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On Finite Sequences Conditionally Uniform Given Minima and Maxima *

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Abstract. Conditionings in a finite sequence $X^{(N)} = (X_1, X_2, \dots, X_N)$ of real random variables by $\max X^{(N)}$ and by $\max X^{(N)}$ together with $\min X^{(N)}$ are considered. If $X^{(N)}$ is conditionally uniform in a very general sense with respect to a reference Borel measure ν then a shorter subsequence $X^{(n)} = (X_1, X_2, \dots, X_n)$, $1 \leqslant n < N$, can be well approximated, in the variation distance, by a mixture of n-powers of restrictions of ν. These finite de Finetti type results can be used to obtain integral representations of infinite sequences which have all their finite sub-sequences conditionally uniform.

1. Introduction

In Statistics, natural kinds of symmetry in a random sequence are specified by means of sufficient statistics and kernels, see Diaconis(1988) or Diaconis and Freedman(1984). The distributions of sequences under a symmetry assumption form usually a convex set and then integral representations of the distributions are available in numerous de Finetti-type theorems. Majority of results in this field concerns models with sufficient statistics of the form $\sum_{i=1}^{n} \Psi(X_i)$ where extreme distributions are products of measures in an exponential family. Relationships between predictive and classical sufficiency has been recently clarified in Fortini, *et al.*(2000).

Though there exist very general results on sequences with symmetries, each particular case requires an additional work to identifying explicitly the form of mixtures and mixing probabilities. In this note we present finite and infinite versions of de Finetti-type theorems for sequences of real random variables which are conditionally uniform given the maxima, or given the minima and maxima. The adjective 'uniform' is specified in a very general sense w.r.t. a reference Borel measure. This approach provides unification and generalization of problems examined separately before, notably in Iglesias, *et al.*(1998), Gnedin(1996), Rachev and Rüschendorf(1991), Ressel(1985).

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The connection to Fortini, *et al.*(2000) is through the characterization of de Finetti-type theorems via sufficiency, but our approach is different from theirs. In a statistical language, results of this work characterize models obtained by truncating known distributions with truncating parameters assumed to be unknown.

First, aggregates of product measures modeling the conditional uniformity are examined in Section 2. Using the aggregates we obtain the corresponding finite de Finetti-type theorems, when conditioned on the maxima in Section 3 and when conditioned on minima and maxima in Section 5. The infinite versions are worked out in Section 4 and Section 6.

2. Aggregates of product measures

Let $(A, \mathcal{A}, \lambda)$ be a measure space, $0 < \lambda(A) < \infty$ and n a positive integer number. An element a of A is fixed assuming that the singleton $\{a\}$ is $\mathcal{A}$-measurable. The Dirac measure sitting in a is denoted by δ_a and the restriction of λ to a set $B \in \mathcal{A}$ by $\lambda_{|B}$, that means $\lambda_{|B}(C) = \lambda(B \cap C)$ for $C \in \mathcal{A}$.

In the first part of this section we examine the following measure on the product space A^n

$$\alpha_n = \sum_{\emptyset \neq I \subset \hat{n}} \lambda(\{a\})^{|I|-1}\, \delta_a^I \times \lambda_{|A-\{a\}}^{\hat{n}-I}$$

where $\hat{n} = \{1, 2, \ldots, n\}$, $|I|$ is the cardinality of I, δ_a^I is the product measure on A^I and $0^0 = 1$ by convention. Note that the summands are mutually singular product measures. Let us observe that the α_n measure of $(A - \{a\})^n$ is zero and α_n restricted to $A^k \times \{a\} \times A^{n-1-k}$ is equal to $\lambda^k \times \delta_a \times \lambda^{n-1-k}$, $0 \leqslant k \leqslant n-1$. Roughly speaking, α_n is constructed by pasting the latter n measures together. We set $t_n = \alpha_n(A^n)$.

EXAMPLE 1. Let $A = \{0, 1, 2, \ldots a\}$ where a is a nonnegative integer, let $\mathcal{A}$ be the power set of A and λ the counting measure on A. Then α_n is the counting measure on the set of all n-tuples $(a_1, \ldots, a_n) \in A^n$ having at least one coordinate equal to a, i.e. satisfying $\max_{1 \leqslant j \leqslant n} a_j = a$.

EXAMPLE 2. Let $(A, \mathcal{A})$ be an interval $[0, a]$ of the real line $\mathbb{R}$ (a is any positive real number) endowed with the σ-algebra of its Borel subsets and let λ be the Lebesque measure on this interval. Then α_n is the sum of n product measures supported by the set of all n-tuples $(a_1, \ldots, a_n) \in A^n$ with the maximum equal to a.

REMARK. If in one of these two examples λ is a probability measure and if $X_1, \ldots, X_n$ are *i.i.d.* random variables with the law λ then α_n is, up to a normalizing constant, the conditional distribution of $X_1, \ldots, X_n$ given $\max\{X_i;\ 1 \leqslant i \leqslant n\} = a$.

Having a measure on the product space A^{n+k}, $k \geqslant 1$, its marginal measure on A^n will be denoted by means of the super-index (n). The bar over a nonzero finite measure will normalize it to a probability measure. Our first result estimates the variation distance between a marginal of the normalization of α_{n+k} and a power of the normalized λ.

LEMMA 1. *For every $n \geqslant 1$ and $k \geqslant 1$, the total variation $\| \overline{\alpha}_{n+k}^{(n)} - \overline{\lambda}^n \|$ is at most $2n/(n+k)$, with the equality if and only if $\lambda(\{a\}) = 0$.*

PROOF. Let $r = \lambda(\{a\})$ and $s = \lambda(A - \{a\})$, then obviously $t_n = [(r+s)^n - s^n]r^{-1}$ if $r > 0$ and $t_n = ns^{n-1}$ otherwise. It is also not difficult to see that $\alpha_{n+1}^{(n)} = s\,\alpha_n + \lambda^n$.

Using the identity $t_{n+k} = s^k t_n + t_k(r+s)^n$ one can obtain $\alpha_{n+k}^{(n)} = s^k \alpha_n + t_k \lambda^n$, $k \geqslant 1$, by induction. This yields

$$\left\| \overline{\alpha}_{n+k}^{(n)} - \overline{\lambda}^n \right\| = \frac{s^k t_n}{t_{n+k}} \left\| \overline{\alpha}_n - \overline{\lambda}^n \right\|$$

that, for $r = 0$, is equal to the desired bound $2n/(n+k)$. Here, the variation distance of the probability measures $\overline{\alpha}_n$ and $\overline{\lambda}^n$ was two as they are singular. If $r > 0$ the distance is strictly smaller than two. Expanding powers of $r+s$ in the above ratio and comparing corresponding terms one can see that it is bounded from above by $n/(n+k)$. $\square$

REMARKS. 1. In Rachev and Rüschendorf(1991, Theorem 6.2, p. 1332), the above total variation with λ, as in our Example 1, was bounded by $n(n+2k)/2k(n+k)$; this is tighter than $2n/(n+k)$ if and only if $n < 2k$.

2. If $X_1, \ldots, X_{n+k}$ are *i.i.d.* nonnegative random variables following a law P then the distribution of $X_1, \ldots, X_n$ given $\max\{X_i;\ 1 \leqslant i \leqslant n+k\} = a$ is close in the variation distance to the law of n *i.i.d.* random variables following the law P truncated at the value $a \geqslant 0$ from the right.

Second part of this section is devoted to a similar result for another aggregate of product measures of λ's and δ's. A motivation stems from a study of uniform distributions on ℓ_∞-spheres, see Iglesias, *et al.*(1998). Let a and b be two different elements of A with $\{a\}$ and $\{b\}$ measurable and

$$\gamma_n = \sum \lambda(\{a\})^{|I|-1}\, \delta_a^I \times \lambda(\{b\})^{|J|-1}\, \delta_b^J \times \lambda_{|A-\{a,b\}}^{\hat{n}-(I\cup J)}$$

where the summation is extended over all ordered pairs (I, J) of nonempty and disjoint subsets of $\hat{n}$. Analogously as above, the measure γ_n is pasted together from $n(n-1)$ product measures $\lambda^k \times \delta_a \times \lambda^l \times \delta_b \times \lambda^{n-2-k-l}$ and $\lambda^k \times \delta_b \times \lambda^l \times \delta_a \times \lambda^{n-2-k-l}$, $0 \leqslant k+l \leqslant n-2$. We set $\gamma_n(A^n) = v_n$.

EXAMPLE 3. Let $A = \{a, a+1, \ldots, b\}$ where $a \leqslant b$ are integers, $\mathcal{A}$ be the power set of A and λ be the counting measure on A. Then γ_n is the counting measure on the set of all n-tuples $(a_1, \ldots, a_n) \in A^n$ having at least one coordinate equal to a and at least one coordinate equal to b, i.e. satisfying $\min_{1\leqslant j\leqslant n} a_j = a$ and $\max_{1\leqslant j\leqslant n} a_j = b$.

EXAMPLE 4. Let $(A, \mathcal{A})$ be an interval $[a, b]$ of the real line $\mathbb{R}$ ($a < b$ are real numbers) endowed with the σ-algebra of its Borel subsets and λ be the Lebesque measure on this interval. Then α_n is the sum of $n^2 - n$ product measures and its support is again the set of all n-tuples $(a_1, \ldots, a_n) \in A^n$ with the minimum equal to a and the maximum equal to b.

REMARK. If in one of these two examples above λ is any probability measure and if $X_1, \ldots, X_n$ are *i.i.d.* random variables with the law λ then α_n is, up to a normalizing constant, the conditional law of $X_1, \ldots, X_n$ given $\min\{X_i;\ 1 \leqslant i \leqslant n\} = a$ and $\max\{X_i;\ 1 \leqslant i \leqslant n\} = b$.

LEMMA 2. *Let $\lambda(A - \{a\}) > 0$ and $\lambda(A - \{b\}) > 0$. For every $n \geqslant 1$ and $k \geqslant 1$*

$$\left\| \overline{\gamma}_{n+k}^{(n)} - \overline{\lambda}^n \right\| \leqslant \frac{2n(n+2k-1)}{(n+k)(n+k-1)}$$

the equality being the case if and only if $\lambda(\{a\}) = \lambda(\{b\}) = 0$.

PROOF. The number v_n is the value of the following polynomial in three variables

$$\frac{1}{r_a r_b}\Big[(r_a + r_b + s)^n - (r_a + s)^n - (r_b + s)^n + s^n\Big]$$

when substituting $r_a = \lambda(\{a\})$, $r_b = \lambda(\{b\})$ and $s = \lambda(A - \{a, b\})$. The positive assumptions imply $v_n > 0$. It is not difficult to establish that $\gamma_{n+1}^{(n)} = s\gamma_n + \alpha_n + \beta_n$ where β_n is constructed analogously as α_n when a is switched to b. After verification of the polynomial identity

$$v_{n+k} = s^k v_n + t_{k,b}^{\{a\}} t_{n,a} + t_{k,a}^{\{b\}} t_{n,b} + v_k(r_a + r_b + s)^n$$

where $t_{k,b}^{\{a\}} = [(r_b+s)^k - s^k]r_b^{-1}$, $t_{n,a} = [(r_a+r_b+s)^n - (r_b+s)^n]r_a^{-1}$ and $t_{k,a}^{\{b\}}$ and $t_{n,b}$ are defined correspondingly, one can obtain by induction

$$\gamma_{n+k}^{(n)} = s^k\gamma_n + t_{k,b}^{\{a\}}\alpha_n + t_{k,a}^{\{b\}}\beta_n + v_k\lambda^n \ .$$

The desired total variation takes the form

$$\left(1 - \frac{v_k}{v_{n+k}}(r_a+r_b+s)^n\right) \left\| \mu - \overline{\lambda}^n \right\|$$

where μ is a convex combination of $\overline{\alpha}_n, \overline{\beta}_n$ and $\overline{\gamma}_n$. If $r_a = 0$ and $r_b = 0$ then μ and $\overline{\lambda}^n$ are singular, $v_n = n(n-1)s^{n-2}$ and the above expression is equal to the upper bound. In the opposite case, $r_a + r_b > 0$, it suffices to prove that

$$(r_a+r_b+s)^n \frac{(r_a+r_b+s)^k - (r_a+s)^k - (r_b+s)^k + s^k}{(r_a+r_b+s)^{n+k} - (r_a+s)^{n+k} - (r_b+s)^{n+k} + s^{n+k}} \geqslant \frac{k(k-1)}{(n+k)(n+k-1)} \ .$$

Without loss of generality we can assume that $r_a + r_b + s = 1$. Let us write $x = r_a + s$, $y = r_b + s$, $m = n + k$ and look at the function

$$g(x,y) = m(m-1)\left[1 - x^k - y^k + (x+y-1)^k\right] - k(k-1)\left[1 - x^m - y^m + (x+y-1)^m\right]$$

that should be nonnegative for $0 \leqslant x \leqslant y \leqslant 1$ and $x + y \geqslant 1$. The expression

$$\frac{1}{mk}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) g(x,y) = (m-1)(x^{k-1} - y^{k-1}) - (k-1)(x^{m-1} - y^{m-1})$$

is non positive, due to $(k-1)(1 - z^{m-1}) \leqslant (m-1)(1 - z^{k-1})$, $0 \leqslant z \leqslant 1$. Since $g(x,1) = 0$ for $0 \leqslant x \leqslant 1$ the nonnegativity of g is confirmed. $\square$

REMARK. Let $X_1, X_2, \ldots, X_{n+k}$ be *i.i.d.* random variables with a common law P. Lemma 2 and the previous remark imply that the distribution of $X_1, \ldots, X_n$ given $\min\{X_i;\ 1 \leqslant i \leqslant n+k\} = a$ and $\max\{X_i;\ 1 \leqslant i \leqslant n+k\} = b$ is close in the variation distance to the law of n *i.i.d.* random variables distributed according to the truncation of P at a from left and at $b \geqslant a$ from the right.

3. Conditioning by maximum in a finite sequence

In this section and in the following one let ν be a nonzero Borel measure on the real half-line $[0,\infty)$ so that every closed interval $[0,a]$, $a \geqslant 0$, has finite ν measure. The interval $[0,a]$ with its Borel subsets and the restriction $\nu_{|[0,a]}$ are taken for the measurable space $(A, \mathcal{A}, \lambda)$ and the finite measure $\alpha_{n,a}$ is constructed, $n \geqslant 1$, adding the index a to express explicitly dependence on $a \geqslant 0$. The normalization $\overline{\alpha}_{n,a}$ is a probability measure if $\nu[0,a] > 0$; otherwise, it is treated as a zero measure.

Introducing the following definition we had in mind especially the Lebesgue measure or the counting measure on nonnegative integers on the place of ν, see Example 1 and Example 2, correspondingly.

DEFINITION 1. *A finite sequence* $X^{(n)} = (X_1, \ldots, X_n)$, $n \geqslant 1$, *of nonnegative random variables is said to be* ν-uniform given its maximum *if the distribution* $P_{X^{(n)}}$ *is equal to the mixture*

$$\int_{[0,\infty)} \overline{\alpha}_{n,a}\, P_{\max X^{(n)}}(da) \ .$$

An infinite sequence is ν-uniform given maxima *if every its finite initial segment is so.*

Equivalently, $\overline{\alpha}_{n,a}$ is assumed to be a conditional distribution of $X^{(n)}$ given $\max X^{(n)} = a$. These sequences are obviously exchangeable.

Every *i.i.d.* sequence distributed according to $\overline{\nu}_{|[0,a]}, \overline{\nu}_{|[0,a)}$ or $\overline{\nu}$ is ν-uniform/max provided that $\nu[0,a] > 0$, $\nu[0,a) > 0$ and ν is finite, respectively. Convex combinations of the distributions of ν-uniform/max sequences are ν-uniform/max.

PROPOSITION 1. *An initial segment $X^{(n)} = (X_1, \dots, X_n)$ of every finite random sequence $X^{(N)} = (X_1, \dots, X_N)$, $1 \leqslant n \leqslant N$, that is ν-uniform given its maximum has its distribution approximated by*

$$\left\| P_{X^{(n)}} - \int_{[0,\infty)} \overline{\nu}^n_{|[0,a]} \, P_{\max X^{(N)}}\,(da) \right\| \;\leqslant\; \frac{2n}{N} \ .$$

PROOF. The variation distance does not exceed

$$\int_{[0,\infty)} \left\| \overline{\alpha}^n_{N,a} - \overline{\nu}^n_{|[0,a]} \right\| \, P_{\max X^{(N)}}(da)$$

where values of the integrated function are either zero, as the two measures are zero, or bounded by $2n/N$ owing to Lemma 1. □

4. Conditioning by maxima in infinite sequences

It is straightforward that if $a \geqslant 0$ and $\nu[0,a] > 0$ ($a > 0$ and $\nu[0,a) > 0$) then the measure $\overline{\nu}^\infty_{|[0,a]}$ ($\overline{\nu}^\infty_{|[0,a)}$) is a distribution of infinite sequences that is ν-uniform given maxima. If ν is finite then also $\overline{\nu}^\infty$ is a distribution of this type. In the assertion below we show that mixtures of all these product measures exhaust distributions of this type. We prefer to work with the family of measures $\nu_{|[0,a]}$, $a \geqslant 0$. A measure $\nu_{|[0,a)}$, $a > 0$, is not in this family if and only if $\nu(\{a\}) > 0$ and $\nu(b,a) > 0$ for every $0 < b < a$. Let us denote by S_ν the set of all these positive numbers a and let, in addition, ∞ belongs to S_ν if and only if ν is finite and $\nu(b,\infty) > 0$ for all $b \geqslant 0$. The set S_ν is at most countable.

THEOREM 1. *Let $X = (X_n;\ n \geqslant 1)$ be an infinite sequence of nonnegative random variables that is ν-uniform given maxima. Then there exists a finite measure κ on the real half-line $[0,\infty)$ and nonnegative numbers κ_a for $a \in S_\nu$ such that*

$$P_X \;=\; \int_{[0,\infty)} \overline{\nu}^\infty_{|[0,a]} \; \kappa(da) \;+\; \sum_{a \in S_\nu} \overline{\nu}^\infty_{|[0,a)} \, \kappa_a$$

and $\kappa[0,\infty) + \sum_{a\in S_\nu} \kappa_a = 1$. The measure κ can be chosen to satisfy $\kappa(b,c] = 0$ once $\nu(b,c] = 0$, $0 \leqslant b < c$, and then it is unique with this property. The numbers κ_a are unique as well.

This assertion could be, with a considerable effort, adjusted to the framework of Theorem 1.1 from Diaconis and Freedman(1984). We prefer to have a swift analytical proof identifying explicitly all ingredients of the mixture.

PROOF. Uniqueness of the measure κ and of the numbers κ_a, $a \in S_\nu$, follows from uniqueness in the classical de Finetti theorem. In fact, let B_ν be the Borel set $[0,\infty) - \bigcup\{(b,c];\ \nu(b,c] = 0\}$ and let $\phi_\nu(a) = \max\{b \leqslant a;\ b \in B_\nu\}$, $a \geqslant 0$. The integral from Theorem 1 can be equivalently written as the integral of the same function over $a \in B_\nu$ with respect to the ϕ_ν image of the measure κ.

Let μ_n denote the distribution of maximum of $X_1, \dots, X_n$, $n \geqslant 1$. For every $a \geqslant 0$ the sequence $\mu_n[0,a]$, $n \geqslant 1$, does not increase. The function $a \to \lim_{n\to\infty} \mu_n[0,a]$ is non-decreasing and right-continuous and hence equal to $a \to \mu[0,a]$ for a unique Borel measure μ on $[0,\infty)$. Let $\kappa_a = \lim_{k\to\infty} \mu_k[0,a) - \mu[0,a)$, $a > 0$, and $\kappa_\infty = 1 - \mu[0,\infty)$. Note that $0 \leqslant \kappa_a \leqslant \mu(\{a\})$ for any positive a. We set $\kappa = \mu - \sum_{\infty \neq a \in S_\nu} \kappa_a \delta_a$.

If $\nu(b,c]=0$, $0\leqslant b<c$, then also $\mu_n(b,c]=0$ for any $n\geqslant 1$ whence $\mu(b,c]=0$ and $\kappa(b,c]=0$. By Lemma 1

$$\begin{aligned}\mu_n(b,c] &= P^{(n)}_{X^{(n+k)}}\big([0,c]^n-[0,b]^n\big) = \int_{[0,\infty)} \overline{\alpha}^{(n)}_{n+k,a}\big([0,c]^n-[0,b]^n\big)\,\mu_{n+k}(da)\\ &\leqslant \int_{[0,\infty)} \Big[\big(\overline{\nu}_{|[0,a]}[0,c]\big)^n-\big(\overline{\nu}_{|[0,a]}[0,b]\big)^n\Big]\,\mu_{n+k}(da)+\frac{2n}{n+k}\end{aligned}$$

and since the last integral is zero and k is arbitrary we have indeed $\mu_n(b,c]=0$. As a consequence, $\kappa_a=0$ once $a>0$, $\nu(\{a\})>0$ and $a\notin S_\nu$.

To verify the announced mixture representation of the distribution P_X it suffices to prove that for every $n\geqslant 1$ the distribution $P_{X^{(n)}}$ is equal to the mixture of n-th power of the restricted measures. This is an equality of two distributions that are both ν-uniform given maximum whence if suffices to verify the equality on sets $[0,b]^n$, $b\geqslant 0$. To summarize, we fix $n\geqslant 1$ and $b\geqslant 0$ and want to prove that

$$\mu_n[0,b] = \int_{[0,\infty)} f(a)\,\mu(da) + \sum_{\infty\neq a\in S_\nu} \big(f(a-)-f(a)\big)\,\kappa_a + f(\infty)\,\kappa_\infty$$

where $f(a)=\big(\overline{\nu}_{|[0,a]}[0,b]\big)^n$ for $a\geqslant 0$, $f(a-)=\big(\overline{\nu}_{|[0,a)}[0,b]\big)^n$ for every $a\in S_\nu-\{\infty\}$ (we remark that $f(a-)$ is really $\lim_{c\nearrow a} f(c)$ in this case) and $f(\infty)=\lim_{c\to\infty} f(c)$ ($f(\infty)=0$ if ν is infinite and $f(\infty)=\big(\overline{\nu}[0,b]\big)^n$ if ν is finite). Note also that $0=f(a-)-f(a)$ if $a\in S_\nu\cap[0,b]$.

The above equality is obviously valid if $\nu[0,b]=0$ because then $\mu_n[0,b]=0$ and the function f is identically equal to zero. From now on we can suppose $\nu[0,b]>0$. By Proposition 1 the integrals $\int_{[0,\infty)} f(a)\,\mu_{n+k}(da)$ converge to $\mu_n[0,b]$ once $k\to\infty$ and thus it suffices to show the convergence

$$I_k=\int_{(b,\infty)} f(a)\,\mu_k(da) \to \int_{(b,\infty)} f(a)\,\mu(da) + \sum_{a\in S_\nu\cap(b,\infty)} \big(f(a-)-f(a)\big)\,\kappa_a + f(\infty)\,\kappa_\infty$$

when k approaches ∞. The expression on the right-hand-side will be denoted by K.

Let us observe that the function $f\leqslant 1$ is right-continuous and non-increasing on interval (b,∞). To prove the above convergence, let $\varepsilon>0$ and $b=b_0<\ldots<b_m=\infty$, $m\geqslant 1$, be chosen to have $f(b_{j-1})\leqslant f(b_j-)+\varepsilon$, $1\leqslant j\leqslant m$. The number I_k belongs to an interval with the endpoints

$$\pm\varepsilon + \sum_{j=1}^{m} f(b_j-)\,\mu_k(b_{j-1},b_j) + \sum_{j=1}^{m-1} f(b_j)\mu_k(\{b_j\}) \ .$$

Remembering that $\mu_k(c,a)\to\mu(c,a)+\kappa_a$ for $c<a$, $\mu_k(\{a\})\to\mu(\{a\})-\kappa_a$ for $a>0$, and $\mu_k(c,\infty)\to\mu(c,\infty)+\kappa_\infty$, all limit points of the sequence I_k, $k\geqslant 1$, are between

$$\pm\varepsilon + \sum_{j=1}^{m} f(b_j-)\,\mu(b_{j-1},b_j) + \sum_{j=1}^{m-1} f(b_j)\mu(\{b_j\}) + \big(f(b_j-)-f(b_j)\big)\,\kappa_{b_j} + f(b_m-)\,\kappa_{b_m}$$

and thus between $\pm 2\varepsilon+K$. Note that $\big(f(a-)-f(a)\big)\,\kappa_a>0$ implies $a\in S_\nu$ and $f(a-)-f(a)\leqslant\varepsilon$ for $a\in S_\nu-\{b_1,\ldots,b_m,\infty\}$. Since ε was arbitrary, I_k converges to K. □

REMARKS. 1. If ν is the counting measure on nonnegative integers then Theorem 1 reduces to Proposition 3.2, p. 321, in Iglesias, *et al.*(1998) (with $\kappa=\mu$ and $S_\nu=\emptyset$). See also Gnedin(1996) or Gnedin(1994) and Ressel(1985, Example 4, p. 910).

2. The special case of Theorem 1 for ν being the Lebesque measure on $[0,\infty)$ appeared in Example 7.1, p. 107 , in Fortini, *et al.*(2000) (with $\kappa\sim$ Pareto(α,x_0) and $S_\nu=\emptyset$). Also, in general, as Example 2.5, p. 210, in Diaconis and Freedman(1984). Their statement "The necessary and sufficient condition (for a sequence $X_1,X_2,\ldots$, to be a mixture over θ of sequences of independent

uniform variables with the range $[0,\theta]$) is that given $M_n = \max(X_1,\dots,X_n)$, the X_i's are independent and uniform over $[0, M_n]$, for $i = 1,\dots,n$" is not correct. Instead, the X_i's for $i = 1,\dots,n$ and $i \neq j$ should have been independent and uniform over $[0, M_n]$ *given* M_n *and the maximizer* X_j, for any $j = 1,\dots,n$, cf. Definition 1. This situation was also commented in Bernardo and Smith(1994, p. 206).

3. When a sequence is *i.i.d.* (thus the mixture is trivial) then either κ sits at a point or $\kappa = 0$ and $\kappa_a = 1$ for exacly one a, depending on S_ν being $\emptyset$ or $\{a\}$.

4. If ν is a probability measure on the nonnegative integers or on the nonnegative half line then Theorem 1 provides predictivistic characterizations of truncated to the right distributions. Examples of such characterizations are the truncated geometric or Poisson distributions with their natural parameter assumed to be known.

5. Conditioning by minimum and maximum in a finite sequence

Let ν be from now on a fixed nonzero Borel measure on the real line $\mathbb{R}$. A closed interval $[a,b]$ having positive ν measure, $a < b$, with its Borel subsets and the restriction $\nu_{|[a,b]}$ are taken for $(A, \mathcal{A}, \lambda)$ as in the second part of Section 2 and the finite measure $\gamma_{n,a,b}$ is constructed. If $\nu[a,b] = 0$, $a < b$, we set $\gamma_{n,a,b} = 0$. If $a = b$ we set $\gamma_{n,a,b} = \delta_a^n$ once $\nu(\{a\}) > 0$ and $\gamma_{n,a,b} = 0$ otherwise. Let us denote by $\mathbb{H}$ the closed half-plane of all pairs $(a,b) \in \mathbb{R}^2$ satisfying $a \leqslant b$.

DEFINITION 2. *A finite sequence* $X^{(n)} = (X_1,\dots,X_n)$, $n \geqslant 1$, *of real random variables is said to be* ν-uniform given its minimum and maximum *if its distribution* $P_{X^{(n)}}$ *can be written as the mixture*

$$\int_{\mathbb{H}} \overline{\gamma}_{n,a,b}\, P_{(\min X^{(n)},\, \max X^{(n)})}\,(da, db) \ .$$

An infinite sequence is ν-uniform given minima and maxima *if, similarly as in Definition 1, every its finite initial segment is so.*

REMARK. In a more applied context of lifetime data analysis, Barlow and Tsai(1995) considered ν–uniformity on invariant maxima and sum sets.

PROPOSITION 2. *An initial segment* $X^{(n)} = (X_1,\dots,X_n)$ *of every finite random sequence* $X^{(N)} = (X_1,\dots,X_N)$, $1 \leqslant n < N$, *that is* ν*-uniform given its minimum and maximum has the distribution approximated by*

$$\left\| P_{X^{(n)}} - \int_{\mathbb{H}} \overline{\nu}^n_{|[a,b]}\, P_{(\min X^{(N)},\, \max X^{(N)})}\,(da, db) \right\| \ \leqslant\ \frac{2n(2N-n-1)}{N(N-1)} \ .$$

PROOF. Analogous to the proof of Proposition 1, now using Lemma 2. □

6. Conditioning by minima and maxima in infinite sequences

The measures $\overline{\nu}^\infty_{|[a,b]}$ for $a \leqslant b$ and $\overline{\nu}^\infty_{|[a,b)}$, $\overline{\nu}^\infty_{|(a,b]}$ and $\overline{\nu}^\infty_{|(a,b)}$ for $a < b$, if nonzero, are distributions of infinite sequences that are ν-uniform given minima and maxima. If $\nu_{|[0,+\infty)}$ is a finite Borel measure then also $\overline{\nu}^\infty_{|[a,+\infty)}$ and $\overline{\nu}^\infty_{|(a,+\infty)}$ for $a \in \mathbb{R}$, if nonzero, are distributions of this sort and symmetrically with $\overline{\nu}^\infty_{|(-\infty,b]}$ and $\overline{\nu}^\infty_{|(-\infty,b)}$ for $b \in \mathbb{R}$ if $\nu_{|(-\infty,0]}$ is finite. Finally, if even the measure ν is finite then $\overline{\nu}^\infty$ is ν-uniform given minima and maxima. In the theorem below we claim that mixtures of these infinite-product probability measures coincide with the distributions of all ν-uniform/min&max sequences.

We need a convenient parametrization of the listed measures. We prefer the closed intervals and then the measures $\overline{\nu}^\infty_{|[a,b]}$ are parametrized by $\mathbb{H}$. Further, we write $a \in S^+_\nu$ if $\nu(\{a\}) > 0$ and

$\nu(a,b) > 0$ for all $b > a$, in addition, $-\infty \in S^+_\nu$ if $\nu(-\infty,0)$ is finite and $\nu(-\infty,b)$ is positive for all $b \in \mathbb{R}$. Symmetrically, we write $b \in S^-_\nu$ if $\nu(\{b\}) > 0$ and $\nu(a,b) > 0$ for all $a < b$, in addition, $+\infty \in S^-_\nu$ if $\nu(0,+\infty)$ is finite and $\nu(a,+\infty)$ is positive for all $a \in \mathbb{R}$. The sets S^+_ν and S^-_ν are both at most countable.

THEOREM 2. *Let $X = (X_n;\, n \geqslant 1)$ be an infinite sequence that is ν-uniform given minima and maxima. Then there exist a finite Borel measure κ on the closed half-plane $\mathbb{H}$, finite Borel measures κ^+_a on $(a,+\infty)$ for $a \in S^+_\nu$, finite Borel measures κ^-_b on $(-\infty,b)$ for $b \in S^-_\nu$ and nonnegative numbers $\kappa_{a,b}$ for $a \in S^+_\nu$, $b \in S^-_\nu$ and $a < b$ such that*

$$P_X = \int_{\mathbb{H}} \overline{\nu}^{\infty}_{|[a,b]}\, \kappa(da,db) + \sum_{a\in S^+_\nu} \int_{(a,+\infty)} \overline{\nu}^{\infty}_{|(a,b]}\, \kappa^+_a(db) + \sum_{b\in S^-_\nu} \int_{(-\infty,b)} \overline{\nu}^{\infty}_{|[a,b)}\, \kappa^-_b(da) + \sum_{a\in S^+_\nu} \sum_{a<b\in S^-_\nu} \overline{\nu}^{\infty}_{|(a,b)}\, \kappa_{a,b} .$$

The measure κ can be chosen to satisfy $\kappa\big(\mathbb{R}\times(b,c]\big) = 0$ if $\nu(b,c] = 0$ and $\kappa\big([b,c)\times\mathbb{R}\big) = 0$ if $\nu[b,c) = 0$, $b < c$, and then it is unique with this property. The measures κ^+_a, $a \in S^+_\nu$, can be chosen uniquely to obey $\kappa^+_a[b_1,b_2) = 0$ if $\nu[b_1,b_2) = 0$, $a < b_1 < b_2$, and the measures κ^-_b, $b \in S^-_\nu$, can be chosen uniquely to obey $\kappa^-_b(a_1,a_2] = 0$ if $\nu(a_1,a_2] = 0$, $a_1 < a_2 < b$. The numbers $\kappa_{a,b}$ are unique as well.

A proof of this theorem, analogous to the proof of Theorem 1, can be worked out, see Iglesias, *et al.*(1999). Since it is much more technical and longer we omit it here.

REMARKS. 1. The special case of Theorem 2 with ν being the Lebesque measure on $\mathbb{R}$ was proved in Iglesias, *et al.*(1998). The same situation was treated from the extendibility point of view in Gnedin(1996).

2. Theorem 2 provides a preditivistic justification (in the sense of de Finetti) for distributions truncated to the left and to the right.

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References

BERNARDO, J.M. & SMITH, A.F.M. (1994). *Bayesian Theory.* J. Wiley and Sons, New York.

BARLOW, R. & TSAI, P. (1995). Foundational issues concerning the analysis of censored data. *Lifetime Data Analysis* **1**, 27-34.

DIACONIS, P. (1988). Recent progress on de Finetti's notions of exchangeability. In: *Bayesian Statistics 3*, Bernardo J.B., de Groot M.H., Lindley D.V. & Smith A.F.M. (eds). Oxford University Press, Oxford, 111–125.

DIACONIS, P. & FREEDMAN, D. (1984). Partial exchangeability and sufficiency. In: *Statistics: Applications and New Directions.*, Ghosh J.K. & Roy J. (eds). Indian Statist. Inst., Calcutta, 205–236.

FORTINI, S., LADELLI, L. & REGAZZINI, E. (2000). Exchangeability, Predictive Distributions and Parametric Models. *Sankhya*, Ser. A, **62**, 86-109.

GNEDIN, A.V. (1996). On a class of exchangeable sequences. *Statist. Probab. Letters* **25**, 351-355 (see also **28** 159-164).

GNEDIN, A.V. (1994). A solution to the game of googol. *Ann. Probab.* **22**, 1588-1595.

IGLESIAS, P., PEREIRA, C.A.B. & TANAKA, N.I. (1998). Characterizations of Multivariate Spherical Distributions in ℓ_∞–norm. *TEST*, **7**, 307-324.

IGLESIAS, P. MATÚŠ, F., PEREIRA, C.A.B. & and TANAKA, N.I. (1999) On finite sequences conditionally uniform given minima and maxima. Institute of Information Theory and Automation, Prague, Research Report 1950.

RACHEV, S.T. & RÜSCHENDORF, L. (1991). Approximate Independence of Distributions on Spheres and their Stability Properties. *Ann. Probab.* **19**, 1311-1337.

RESSEL, P. (1985). de Finetti-type theorems: an analytical approach. *Ann. Probab.* **13**, 898-922.

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