$ for a $p \in [1, \infty)$,
- $(v_j) \subset \mathrm{L}^p(\Omega; \mathbb{R}^N)$ norm-bounded.

For many applications it is important to answer:

$$\textbf{What is} \quad \operatorname*{w\text{-}lim}_{j\to\infty} f\big(x, v_j(x)\big) \quad \textbf{?}$$

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It is not $f(\bullet, (\text{w-lim } v_j))$**!** Because of:

- oscillation effects (e.g. $v_j(x) = \sin(jx)$)
- concentration effects (e.g. $v_j(x) = j^{1/p} \mathbb{1}_{(0,1/j)}$)

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If (v_j) equiintegrable (prevents concentrations): Young measures ('37, '42)

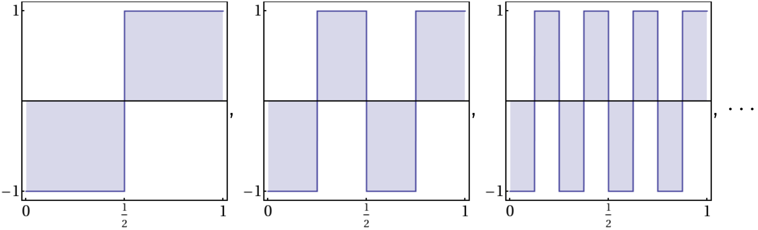
Fundamental Theorem of YM theory: Up to a subsequence, there exists a family of probability measures $(\nu_x)_{x\in\Omega}$ (with some measurability properties) on $\mathbb{R}^N$ and

$$\int_\Omega f(x, v_j(x))\,\mathrm{d}x \quad \to \quad \int_\Omega \int_{\mathbb{R}^N} f(x, \bullet)\,\mathrm{d}\nu_x\,\mathrm{d}x$$

for *all* continuous $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ with p-growth.

Example: Oscillation 1

$v_j =$

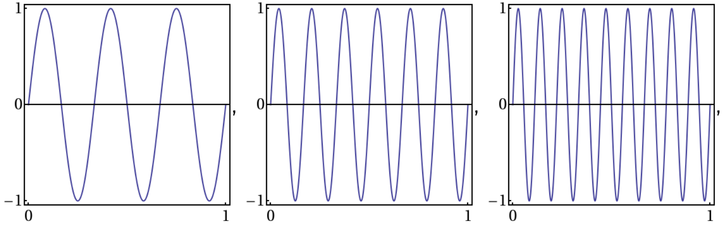


Then:

$$\nu_x = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1} \text{ a.e.}$$

Interpretation: v_j asymptotically is in $+1$ and -1 each with volume fraction $\frac{1}{2}$.

$v_j =$



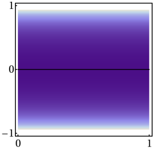
$, \; \dots$

$$v_j := \sin(2j\pi x) \xrightarrow{Y} (\nu_x)_x$$

Then:

$$\nu_x = \frac{1}{\pi\sqrt{1-y^2}} \mathcal{L}^1_y \llcorner (-1,1) \text{ a.e.}$$

$(\nu_x)_x =$



Characterization of gradient Young measures

For several purposes (e.g. relaxation theorems), one is interested in the following:

> *Can one characterize the class of Young measures which can be generated by a sequence of* **gradients** *of* $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)$*-functions?*

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Theorem (Kinderlehrer–Pedregal '91/'94)

A Young measure $\nu = (\nu_x)$ *is a* **gradient** p**-Young measure**, *that is there exists a sequence* $(\nabla u_j) \subset \mathrm{L}^p(\Omega;\mathbb{R}^{d\times d})$ *generating* $\nu = (\nu_x)$*, if and only if the following conditions hold:*

(KP1) $\displaystyle\int |A|^p \,\mathrm{d}\nu_x(A) < \infty.$

(KP2) *The barycenter* $\int A \,\mathrm{d}\nu_x(A)$ *is a gradient, i.e. there exists* $\nabla u \in \mathrm{L}^p(\Omega;\mathbb{R}^{d\times d})$ *with* $\int A \,\mathrm{d}\nu_x(A) = \nabla u(x)$ *a.e.*

(KP3) *For every quasiconvex function* $h\colon \mathbb{R}^{d\times d} \to \mathbb{R}$ *with* $|h(A)| \le c(1+|A|^p)$*, the* **Jensen-type inequality**

$$h(\nabla u(x)) \le \int h(A) \,\mathrm{d}\nu_x(A) \qquad \text{holds for a.e. } x \in \Omega.$$

Characterization of gradient Young measures II

Here: A locally bounded mapping $h\colon \mathbb{R}^{d\times d} \to \mathbb{R}$ is called **quasiconvex** if

$$h(M) \le \fint_{B(0,1)} h(M + \nabla\psi(x))\,\mathrm{d}x$$

for all $M \in \mathbb{R}^{d\times d}$ and all $\psi \in \mathrm{C}_c^\infty(B(0,1);\mathbb{R}^d)$ (compactly supported).

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Applications of Kinderlehrer–Pedregal Theorem:

- Relaxation theorems (no minimizer for original problem $\rightsquigarrow$ extend to minimization problem on Young measures).
- Theoretical investigations.
- Properties of quasiconvexity.

YM characterization under side constraints

Question 3:

Can one characterize the class of Young measures which can be generated by a sequence (∇u_j) *of gradients of* $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)$*-functions* **satisfying the side constraint**

$$\det \nabla u_j > 0 \quad \text{a.e.} \qquad ? \tag{$\star$}$$

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Theorem (K-R-W#1 2013)

Let $\mathbf{p} < \mathbf{d}$ *(the "soft" case). A Young measure* $\nu = (\nu_x)$ *is an* **orientation-preserving gradient** p**-Young measure**, *that is there exists a sequence* $(\nabla u_j) \subset \mathrm{L}^p(\Omega;\mathbb{R}^{d\times d})$ *generating* $\nu = (\nu_x)$ **and satisfying** $(\star)$, *if and only if the following conditions hold:*

(KP1)–(KP3) *from the Kinderlehrer–Pedregal Theorem.*

(SUPP) $\operatorname{supp} \nu_x \in \{\, M \in \mathbb{R}^{d\times d} \;:\; \det M \geq 0 \,\}$

"Geometry" of the set $\{ M \in \mathbb{R}^{d\times d} \ : \ \det M = 0 \}$ in matrix space I

- **Observation:** Let $M_0 \in \mathbb{R}^{d\times d}$ with $\det M_0 < 0$. Then, M can be written as the barycenter of a probability measure $\mu \in \mathbf{M}^1(\mathbb{R}^{d\times d})$ with

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- Indeed, if (modified singular value decomposition)

$$M_0 = \begin{pmatrix} -\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix} \qquad \text{with} \qquad 0 \le \sigma_1 \le \sigma_2 \le \cdots \le \sigma_d,$$

then (trivially)

$$M_0 = \frac{1}{2}\begin{pmatrix} 0 & & & & \\ & 2\sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_d \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2\sigma_1 & & & & \\ & 0 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_d \end{pmatrix} =: \frac{1}{2}M_1 + \frac{1}{2}M_2.$$

- $\det M_1 = \det M_2 = 0$ and $\mu := \frac{1}{2}\delta_{M_1} + \frac{1}{2}\delta_{M_2}$.

"Geometry" of the set $\{ M \in \mathbb{R}^{d\times d} \;:\; \det M = 0 \}$ in matrix space II

- **More intricate question:** Can choose μ as a gradient Young measure or even a p-laminate (a special type of gradient p-Young measure)?
- A probablity measure ν is a **finite-order laminate** if either

$$\nu = \delta_A, \quad A \in \mathbb{R}^{d\times d}, \qquad \text{or} \qquad \nu = \theta\nu_1 + (1-\theta)\nu_2,$$

where ν_1, ν_2 are p-laminates (recursively), $\theta \in (0,1)$, and

$$\operatorname{rank}\left(\int A \,\mathrm{d}\nu_1(A) - \int A \,\mathrm{d}\nu_2(A)\right) = 1$$

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- A ***p*-laminate** is a probability measure ν with $\int |\bullet|^p \,\mathrm{d}\nu < \infty$ and such that there exist finite-order laminates ν_j such that

$$\nu_j \overset{*}{\rightharpoonup} \nu \qquad \text{as measures.}$$

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- By the Kinderlehrer–Pedregal Theorem: *p*-laminates ν can be generated by a sequence of gradients! That is, there exists $(\nabla w_j) \subset \mathrm{L}^p(\Omega; \mathbb{R}^d)$ with

$$\int_\Omega f(x, \nabla w_j(x)) \,\mathrm{d}x \quad \to \quad \int_\Omega \int_{\mathbb{R}^N} f(x, \bullet) \,\mathrm{d}\nu \,\mathrm{d}x$$

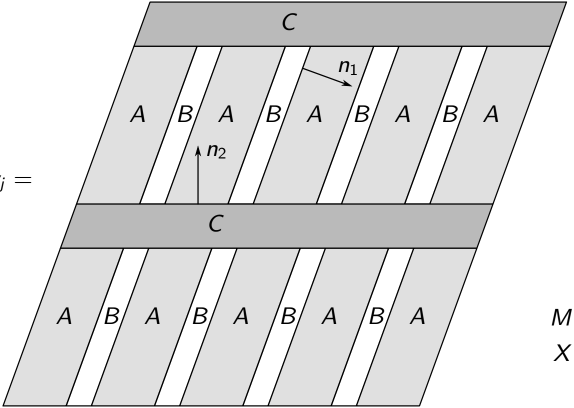
for *all* continuous $f \colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ with *p*-growth.

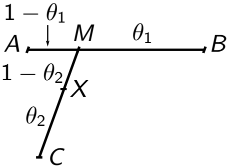
Lamination example

2^{nd} order laminate: $\nu = \theta_2\big(\theta_1\delta_A + (1-\theta_1)\delta_B\big) + (1-\theta_2)\delta_C$

where $\text{rank}(A - B) = 1$, $\text{rank}[(\theta_1 A + (1-\theta_1)B) - C] = 1$.

$w_j =$





$$M = \theta_1 A + (1-\theta_1)B$$
$$X = \theta_2 M + (1-\theta_2)C$$

"Geometry" of the set $\{ M \in \mathbb{R}^{d\times d} \ : \ \det M = 0 \}$ in matrix space III

Proposition

Let $M_0 \in \mathbb{R}^{d\times d}$ with $\det M_0 < 0$. *Then, there exists a homogeneous Young measure $\nu \in \mathbf{M}^1(\mathbb{R}^{d\times d})$ that is a p-laminate of infinite order for every $p \in [1, d)$ and such that the following assertions hold:*

(i) $\displaystyle\int A \,\mathrm{d}\nu(A) = M_0,$

(ii) $\operatorname{supp}\nu \subset \{ M \in \mathbb{R}^{d\times d} \ : \ \det M = 0 \}$,

(iii) $\displaystyle\int |\bullet|^p \,\mathrm{d}\nu \le C_p |M_0|^p,$

(iv) $\displaystyle\int |A - M_0|^p \,\mathrm{d}\nu(A) \le C_p |\det M_0|^{p/d},$

where $C_p = C(d, p)$.

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Note: This cannot hold for $p \geq d$: In this case we would have

$$0 > \det M_0 \overset{\det \text{ Null-Lagrangian}}{=} \int \det \,\mathrm{d}\nu = 0.$$

Proof of Proposition I

- Without loss of generality $M_0 = \begin{pmatrix} -\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix}$ with $\sigma_i > 0$ (modified singular value decomposition).

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- $\gamma := \sqrt{\sigma_1 \sigma_2}$.
- Decompose M_0 twice along rank-one lines as follows:

$$
\begin{aligned}
M_0 &= \frac{1}{2}\Big[M_0 + \gamma \underbrace{(\mathrm{e}_1 \otimes \mathrm{e}_2)}_{\text{rank}=1}\Big] + \frac{1}{2}\big[M_0 - \gamma(\mathrm{e}_1 \otimes \mathrm{e}_2)\big] \\
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Proof of Proposition I

- Without loss of generality $M_0 = \begin{pmatrix} -\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix}$ with $\sigma_i > 0$ (modified singular value decomposition).
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- Decompose M_0 twice along rank-one lines as follows:

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- Compute: $\det M_{1,G1} = \det M_{1,G2} = 0$ and $\det M_{1,B1} = \det M_{1,B2} < 0$.
- Set $r := 2^{\frac{p}{d}-1}$, hence $r < 1$. Then:

$$
|\det M_{1,B1}| = |\det M_{1,B2}| = 2|\det M_0| = (2r)^{d/p}\,|\det M_0|.
$$

Proof of Proposition II

- So far: $M_0 = \frac{1}{4}M_{1,B1} + \frac{1}{4}M_{1,G1} + \frac{1}{4}M_{1,G2} + \frac{1}{4}M_{1,B2}$.

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- Apply the same scheme recursively to the "bad" matrices $M_{1,B1}$, $M_{1,B2}$.
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- Exemplary estimate:

$$\begin{aligned}\int |A - M_0|^p \, \mathrm{d}\nu_j(A) &= \sum_{i=1}^{j} \sum_{k=1}^{2^i} \frac{1}{4^i} |M_{i,Gk} - M_0|^p + \sum_{k=1}^{2^j} \frac{1}{4^j} |M_{j,Bk} - M_0|^p \\ &\le \cdots \\ &\le \left[\frac{2^{1/2}}{(2r)^{1/p} - 1}\right]^p \cdot |\det M_0|^{p/d} \cdot \left[\sum_{i=1}^{j} \frac{2^i (2r)^i}{4^i} + \frac{2^j (2r)^j}{4^j}\right] \\ &\le \left[\frac{2^{1/2}}{(2r)^{1/p} - 1}\right]^p \cdot |\det M_0|^{p/d} \cdot \left[\frac{1}{1-r} + r^j\right] \\ &\le C_p |\det M_0|^{p/d} \qquad \text{since } r < 1!\end{aligned}$$

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\end{aligned}$$

- Sought p-laminate is $\nu := \text{w*-}\lim_{j\to\infty} \nu_j$.

A corollary on approximation

Corollary

Let $\Omega \subset \mathbb{R}^d$ *be open and bounded and* $1 < p < d$*. Let* $u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$ *be weakly orientation-preserving,*

$$\det \nabla u \geq 0 \quad a.e.$$

Then, there exists a sequence $(v_j) \subset \mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$ *that is strictly orientation-preserving,*

$$\det \nabla v_j > 0 \quad a.e. \qquad \text{for all } j \in \mathbb{N},$$

and such that $\|v_j - u\|_{1,p} \to 0$ *as* $j \to \infty$*.*

Extensions to *prescribed* Jacobians (K-R-W#2 2013)

Recently: We also considered the more general constraint

$$J_1(x) \leq \det \nabla u_j(x) \leq J_2(x) \quad \text{for all } j \in \mathbb{N} \text{ and a.e. } x \in \Omega.$$

where

1. $J_1 \colon \Omega \to [-\infty, +\infty)$, $J_2 \colon \Omega \to (-\infty, +\infty]$ measurable,
2. $J_1(x) \leq J_2(x)$ for a.e. $x \in \Omega$, and
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Examples: $\det \nabla u_j(x) = r > 0$, $\det \nabla u_j(x) = 1$, ...

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Question 4: Can we carry out our program also for this more general case?

Answer: Yes (theorem omitted).

Corollaries

Corollary

Let $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain,* $1 < p < d$, $J : \Omega \to \mathbb{R}$ *be measurable with*

$$\int_\Omega |J(x)|^{p/d} \,\mathrm{d}x < \infty,$$

and let $g \in \mathrm{W}^{1-1/p,p}(\partial\Omega; \mathbb{R}^d)$*. Then, there exists* $v \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$ *such that*

$$\begin{cases} \det \nabla v(x) = J(x) & \text{for a.e. } x \in \Omega, \\ v|_{\partial\Omega} = g & \text{in the sense of trace.} \end{cases}$$

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Corollary

Let $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain,* $1 < p < d$ *and* $g \in \mathrm{W}^{1-1/p,p}(\partial\Omega; \mathbb{R}^d)$*. Then, there exists an incompressible map* $v \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$ *with the given boundary data (in the sense of trace), i.e.*

$$\det \nabla v(x) = 1 \ \text{ almost everywhere,}$$

$$v|_{\partial\Omega} = g.$$

One of Ball's open problems in nonlinear elasticity

Problem (Problem 1 in Ball '02)

Prove the existence of energy minimizers for elastostatics for quasiconvex stored-energy functions satisfying

$$f(A) \to +\infty \quad \text{as } \det A \downarrow 0.$$

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- Different notions of quasiconvexity $\rightsquigarrow$ not clear which one is "right".
- One "popular" notion: A locally bounded Borel function $h\colon \mathbb{R}^{d\times d} \to (-\infty, +\infty]$ is $\mathrm{W}^{1,p}$-**quasiconvex** if

$$h(A_0) \leq \fint_{B(0,1)} h(\nabla v(x))\,\mathrm{d}x$$

for all $A_0 \in \mathbb{R}^{d\times d}$ and all $v \in \mathrm{W}^{1,p}(B(0,1);\mathbb{R}^d)$ with $v(x) = A_0 x$ on $\partial B(0,1)$ (in the sense of trace).

$\mathrm{W}^{1,p}$-quasiconvexity and "realistic" growth

Lemma

Let $1 < p < d$ and assume that h is $\mathrm{W}^{1,p}$-quasiconvex (or closed $\mathrm{W}^{1,p}$-quasiconvex). If there exists one $r > 0$ and a constant $M = M(r) \geq 0$ such that the very weak growth constraint

$$h(A) \leq M(1 + |A|^p) \qquad \text{for all } A \in \mathbb{R}^{d \times d} \text{ with } \det A = r$$

holds, then $h(A) < \infty$ for every $A \in \mathbb{R}^{d \times d}$.

Remark: No universal growth bounds are required!

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Big "work in progres" issue: We do not currently know any non-trivial examples of $\mathrm{W}^{1,p}$-orientation-preserving quasiconvex functions.

A class of integrands

Assume:

- $1 < p < d$.
- **Singular p/d-growth modulus:** convex function $\kappa\colon (0,\infty) \to [0,\infty)$ with $\kappa(s) \to +\infty$ as $s \to 0$ and $\limsup_{s\to+\infty} \frac{\kappa(s)}{s^{p/d}} < \infty$.
- **Example:** $\kappa(s) := \begin{cases} \frac{1}{s} & s > 0, \\ +\infty & s \leq 0. \end{cases}$

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- **Example:** $\kappa(s) := \begin{cases} \frac{1}{s} & s > 0, \\ +\infty & s \leq 0. \end{cases}$

- $\Omega \subset \mathbb{R}^d$ bounded open Lipschitz domain.
- Let $f\colon \Omega \times \mathbb{R}^{d\times d} \to [0,\infty]$ be a Carathéodory integrand.
- Assume the **elastic coercivity/growth estimates**:

$$\frac{1}{M}\big(|A|^p + \kappa(\det A)\big) \leq f(x,A) \leq M\big(1 + |A|^p + \kappa(\det A)\big)$$

for a constant $M > 0$; here we set $\kappa(s) = \infty$ for $s \leq 0$.

A lower semicontinuity theorem

Theorem (K-R-W#3 2014)

Under the assumptions on the previous slide, let additionally

$f(x, \bullet)$ *be* $\mathrm{W}^{1,p}$*-orientation-preserving quasiconvex for almost every* $x \in \Omega$.

Then, the functional

$$\mathcal{F}[u] := \int_\Omega f(x, \nabla u(x)) \,\mathrm{d}x \qquad u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^d) \text{ with } \det \nabla u > 0 \text{ a.e.},$$

is $\mathrm{W}^{1,p}$*-weakly lower semicontinuous along sequences* $u_j \rightharpoonup u$ *in* $\mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$ *satisfying the additional constraint*

$$\det \nabla u > 0 \quad \text{a.e.} \tag{J_∞}$$

Converse: $\mathrm{W}^{1,p}$*-orientation-preserving quasiconvex is also* **necessary** *for* $\mathrm{W}^{1,p}$*-weak lower semicontinuity.*

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Remarks:

- The main difficulty is taming the "elastic" growth.
- (J_∞) is not unusual, can be established using standard methods.

An existence theorem for minimizers

Theorem

Let $\Omega \subset \mathbb{R}^3$ and $2 \le p < 3$. Under the assumptions of the previous theorem and with a boundary value function $g \in \mathrm{W}^{1-1/p,p}(\partial\Omega; \mathbb{R}^3)$, the minimization problem

$$\mathcal{F}[u] := \int_\Omega f(x, \nabla u(x))\,\mathrm{d}x \;\to\; \min$$

over the set

$$\mathcal{A} = \left\{ u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^3) \;:\; \begin{array}{l} \det \nabla u > 0 \text{ a.e., } u|_{\partial\Omega} = g, \\ \|\det \nabla u\|_r \le M,\ \|\operatorname{cof} \nabla u\|_s \le M \end{array} \right\},$$

where $r > 1$, $s \ge p/(p-1)$, and $M > 0$ are fixed, has at least one solution.

Example (Ball '02): Assume $f : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ can be written as

$$f(x, A) = W(x, A) + F(x, A),$$

where $\mathrm{W}(x, \bullet)$ is polyconvex and satisfies

$$W(x, A) \ge C^{-1}(|A|^p + |\mathrm{cof}\, A|^s + |\det A|^r) - C$$

for some $r > 1$, $2 \le p < 3$, $s \ge p/(p-1)$, and $C > 0$, whereas $F(x, \bullet)$ is $\mathrm{W}^{1,p}$-orientation preserving quasiconvex at a.e. $x \in \Omega$ and has elastic growth.

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- In $\mathrm{W}^{1,p}$ with $p \geq d$, the Jacobian determinant is "rigid".
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- Matrix geometry & convex integration give characterization of Young measures and several interesting corollaries.
- $\mathrm{W}^{1,p}$-quasiconvexity is not suitable for "realistic", elastic growth.
- Better: Orientation-preserving $\mathrm{W}^{1,p}$-quasiconvexity $\rightsquigarrow$ necessary and sufficient for lower semicontinuity.

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Outlook:

- Construct orientation-preserving $\mathrm{W}^{1,p}$-quasiconvex functions.

Thank you for your attention!

References:

[K-R-W#1] K. Koumatos, F. R., and E. Wiedemann: *Orientation-preserving Young measures*, arXiv:1307.1007.

[K-R-W#2] K. Koumatos, F. R., and E. Wiedemann: *Differential inclusions and Young measures involving prescribed Jacobians*, arXiv:1312.1820.

[K-R-W#3] K. Koumatos, F. R., and E. Wiedemann: *Lower semicontinuity under the positive Jacobian constraint and applications to nonlinear elasticity*, in preparation.