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Adhesivity of polymatroids

František Matúš[*]

Abstract. Two polymatroids are adhesive if a polymatroid extends both in such a way that two ground sets become a modular pair. Motivated by entropy functions, the class of polymatroids with adhesive restrictions and a class of selfadhesive polymatroids are introduced and studied. Adhesivity is described by polyhedral cones of rank functions and defining inequalities of the cones are identified, among them known and new non-Shannon type information inequalities for entropy functions. The selfadhesive polymatroids on a four-element set are characterized by Zhang-Yeung inequalities.

1. Introduction

Let N be a finite set and g a real-valued function on the power set 2^N of N. The pair (N,g) is a *polymatroid* [6, 3, 12] if the function g is normalized, $g(\emptyset)=0$, nondecreasing, $g(I)\leqslant g(J)$ for $I\subseteq J\subseteq N$, and submodular

$$\varDelta_{I,J}\, g \triangleq g(I)+g(J)-g(I\cup J)-g(I\cap J)\geqslant 0\,,\qquad I,J\subseteq N\,.$$

Here, N is the ground set, g the rank function and $g(N)$ the rank of the polymatroid. By frequent abuse of terminology, g itself is also referred to as a polymatroid. If $\varDelta_{I,J}\, g=0$ then I,J is a *modular* pair of g. An integer-valued polymatroid with $g(i)\leqslant 1$, $i\in N$, is a *matroid* [16, 14]; the singletons and elements of N are not distinguished.

When two polymatroids (N,g) and (M,h) satisfy $N\subseteq M$ and $g(I)=h(I)$ for $I\subseteq N$ then the former is the restriction of the latter to N and the latter is an extension of the former to M. We say that two polymatroids (N,g) and (M,h) are *adhesive*, or *adhere*, if a polymatroid $(N\cup M,f)$ extends both and makes the pair N,M modular, $\varDelta_{N,M}\, f=0$. Such a polymatroid f is called an *adhesive extension* of g and h.

Given a finite set N and $I\subseteq N$, let M be a set of the same cardinality as N such that $N\cap M=I$ and $\pi\colon N\to M$ a bijection satisfying $\pi(i)=i$ for $i\in I$. For a polymatroid

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[*]F. Matúš is with Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Prague, Czech Republic; `matus@utia.cas.cz`

(N,g) the mapping $h\colon J \mapsto g(\pi^{-1}(J))$, $J \subseteq M$, defines a polymatroid (M,h) called the π-copy of g with the overlap I. If g and h adhere the polymatroid (N,g) is *selfadhesive at* $I \subseteq N$. Obviously, this notion does not depend on a specific choice of M and π. A polymatroid is *selfadhesive* if it is selfadhesive at each $I \subseteq N$.

This work focuses on two new classes of polymatroids: the polymatroids that have any two restrictions adhesive and the selfadhesive polymatroids. The former contains the latter by Lemma 1 in Section 2.

Our main motivation for an investigation of these classes is their relation to entropy functions. Given a random vector $\xi = (\xi_i)_{i\in N}$, the entropy function h_ξ maps $I \subseteq N$ to the Shannon entropy of the subvector $\xi_I = (\xi_i)_{i\in I}$ and is a polymatroid by basic properties of entropy [2]. A polymatroid is *entropic* if it equals the entropy function of a random vector taking a finite number of values. By Lemma 2 in Section 2, the restrictions of an entropic polymatroid adhere and, by its Corollary 1, the entropic polymatroids are even selfadhesive. For a discussion on entropic polymatroids see [10].

In recent years it has been discovered that the entropy functions, or entropic polymatroids, satisfy a number of unexpected linear inequalities [20, 7, 19, 1], reviewed also in [17, 18]. The most remarkable first prototype of these information theoretical inequalities is *Zhang-Yeung inequality* [20, Theorem 3, p. 1444]

$$(1)\qquad \begin{aligned} &3\big[g(ik)+g(il)+g(kl)\big]+g(jk)+g(jl)\\ &\qquad\qquad -g(i)-2\big[g(k)+g(l)\big]-g(ij)-4g(ikl)-g(jkl)\geqslant 0 \end{aligned}$$

where i,j,k,l are different singletons of N and the signs for unions of singletons are omitted. A revision of the existing proofs of such inequalities reveals that the main arguments can be abstracted to the level of polymatroids with properties of adhesivity, and thus a detailed investigation of the two above classes of polymatroids can provide a unified and systematic way to treatment of linear information theoretical inequalities.

This work is organized as follows. Section 2 collects preliminaries and basic observations on adhesive restrictions and selfadhesivity. A generalization of the proper amalgamation of matroids to polymatroids is presented in Theorem 1. This idea together with parallel extensions settles existence of adhesive extensions if the ground sets of polymatroids are small: each polymatroid with a four-element ground set has adhesive restrictions and each polymatroid with a three-element ground set is selfadhesive.

Section 3 is devoted to descriptions of adhesive restrictions in terms of polyhedral cones and their defining inequalities. Two equivalent conditions for a polymatroid to have two restrictions adhesive are presented in Theorem 2. The second one opens a systematic constructive way to describe these polymatroids by means of a finite set of linear inequalities, discussed in Remark 3 and illustrated in Example 2. Known information theoretical inequalities of non-Shannon type, and a new one, are interpreted as applications of Theorem 2 to the entropic polymatroids, see Corollaries 4 and 5 and Remarks 4 and 6.

Section 4 deals with the polymatroids having a four-element ground set. Among them, the polymatroids that are selfadhesive at a two-element set are described by six instances of Zhang-Yeung inequality, see Theorem 3. As a corollary, the selfadhesive polymatroids over a set of four elements are nothing but the polymatroids that satisfy all instances of Zhang-Yeung inequality, endowing thereby its new nontrivial geometric interpretation.

2. Preliminaries, parallel extensions and generalized proper amalgams.

The following lemmas support simple statements of the introduction.

Lemma 1. *If $I,J\subseteq N$ and a polymatroid (N,g) is selfadhesive at $I\cap J$ then the restrictions of g to I and J adhere.*

Proof. By assumption, for a bijection $\pi\colon N\to M$ such that $I\cap J=N\cap M$, a polymatroid $(N\cup M,f)$ with $\mathit{\Delta}_{N,M}\, f=0$ extends g and the π-copy of g with the overlap $I\cap J$. Then, the function h defined by

$$h(K)=f((K\cap I)\cup\pi(K\cap J))\,,\qquad K\subseteq I\cup J\,,$$

is a polymatroid on $I\cup J$ that extends the restrictions of g to I and J. The submodularity of f and the assumption $\mathit{\Delta}_{N,M}\, f=0$ imply $\mathit{\Delta}_{I,\pi(J)}\, f=0$. Since $\mathit{\Delta}_{I,J}\, h=\mathit{\Delta}_{I,\pi(J)}\, f$ it follows that h is an adhesive extension of its restrictions to I and J, thus the restrictions of g to I and J adhere. □

As a consequence, two restrictions of any selfadhesive polymatroid adhere.

Lemma 2. *Two restrictions of an entropic polymatroid have an adhesive extension that is entropic.*

Proof. Let I,J be two subsets of N, $M=I\cup J$, and h_ξ the entropy function of a random vector $\xi=(\xi_i)_{i\in N}$. If ξ_i ranges in a finite set X_i let $P(x_K)$ denote the probability that ξ_K attains the value $x_K=(x_k)_{k\in K}$ with $x_k\in X_k$, $K\subseteq N$. Define $Q(x_M)$ equal to zero if $P(x_{I\cap J})=0$ and $P(x_I)P(x_J)/P(x_{I\cap J})$ otherwise; here x_I, x_J and $x_{I\cap J}$ are the coordinate projections of x_M. Then, Q is equal to the distribution of a random vector $\eta=(\eta_i)_{i\in M}$. By the construction, ξ_I and η_I are equidistributed, and hence the restrictions of h_ξ and h_η to I coincide. This holds also when I is replaced by J, by symmetry. Since $\eta_{I\setminus J}$ and $\eta_{J\setminus I}$ are conditionally independent given $\eta_{I\cap J}$ it follows that $\mathit{\Delta}_{I,J}\, h_\eta$ equals 0. Thus, h_η is an adhesive extension of the restrictions of h_ξ to I and J, obviously entropic. □

Corollary 1. *The entropic polymatroids are selfadhesive.*

Proof. If $\xi=(\xi_i)_{i\in N}$ is a random vector and π a bijection between N and M with $\pi(i)=i$ for $i\in N\cap M$ then let the vector $\eta=(\eta_i)_{i\in N\cup M}$ consist of $\eta_i=\xi_i$ when $i\in N$ and $\eta_i=\xi_{\pi^{-1}(i)}$ when $i\in M\setminus N$. The assertion follows from Lemma 2 applied to the restrictions of h_η to N and M. □

For a polymatroid (N,g) and bijection $\pi\colon N\to M$ such that $\pi(i)=i$, $i\in N\cap M$, let

$$g^\pi(K)\triangleq g((K\cap N)\cup\pi^{-1}(K\cap M))\,,\qquad K\subseteq N\cup M\,.$$

It is easy to see that $(N\cup M,g^\pi)$ is a polymatroid; note that it can be constructed by parallel extensions, with elements $\pi(i)$ parallel to i, $i\in N\setminus M$ [6, p. 245]. Since the restriction of g^π to M is the π-copy of g the polymatroid g is selfadhesive at $I=N\cap M$ if and only if the restrictions of g^π to N and M adhere. Note that g^π itself is an adhesive extension of its restrictions to N and M if and only if $g(I)=g(N)$.

Lemma 3. *Each polymatroid (N,g) is selfadhesive at any $N\setminus i$, $i\in N$.*

Proof. Let (N,r) be the matroid with $r(J)$ equal to 1 or 0 according as $J \subseteq N$ contains i or not. This matroid is obviously selfadhesive. It is easy to see that (N,h) with h equal to $g - [\, g(N) - g(N \setminus i)\,]\, r$ is a polymatroid and $h(N) = h(N \setminus i) = g(N \setminus i)$. Then, a parallel extension h^π with $\pi(N \setminus i) = N \setminus i$ makes h and its π-copy to adhere. Since conical combinations of polymatroids that are selfadhesive at some set are obviously selfadhesive at the set it follows that $g = h + [\, g(N) - g(N \setminus i)\,]\, r$ is selfadhesive at $N \setminus i$. $\square$

A matroid with the ground set $N \cup M$ that extends two matroids (N,g) and (M,h) has been called in literature an *amalgam* of g and h, see [14, Section 12.4] including a number of references on p. 424. In particular, sometimes the *proper amalgam* of matroids can be constructed and provides an instance of adhesive extensions; this construction goes back to an unpublished result of A.W. Ingleton, presented in an unpublished review by J.H. Mason and later in [14]. An extension of the construction of proper amalgams to polymatroids is presented in Theorem 1, supplied by a self-contained proof.

Theorem 1. *If (N,g) and (M,h) are two polymatroids with coinciding restrictions to $I = N \cap M$ and the inequality*

$$\Delta_{J_1,J_2}\, g + \Delta_{K_1,K_2}\, h \geqslant \Delta_{I\cap J_1, I\cap J_2}\, g \tag{2}$$

holds for all flats J_1, J_2 of g and K_1, K_2 of h satisfying $I \cap J_1 = I \cap K_1$ and $I \cap J_2 = I \cap K_2$ then the function f^* given by

$$f^*(K) \triangleq \min \left\{ g(L\cap N) + h(L \cap M) - g(L \cap I) \colon K \subseteq L \subseteq N \cup M \right\}, \tag{3}$$

$K \subseteq N \cup M$, is an adhesive extension of g and h.

Recall that a subset J of N is a flat of a polymatroid (N,g) if $g(K) > g(J)$ for $K \supset J$. The closure $\overline{J}^g$ of J is the union of those $K \supseteq J$ that satisfy $g(K) = g(J)$. By the definition of polymatroids, $g(\overline{J}^g) = g(J)$ and $\overline{J}^g$ is a flat, the inclusion smallest one containing J.

Proof. Since the restrictions to I coincide the function f given by

$$f(L) \triangleq g(L \cap N) + h(L \cap M) - g(L \cap I)\,, \quad L \subseteq N \cup M\,,$$

where $g(L \cap I)$ can be replaced by $h(L \cap I)$, extends g and h. Then, $f^*(K) = f(L)$ for $K \subseteq M$ and $K \subseteq L \subseteq N \cup M$ implies

$$f(K) \geqslant f^*(K) = f(L) \geqslant h(L \cap M) \geqslant h(K) = f(K)\,.$$

Therefore, f^* extends h and, by symmetry, also g. It follows that

$$\Delta_{N,M}\, f^* = g(N) + h(M) - f(N \cup M) - g(I) = 0\,,$$

and thus N, M is a modular pair of f^*. Since $f^*(\emptyset) = 0$ and f^* is nondecreasing by construction it suffices to prove that the inequalities (2) imply the submodularity of f^*.

To this end, an auxiliary construction is needed. For $L \subseteq N \cup M$ let $\overline{L}^{gh}$ be the union over $t \geqslant 0$ of the sets L_t defined recursively by $L_0 = L$ and

$$L_{t+1} = \begin{cases} L_t \cup \overline{L_t \cap N}^{g}, & t \text{ even}, \\ L_t \cup \overline{L_t \cap M}^{h}, & t \text{ odd}. \end{cases}$$

By finiteness of the union, $\overline{L}^{gh} \cap N$ is a flat of g and $\overline{L}^{gh} \cap M$ is a flat of h. The inequality $f(\overline{L}^{gh}) \leqslant f(L)$ for $L \subseteq N \cup M$ follows by finite induction from

$$\begin{aligned} f(L_1) &= g(\overline{L \cap N}^g) + h((L \cup \overline{L \cap N}^g) \cap M) - h(\overline{L \cap N}^g \cap I) \\ &= g(L \cap N) + h((L \cap M) \cup (\overline{L \cap N}^g \cap I)) - h(\overline{L \cap N}^g \cap I) \\ &\leqslant g(L \cap N) + h(L \cap M) - h((L \cap M) \cap (\overline{L \cap N}^g \cap I)) = f(L_0)\,, \end{aligned}$$

using $g(\overline{J}^g) = g(J)$ and the submodularity of h, and from $f(L_2) \leqslant f(L_1)$, obtained by symmetry.

This observation implies that $f^*(K)$ with $K \subseteq N \cup M$ equals $f(L)$ for some $L \supseteq K$ such that $L \cap N$ and $L \cap M$ are flats. Then, for any subsets K', K'' of $N \cup M$, and the corresponding $L' \supseteq K'$ and $L'' \subseteq K''$,

$$\varDelta_{K',K''}\, f^* = f(L') + f(L'') - f^*(K' \cup K'') - f^*(K' \cap K'') \geqslant \varDelta_{L',L''}\, f\,.$$

Hence, the nonnegativity of $\varDelta_{K',K''}\, f^*$, and thus the submodularity of f^*, follows from the simple identity

$$\varDelta_{L',L''}\, f = \varDelta_{N\cap L', N\cap L''}\, g + \varDelta_{M\cap L', M\cap L''}\, h - \varDelta_{I\cap L', I\cap L''}\, g$$

and (2) with the flats $J_1 = N \cap L'$ and $J_2 = N \cap L''$ of g and the flats $K_1 = M \cap L'$ and $K_2 = M \cap L''$ of h. $\square$

Remark 1. The inequalities (2) are obviously satisfied if $g(I) = \sum_{i\in I} g(i)$ which makes the right-hand sides to vanish. In other words, if the polymatroids g, h restrict to a modular polymatroid on I then they adhere. In particular, polymatroids adhere at singletons and, trivially, at $\emptyset$ in which case the construction (3) reduces to that of the direct sum.

It follows from Remark 1 that a polymatroid with the ground set of cardinality at most three has all pairs of restrictions adhesive. By the following corollary, this remains valid also if the ground sets have cardinality at most four.

Corollary 2. *Any polymatroids (N, g) and (M, h) with $N = \{1, 2, 3\}$, $M = \{1, 2, 4\}$ and the coinciding restrictions to $I = \{1, 2\}$ adhere.*

As earlier in the introduction, the sign for union between singletons is sometimes omitted.

Proof. The assertion holds by Theorem 1 if $\varDelta_{13,23}\, g \geqslant \varDelta_{1,2}\, g$ or $\varDelta_{14,24}\, h \geqslant \varDelta_{1,2}\, g$, each of the inequalities implying that the assumptions of the theorem are satisfied. Otherwise, the number

$$\alpha_g \triangleq \varDelta_{1,2}\, g - \varDelta_{13,23}\, g = \varDelta_{1,3}\, g - \varDelta_{12,23}\, g = \varDelta_{2,3}\, g - \varDelta_{12,13}\, g$$

is positive. Where (N, r_N) is the matroid given by $r_N(J) = 1$ for the nonempty subsets J of N, it is easy to see that $g' \triangleq g - \alpha_g r_N$ is a polymatroid. Analogously, $h' \triangleq h - \alpha_h r_M$ is a polymatroid. By symmetry, suppose that $\alpha_g \leqslant \alpha_h$. Then the polymatroids g' and

$$h'' \triangleq h - \alpha_g r_M = h' + [\alpha_h - \alpha_g] r_M$$

are adhesive by Theorem 1 since $\varDelta_{13,23}\, g' = \varDelta_{13,23}\, g = \varDelta_{1,2}\, g'$. Obviously, r_N and r_M adhere, and therefore $g = g' + \alpha_g r_N$ and $h = h'' + \alpha_g r_M$ are adhesive as well. $\square$

Corollary 3. *A polymatroid (N, g) is selfadhesive at $I \subseteq N$ if*

$$2\, \varDelta_{J_1,J_2}\, g \geqslant \varDelta_{I\cap J_1, I\cap J_2}\, g \tag{4}$$

holds for all flats J_1 and J_2.

Proof. Let (M,h) be a π-copy of g with the overlap $I = N \cap M$. If J_1, J_2 are flats of g and K_1, K_2 flats of h such that $I \cap J_1 = I \cap K_1$ and $I \cap J_2 = I \cap K_2$ then $\pi^{-1}(K_1), \pi^{-1}(K_2)$ are flats of g satisfying $I \cap K_1 = I \cap \pi^{-1}(K_1)$ and $I \cap K_2 = I \cap \pi^{-1}(K_2)$. The inequality (4) and its instance with J_1, J_2 replaced by $\pi^{-1}(K_1), \pi^{-1}(K_2)$ imply

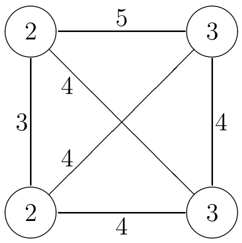
$$2\,\varDelta_{J_1,J_2}\, g + 2\,\varDelta_{\pi^{-1}(K_1),\pi^{-1}(K_2)}\, g \geqslant 2\,\varDelta_{I\cap J_1, I\cap J_2}\, g$$

which rewrites to (2). Hence, an adhesive extension of g and h exists by Theorem 1, whence g is selfadhesive at I. $\square$

By Corollary 3, any polymatroid is selfadhesive at $\emptyset$ and singletons, thus selfadhesive when its ground set has the cardinality at most two. This remains true also in the case of cardinality three due to Lemma 3. The case of cardinality four is nontrivial and is treated in Section 4. Note that Corollary 3 is in general not applicable even on the ground set $N = \{\mathit{1},\mathit{2},\mathit{3}\}$ since (4) is violated, for example, when $I = \mathit{23}$, g is the polymatroid given by $g(I) = g(N) = 3$, $g(\mathit{1}) = 1$ and $g(J) = 2$, $J \neq \emptyset$, otherwise, and the flats are $J_1 = \mathit{12}$ and $J_2 = \mathit{13}$.

The following final example not only illustrates Corollary 3 of Theorem 1 but exposes also one of the special polymatroids occurring in the proof of Lemma 7 in Section 4.

Example 1. Consider the polymatroid with the ground set $N = \{\mathit{1},\mathit{2},\mathit{3},\mathit{4}\}$, visually organized as $\substack{\mathit{1}\,\mathit{3}\\ \mathit{2}\,\mathit{4}}$, and the rank function g given by $g(I) = 5$ if I has at least three elements and by the labeled graph



where vertex labels denote values $g(i)$ for $i \in N$ and edge labels values $g(ij)$ for $i,j \in N$ different. For example, $g(\mathit{3}) = 3$ and $g(\mathit{13}) = 5$. This polymatroid is selfadhesive at $I = \mathit{34}$. In fact, it suffices to verify (4) for all flats J_1 and J_2 of g, by Corollary 3. This is obvious when $I \cap J_1$ and $I \cap J_2$ are in inclusion because the right-hand side of (4) vanishes. Otherwise, by symmetry, $I \cap J_1 = \mathit{3}$ and $I \cap J_2 = \mathit{4}$. Therefore, $\varDelta_{I\cap J_1, I\cap J_2}\, g = 2$, J_1 is $\mathit{3}$ or $\mathit{23}$ and J_2 is $\mathit{4}$ or $\mathit{14}$ or $\mathit{24}$. In all cases $\varDelta_{J_1,J_2}\, g \geqslant 2$, and (4) holds.

3. Cones of polymatroids with adhesive restrictions

This section is devoted to a systematic description of all inequalities satisfied by the polymatroids with adhesive restrictions. To formulate the main result, a number of preliminary notations and observations is necessary. As before, the sign for unions of subsets of N is omitted when convenient.

For $I \subseteq N$ let $\delta_I \in \mathbb{R}^{2^N}$ have the I-coordinate equal to 1 and the remaining coordinates equal to 0. Writing $\varDelta_{I,J} = \delta_I + \delta_J - \delta_{I\cup J} - \delta_{I\cap J}$, the expression $\varDelta_{I,J}\, g$ encountered previously is interpreted as a scalar product on $\mathbb{R}^{2^N}$.

Using the identity

$$\varDelta_{IL,JKL} = \varDelta_{IL,KL} + \varDelta_{IKL,JKL}\,, \qquad I,J,K,L \subseteq N \text{ disjoint.} \tag{5}$$

it is easy to show that a function g is submodular, thus $\Delta_{I,J}\, g \geqslant 0$ for $I, J \subseteq N$, if and only if this inequality holds for $I, J \subseteq N$ with $I \setminus J$ and $J \setminus I$ singletons. In these cases, for $K \subseteq N$ and $i, j \in N \setminus K$ different, the alternative notation $\Delta_{ij|K}$ for $\Delta_{iK,jK}$ will be frequently advantageous. Likewise, for $I, J, K \subseteq N$ disjoint, $\Delta_{I,J|K}$ stands for $\Delta_{IK,JK}$.

The notation $\Delta_{i|K} \triangleq \delta_{iK} - \delta_K$ for $K \subseteq N$ and $i \in N \setminus K$ is used to express that a function g is nondecreasing, by $\Delta_{i|K}\, g \geqslant 0$. If g is submodular it is nondecreasing if $\Delta_{i|N\setminus i}\, g \geqslant 0$ for $i \in N$ (using that $\Delta_{i|K} = \Delta_{i|jK} + \Delta_{ij|K}$ for $K \subseteq N$ and $i, j \in N \setminus K$ different).

Let $\mathcal{E}_N$ be the set of all pairs $(ij|K)$ with $i, j \in N$ and $K \subseteq N \setminus ij$ such that $i = j$ implies $K = N \setminus i$. With the above notations a function g is a polymatroid if and only if $\Delta_{ij|K}\, g \geqslant 0$ for $(ij|K) \in \mathcal{E}_N$ and $g(\emptyset) = 0$. The set of (rank functions of) polymatroids with the ground set N is a polyhedral cone [13, 6] in the Euclidean space $\mathbb{R}^{2^N}$, denoted here by $\boldsymbol{H}_N$.

For $I, J \subseteq N$ let $\mathcal{E}^{I,J}_{I\cup J}$ be the family of $(ij|K) \in \mathcal{E}_{I\cup J}$ with $i \in I \setminus J$, $j \in J \setminus I$ and $I \cap J \subseteq K \subseteq I \cup J$. If g is submodular then $\Delta_{I,J}\, g = 0$ is equivalent to $\Delta_{ij|K}\, g = 0$ for $(ij|K) \in \mathcal{E}^{I,J}_{I\cup J}$. Let $\boldsymbol{H}^{I,J}_{I\cup J}$ be the set of $f \in \boldsymbol{H}_{I\cup J}$ satisfying $\Delta_{I,J}\, f = 0$.

Let Π^I_N denote the coordinate projection of $\mathbb{R}^{2^N}$ to $\mathbb{R}^{2^I}$; it is easy to see that $\boldsymbol{H}_N$ projects onto $\boldsymbol{H}_I$. The coordinate projection of $\mathbb{R}^{2^N}$ to $\mathbb{R}^{2^I \cup 2^J}$ is denoted by $\Pi^{I,J}_N$.

Finally,

$$\Upsilon^{I,J}_{c,d} \triangleq \sum_{(ij|K)\in\mathcal{E}_{I\cup J}\setminus(\mathcal{E}_I\cup\mathcal{E}_J)} c_{ij|K}\, \Delta_{ij|K} - \sum_{(ij|K)\in\mathcal{E}^{I,J}_{I\cup J}} d_{ij|K}\, \Delta_{ij|K} \tag{6}$$

where c, d are two vectors with the real coordinates $c_{ij|K}$ and $d_{ij|K}$, correspondingly. If all the coordinates are nonnegative the pair (c, d) is termed nonnegative as well.

Theorem 2. *For a polymatroid (N, g) and $I, J \subseteq N$ the assertions (i)–(iii) are equivalent.*

(i) The restrictions of g to I, J adhere.

(ii) $\Pi^{I,J}_N$ maps g into the $\Pi^{I,J}_{I\cup J}$-image of $\boldsymbol{H}^{I,J}_{I\cup J}$.

(iii) $\Upsilon^{I,J}_{c,d}\, g \geqslant 0$ for all (c, d) nonnegative such that $\Upsilon^{I,J}_{c,d}\, \delta_L = 0$ whenever $L \subseteq I \cup J$, $L \not\subseteq I$ and $L \not\subseteq J$.

A proof is postponed to the end of this section.

Remark 2. By Theorem 2, the restrictions of g to I, J adhere if and only if the minimum of $\Upsilon^{I,J}_{c,d}\, g$ over nonnegative (c, d) satisfying the equalities of *(iii)* is zero, which is an optimization problem of the linear programming.

Remark 3. In Theorem 2 *(iii)*, the set of nonnegative pairs (c, d) that satisfy $\Upsilon_{c,d}\delta_L = 0$ for $L \in 2^{I\cup J} \setminus (2^I \cup 2^J)$ is a pointed cone. Obviously, $\Upsilon^{I,J}_{c,d}\, g \geqslant 0$ holds for (c, d) from this cone if and only if it holds for a finite set of generators of this cone. This observation shows how to find systematically all inequalities for polymatroids with adhesive restrictions, which is illustrated in the following example.

Example 2. For $N = \{1, 2, 3\}$, $I = \{1, 2\}$ and $J = \{2, 3\}$, the definition (6)

$$\begin{aligned}\Upsilon^{I,J}_{c,d} = {} & c_{12|3}\, \Delta_{12|3} + c_{13|\emptyset}\, \Delta_{13|\emptyset} + c_{13|2}\, \Delta_{13|2} + c_{23|1}\, \Delta_{23|1} \\ & + c_{1|23}\, \Delta_{1|23} + c_{2|13}\, \Delta_{2|13} + c_{3|12}\, \Delta_{3|12} - d_{13|2}\, \Delta_{13|2}\end{aligned}$$

involves the vector c with seven coordinates and the vector d with only one coordinate. By Theorem 2, the restrictions of a polymatroid g to I, J adhere if and only if $\Upsilon^{I,J}_{c,d}\, g \geqslant 0$ for the nonnegative pairs (c,d) that satisfy

$$\begin{aligned} \Upsilon^{I,J}_{c,d}\, \delta_{13} &= \quad c_{12|3} - c_{13|\emptyset} \qquad + c_{23|1} \qquad - c_{2|13} \qquad\qquad = 0 \\ \Upsilon^{I,J}_{c,d}\, \delta_{N} &= -c_{12|3} \qquad -c_{13|2} - c_{23|1} + c_{1|23} + c_{2|13} + c_{3|12} + d_{13|2} = 0\,, \end{aligned}$$

considering $L = 13$ and $L = 123$. The polyhedral cone in $\mathbb{R}^8$ defined by these two equations and nonnegativity of (c,d) has eleven extreme rays. Three of them are generated by the vectors with $d_{13|2}$ nonzero, namely

$$(0,0,1,0,0,0,0,1)\,, \quad (1,1,0,0,0,0,0,1) \quad \text{and} \quad (0,1,0,1,0,0,0,1)\,,$$

while eight of them have $d_{13|2} = 0$, in which cases $\Upsilon^{I,J}_{c,d}\, g \geqslant 0$ is trivially true for polymatroids. This and Remark 3 imply that the polymatroids with the adhesive restrictions to I, J are characterized by three inequalities $\Upsilon^{I,J}_{c,d}\, g \geqslant 0$ with (c,d) corresponding to the above three vectors. These inequalities mean the nonnegativity of the scalar products of

$$\varDelta_{13|2} - \varDelta_{13|2}\,, \quad \varDelta_{12|3} + \varDelta_{13|\emptyset} - \varDelta_{13|2} \quad \text{and} \quad \varDelta_{13|\emptyset} + \varDelta_{23|1} - \varDelta_{13|2}$$

with g. Since the above expressions equal 0, $\varDelta_{12|\emptyset}$ and $\varDelta_{23|\emptyset}$, all polymatroids on this small set N have adhesive restrictions to I, J. Note that this conclusion was made earlier as a consequence of Theorem 1.

The following assertion is actually an instance of the necessary conditions for adhesivity of restrictions from *(iii)* of Theorem 2.

Corollary 4. *If $N = \{1,\ldots,n,u,v,0\}$ has the cardinality $n+2+1$ at least 4 and a polymatroid (N,g) has adhesive restrictions to $I = \{1,\ldots,n,u,v\}$ and $J = \{u,v,0\}$ then the scalar product of*

$$\varDelta_{uv,0} + n\varDelta_{uv|0} - n\varDelta_{u,v} + \sum_{j=1}^{n} \varDelta_{uv|j} + \sum_{j=1}^{n-1} \varDelta_{\{1,\ldots,j\},\{j+1\}} \tag{7}$$

and g is nonnegative.

Note that for $n = 1$ the expression (7) rewrites to $\varDelta_{u0|v} + \varDelta_{v0|u} + \varDelta_{uv|1}$, and thus the assertion of Corollary 4 is trivial.

Remark 4. By Lemma 2, Corollary 4 applies to entropic polymatroids and gives rise to a linear inequality for Shannon entropies of subcollections of $n+3$ random variables. This information inequality appeared in [7, Theorem 1], and [19, Theorem 3], and is equivalent to [20, Theorem 5] when the random variables indexed by 0 and 1 coincide. This case with $n = 2$ is exactly Zhang-Yeung inequality (1), for its information theoretical interpretation see [19]. It has been argued [18] that new inequalities for entropic functions, in particular those of Corollary 4, are of importance for converse coding theorems and in various communication network models. For some details and applications see [9, Section 4].

Proof of Corollary 4. It is a matter of elementary calculations to verify that (7) equals

$$\begin{aligned} & n\big[\delta_{u0} - \delta_{u} + \delta_{v0} - \delta_{v}\big] + \sum_{j=1}^{n} \big[\delta_{uj} + \delta_{vj} - \delta_{uvj}\big] \\ & \quad -(n+1)\big[\delta_{uv0} - \delta_{uv}\big] - (n-1)\delta_{0} - \delta_{\{1,\ldots,n\}} \end{aligned} \tag{8}$$

which, in turn, equals

$$\Delta_{uv,0|\{1,\ldots,n\}} + \sum_{j=1}^{n} \left[\Delta_{uv|j0} + \Delta_{j0|u} + \Delta_{j0|v}\right] + \sum_{j=1}^{n-1} \Delta_{\{1,\ldots,j\},\{j+1\}|0} \qquad -\Delta_{\{1,\ldots,n\},0|uv} - \sum_{j=1}^{n} \Delta_{0j|uv}\,. \tag{9}$$

The expression (9) rewrites by means of (5) to the form of (6) having nonnegative (c,d). The scalar product of (7), or (8), with δ_L vanishes for $L \in 2^{I\cup J} \setminus (2^I \cup 2^J)$. It follows from Theorem 2 that the scalar product of (7) with a polymatroid g whose restrictions to I, J adhere is nonnegative. □

Corollary 5. *If the restrictions of a polymatroid on $N = \{1,2,u,v,0\}$ to $I = \{1,2,u,v\}$ and $J = \{u,v,0\}$ adhere then the scalar product of*

$$\Delta_{2v|1} + \Delta_{2v|u} + \Delta_{1,u} - \Delta_{2,v} + \Delta_{uv|0} + \Delta_{u0|v} + \Delta_{v0|u} \tag{10}$$

with the rank function of the polymatroid is nonnegative.

Proof. After a verification that (10) equals

$$\Delta_{10|v} + \Delta_{20|u} + \Delta_{20|v} + \Delta_{1u|0} + \Delta_{10|2} + \Delta_{2v|10} + \Delta_{uv|20} + \Delta_{u0|12v} + \Delta_{2v,0|1u} \qquad -\left[\Delta_{20|uv} + 2\Delta_{12,0|uv}\right]$$

an argumentation is similar to the previous proof. □

When applied to the entropic polymatroids, Corollary 5 provides a new linear information inequality, see Remark 4.

Remark 5. As a consequence of Theorem 2, the set of (rank functions of) polymatroids with the same ground set N for which all pairs of restrictions adhere is a polyhedral cone, denoted by $\boldsymbol{H}_N^{\mathsf{ar}}$. Note that only the pairs of restrictions to I, J with $I \cup J = N$ matter because adhesivity of the restrictions to I, J follows from adhesivity of the restrictions to $I, J \cup (N \setminus I)$ and (5).

Remark 6. The cone $\boldsymbol{H}_N^{\mathsf{ar}}$, contained in $\boldsymbol{H}_N$, is described by inequalities defining the $\Pi_{I\cup J}^{I,J}$-images of $\boldsymbol{H}_{I\cup J}^{I,J}$, collected over $I, J \subseteq N$. These inequalities can be also found algorithmically by Fourier-Motzkin elimination [21] that is implemented in freely available software packages like PORTA or POLYMAKE. Over the set $N = \{1,2,u,v,0\}$ of five elements, such computations with PORTA[†] witness that a polymatroid has adhesive restrictions to $I = \{1,2,u,v\}$ and $J = \{u,v,0\}$ if and only if

$$\Delta_{12,uv,0}\, g \geqslant 0\,,\;\; \Delta^*_{1,2,u,v,0}\, g \geqslant 0\,,\;\; \Delta^*_{2,1,u,v,0}\, g \geqslant 0\,,\;\; \Delta^*_{1,2,v,u,0}\, g \geqslant 0\,,\;\; \text{and}\;\; \Delta^*_{2,1,v,u,0}\, g \geqslant 0$$

where $\Delta_{12,uv,0}$ denotes the expression (7) with $n = 2$, thus

$$\Delta_{uv|1} + \Delta_{uv|2} + \Delta_{1,2} - \Delta_{u,v} + \Delta_{uv|0} + \Delta_{u0|v} + \Delta_{v0|u}\,, \tag{11}$$

and $\Delta^*_{1,2,u,v,0}$ the expression (10). Note the symmetries $1 \leftrightarrow 2$ and $u \leftrightarrow v$. Another experiments indicate that the cone $\boldsymbol{H}_N^{\mathsf{ar}}$ with this N is characterized by $\Delta_{ij,kl,m}\, g \geqslant 0$ and $\Delta^*_{i,j,k,l,m}\, g \geqslant 0$ for all choices of five different singletons i,j,k,l,m of N.

[†]http://www.zib.de/Optimization/Software/Porta; we worked with Version 1.3 written by T. Christof and revised by A. Loebel and M. Stoer.

Proof of Theorem 2. The adhesivity of the restrictions of g to I, J is equivalent to the existence of a polymatroid f in $\boldsymbol{H}^{I,J}_{I\cup J}$, satisfying $\Delta_{I,J}\, f = 0$, such that the restrictions of g and f to I and J coincide. The latter means that the projection $\Pi^{I,J}_N(g)$ belongs to $\Pi^{I,J}_{I\cup J}(\boldsymbol{H}^{I,J}_{I\cup J})$, proving the equivalence of *(i)* and *(ii)*.

The projection $\Pi^{I\cup J}_N(g)$ belongs to $\boldsymbol{H}^{I,J}_{I\cup J}$ if and only if $\Delta_{ij|K}\, g \geqslant 0$ for $(ij|K) \in \mathcal{E}_{I\cup J}$ and $-\Delta_{ij|K}\, g \geqslant 0$ for $(ij|K) \in \mathcal{E}^{I,J}_{I\cup J}$. Hence by Farkas lemma [15, Corollary 22.3.1], $\Pi^{I,J}_N = \Pi^{I,J}_{I\cup J}\Pi^{I\cup J}_N$ maps g into $\Pi^{I,J}_{I\cup J}(\boldsymbol{H}^{I,J}_{I\cup J})$ if and only if the inequality

$$\sum_{(ij|K)\in\mathcal{E}_{I\cup J}\setminus(\mathcal{E}_I\cup\mathcal{E}_J)} c_{ij|K}\, \Delta_{ij|K}\, g + \sum_{(ij|K)\in\mathcal{E}_I\cup\mathcal{E}_J} e_{ij|K}\, \Delta_{ij|K}\, g - \sum_{(ij|K)\in\mathcal{E}^{I,J}_{I\cup J}} d_{ij|K}\, \Delta_{ij|K}\, g \geqslant 0$$

holds with any nonnegative numbers $c_{ij|K}$, $e_{ij|K}$ and $d_{ij|K}$ satisfying the equality

$$\sum_{(ij|K)\in\mathcal{E}_{I\cup J}\setminus(\mathcal{E}_I\cup\mathcal{E}_J)} c_{ij|K}\, \Delta_{ij|K}\, \delta_L + \sum_{(ij|K)\in\mathcal{E}_I\cup\mathcal{E}_J} e_{ij|K}\, \Delta_{ij|K}\, \delta_L - \sum_{(ij|K)\in\mathcal{E}^{I,J}_{I\cup J}} d_{ij|K}\, \Delta_{ij|K}\, \delta_L = 0$$

for $L \in 2^{I\cup J} \setminus (2^I \cup 2^J)$. Here, each term $\Delta_{ij|K}\, \delta_L$ in the middle sum vanishes so that the equality is equivalent to $\Upsilon^{I,J}_{c,d}\, \delta_L = 0$ and does not depend on $e_{ij|K}$'s. The above inequality rewrites to

$$\Upsilon^{I,J}_{c,d}\, g + \sum_{(ij|K)\in\mathcal{E}_I\cup\mathcal{E}_J} e_{ij|K}\, \Delta_{ij|K}\, g \geqslant 0$$

where the sum is nonnegative because g is a polymatroid . Having no restrictions on $e_{ij|K}$ but the nonnegativity, it suffices to consider them equal to zero here. It follows that *(ii)* is equivalent to *(iii)*. □

4. Selfadhesive polymatroids over a four-element set

The results on polymatroids with adhesive restrictions presented in the previous section have natural counterparts dealing with selfadhesive polymatroids. For example, the (rank functions of the) polymatroids with a ground set N that are selfadhesive at some $I \subseteq N$ form a convex cone, denoted by $\boldsymbol{H}^{\mathsf{sa}}_{N,I}$. This cone obtains as the coordinate projection on $\mathbb{R}^{2^N}$ of the polyhedral cone of all $f \in \boldsymbol{H}^{N,M}_{N\cup M}$ satisfying $f(J) = f(\pi^{-1}(J))$, $J \subseteq M$, where π is a bijection between N and M and $N \cap M = I$. Let $\boldsymbol{H}^{\mathsf{sa}}_N$ denote the cone $\bigcap_{I\subseteq N} \boldsymbol{H}^{\mathsf{sa}}_{N,I}$ of all selfadhesive polymatroids with the ground set N.

This section presents a description of the cones $\boldsymbol{H}^{\mathsf{sa}}_{N,I}$ and $\boldsymbol{H}^{\mathsf{sa}}_N$ when the cardinality of N is four, thus $N = ijkl$ for different singletons i, j, k, l. Lemma 3 and Corollary 3 of Section 2 imply $\boldsymbol{H}^{\mathsf{sa}}_{N,I} = \boldsymbol{H}_N$ for all subsets I of N of the cardinality different from two. In a remaining case $I = kl$, a polymatroid $g \in \boldsymbol{H}^{\mathsf{sa}}_{N,kl}$ and its π-copy with $M = kl\hat{\imath}\hat{\jmath}$, $\pi(i) = \hat{\imath}$ and $\pi(j) = \hat{\jmath}$ have an adhesive extension f. Then, the restrictions of f to $ijkl$ and $kl\hat{\imath}$ adhere. Corollary 4 with $n = 2$ implies that $\Delta_{ij,kl,\hat{\imath}}\, f \geqslant 0$, see also (11). Since $f(J) = g(\pi^{-1}(J))$ for $J \subseteq kl\hat{\imath}$ it follows that $\nabla_{i,j}\, g \geqslant 0$ where

$$\nabla_{i,j} \triangleq \big[\Delta_{kl|i} + \Delta_{kl|j} + \Delta_{ij} - \Delta_{kl}\big] + \Delta_{ik|l} + \Delta_{il|k} + \Delta_{kl|i}\ . \tag{12}$$

Here, $\nabla_{i,j}\, g$ rewrites immediately to the left-hand side of Zhang-Yeung inequality (1).

Theorem 3. *A polymatroid $(ijkl, g)$, is selfadhesive at kl if and only if the scalar products of $\nabla_{i,j}$, $\nabla_{j,i}$, $\nabla_{k,i}$, $\nabla_{k,j}$, $\nabla_{l,i}$ and $\nabla_{l,j}$ with g are nonnegative.*

Since the transposition $k \leftrightarrow l$ does not influence (1), or (12), Zhang-Yeung inequality admits twelve instances when $N = ijkl$. Let $\boldsymbol{H}_N^{\triangledown}$ denote the subcone of $\boldsymbol{H}_N$ defined by means of these inequalities.

Corollary 6. *A polymatroid with the ground set of cardinality four is selfadhesive if and only if it satisfies all Zhang-Yeung inequalities, in symbols $\boldsymbol{H}_N^{\mathsf{sa}} = \boldsymbol{H}_N^{\triangledown}$.*

The rest of this section is occupied by a proof of Theorem 3. In one direction, assuming the selfadhesivity of g at kl, Corollary 4 with $n = 2$ was used above to show $\triangledown_{i,j}\, g \geqslant 0$. By symmetry, $\triangledown_{j,i}\, g \geqslant 0$. Similarly, Corollary 5 implies nonnegativity of the remaining four scalar products. For the opposite direction, the cone of polymatroids that render the six scalar products nonnegative is analyzed. It suffices to prove that each polymatroid of a generating set of the cone is selfadhesive at kl. Relying on some results of [11], the extreme rays of the cone can be explicitly described. Some polymatroids of a generating set of the cone satisfy Ingleton inequalities whence they are selfadhesive by Corollary 7. Remaining polymatroids of the generating set are not numerous, and are examined in Lemmas 5 and 6. Their selfadhesivity at kl is verified case-by-case, see the proof of Lemma 7.

First, some well-known facts on the cone $\boldsymbol{H}_N$ of the dimension 15 are recalled; for references see e.g. [8]. By [13, Proposition 10.4.12], a matroid generates an extreme ray of $\boldsymbol{H}_N$ if and only if its restriction to the set of its nonloops is connected; an element $i \in N$ is a loop of a polymatroid (N, g) if $g(i) = 0$, and a polymatroid is connected if $g(N) = g(I) + g(N \setminus I)$ holds only for I or $N \setminus I$ empty. In particular, extreme rays of $\boldsymbol{H}_N$ are generated by the matroids (N, r_t^I), $I \subset N$ and $1 \leqslant t \leqslant |N \setminus I|$, given by

$$r_t^I(J) = \min\{t, |J \setminus I|\}\,, \qquad J \subseteq N\,.$$

Here, I is the set of loops and t is the rank; in $r_t^{\emptyset}$ the superindex is omitted. The cone $\boldsymbol{H}_N$ with a four-element set N has 41 extreme rays [11, p. 272], among them six are generated by the polymatroids

$$f_{ij}(K) = \begin{cases} 3\,, & K \in \{ik, jk, il, jl, kl\}, \\ \min\{4, 2|K|\}\,, & \text{otherwise}, \end{cases}$$

where $i, j \in N$ are different and $kl = N \setminus ij$.

Second, a crucial role in the proof of Theorem 3 is played by the expression

$$\square_{ij} \triangleq \Delta_{kl|i} + \Delta_{kl|j} + \Delta_{ij} - \Delta_{kl}$$

that occurred within the brackets of (12), and earlier in (11) and (10). The nonnegativity of $\square_{ij}\, g$, thus *Ingleton inequality* [4, 5]

$$g(ik) + g(jk) + g(il) + g(jl) + g(kl) - g(ij) - g(k) - g(l) - g(ikl) - g(jkl) \geqslant 0\,,$$

is related to linearity of matroids; proofs are in [5, p. 159] and [14, p. 177].

Let $\boldsymbol{H}_N^{\square}$ denote the subcone of $\boldsymbol{H}_N$ defined by the six Ingleton inequalities $\square_{ij}\, g \geqslant 0$, $i, j \in N$ different. By (12), $\boldsymbol{H}_N^{\square} \subseteq \boldsymbol{H}_N^{\triangledown}$.

Lemma 4. *The cone $\boldsymbol{H}_N^{\square}$ is generated by entropic polymatroids.*

Proof. By [11, Lemma 4.1], the extreme rays of $\boldsymbol{H}_N^{\square}$ coincide with the extreme rays of $\boldsymbol{H}_N$ different from those generated by the functions f_{ij}, $i, j \in N$ different. Each of the rays contains an entropic polymatroid, constructed explicitly e.g. in [11, Example, p. 275]. $\square$

It follows by Corollary 1 that each polymatroid satisfying all Ingleton inequalities is selfadhesive.

Corollary 7. $\boldsymbol{H}_N^{\square} \subseteq \boldsymbol{H}_N^{\mathsf{sa}}$

Knowing that $\boldsymbol{H}_N^{\mathsf{sa}}$ and $\boldsymbol{H}_N^{\triangledown}$ contain $\boldsymbol{H}_N^{\square}$, it suffices to examine situations when an Ingleton inequality is violated. To this end, the cones

$$\boldsymbol{H}_N^{\square}(ij) \triangleq \{g \in \boldsymbol{H}_N \colon \ \square_{ij}\, g \leqslant 0\} \quad \text{and} \quad \boldsymbol{H}_N^{\triangledown}(i,j) \triangleq \{g \in \boldsymbol{H}_N^{\square}(ij) \colon \triangledown_{i,j}\, g \geqslant 0\}$$

are considered. By [11, Lemma 6.1], the extreme rays of $\boldsymbol{H}_N^{\square}(ij)$ are generated by the 15 linearly independent polymatroids listed in the first and third lines of

$$\begin{array}{cccccccccc} & f_{ij} & r_1^{ijk} & r_1^{ijl} & r_1^{ikl} & r_1^{jkl} & r_1 & r_3 & r_1^i & r_1^j \\ & -\square_{ij} & \varDelta_{l|ijk} & \varDelta_{k|ijl} & \varDelta_{j|ikl} & \varDelta_{i|jkl} & \varDelta_{ij|\emptyset} & \varDelta_{kl|ij} & \varDelta_{kl|i} & \varDelta_{kl|j} \\ (13) & & r_1^{ik} & r_1^{jk} & r_1^{il} & r_1^{jl} & r_2^k & r_2^l & & \\ & & \varDelta_{jl|k} & \varDelta_{il|k} & \varDelta_{jk|l} & \varDelta_{ik|l} & \varDelta_{ij|l} & \varDelta_{ij|k} & & \end{array}$$

where $kl = N \setminus ij$. These polymatroids span the linear subspace of $\mathbb{R}^{2^N}$ given by $g(\emptyset) = 0$ and their coordinate functionals are listed in the second and fourth lines of (13), see the proof of [11, Lemma 6.1]. For example, $-\square_{ij}\, f_{ij} = 1$, $-\square_{ij}\, r_1^{ijk} = 0$, etc.

Lemma 5. *The set of extreme rays of $\boldsymbol{H}_N^{\triangledown}(i,j)$ consists of the three rays generated by*

$$f_{ij} + r_1^i \qquad f_{ij} + r_1^{jk} \qquad f_{ij} + r_1^{jl} \tag{14}$$

and of the extreme rays of $\boldsymbol{H}_N^{\square}(ij)$ but the first one, generated by f_{ij}.

Proof. Knowing the structure of $\boldsymbol{H}_N^{\square}(ij)$, the polymatroids of $\boldsymbol{H}_N^{\triangledown}(i,j)$ are nonnegative combinations of the independent polymatroids of (13). Expressed in coefficients of the combinations, $\boldsymbol{H}_N^{\triangledown}(i,j)$ turns out to be the cone in $\mathbb{R}^{15}$ given by the nonnegativity of $x_1, \ldots, x_{15}$ and $-x_1 + x_8 + x_{11} + x_{13} \geqslant 0$ where the inequality results from $\triangledown_{i,j}\, g \geqslant 0$ by (12). The extreme rays of this cone are easy to find, and recast to the assertion. $\square$

Lemma 6. *The set of extreme rays of the cone $\boldsymbol{H}_N^{\triangledown}(i,j) \cap \boldsymbol{H}_N^{\triangledown}(j,i)$ consists the nine rays generated by*

$$\begin{array}{ccc} f_{ij} + r_1^i + r_1^j & f_{ij} + r_1^{jk} + r_1^j & f_{ij} + r_1^{jl} + r_1^j \\ f_{ij} + r_1^i + r_1^{ik} & f_{ij} + r_1^{jk} + r_1^{ik} & f_{ij} + r_1^{jl} + r_1^{ik} \\ f_{ij} + r_1^i + r_1^{il} & f_{ij} + r_1^{jk} + r_1^{il} & f_{ij} + r_1^{jl} + r_1^{il} \end{array} \tag{15}$$

and of the extreme rays of $\boldsymbol{H}_N^{\square}(ij)$ but the first one, generated by f_{ij}.

Proof. An argumentation is analogous to the previous proof. The cone given by the nonnegativity of $x_1, \ldots, x_{15}$, $-x_1 + x_8 + x_{11} + x_{13}$ and $-x_1 + x_9 + x_{10} + x_{12}$ arises. $\square$

Remark 7. By [11, Lemma 6.2], any two of the cones $\boldsymbol{H}_N^{\square}(ij)$ intersect in a subset of $\boldsymbol{H}_N^{\square}$ and $\boldsymbol{H}_N$ is disjoint union of $\boldsymbol{H}_N^{\square}$ and the six cones $\{g \in \boldsymbol{H}_N \colon \ \square_{ij}\, g < 0\}$. It follows that intersection of $\boldsymbol{H}_N$ with the halfspace given by $\triangledown_{i,j}\, g \geqslant 0$ is union of $\boldsymbol{H}_N^{\square}$, $\boldsymbol{H}_N^{\square}(ik)$, $\boldsymbol{H}_N^{\square}(il)$, $\boldsymbol{H}_N^{\square}(jk)$, $\boldsymbol{H}_N^{\square}(jl)$, $\boldsymbol{H}_N^{\square}(kl)$ and $\boldsymbol{H}_N^{\triangledown}(i,j)$. Knowing the extreme rays of all these seven cones, the intersection is generated by the three polymatroids (14) and the 40 extreme rays of $\boldsymbol{H}_N$ different from the one generated by f_{ij}. It is also not difficult to see that the intersection has exactly 43 extreme rays. This reasoning works also when intersecting $\boldsymbol{H}_N$ with any set of the halfspaces given by inequalities $\nabla_{i,j}\, g \geqslant 0$, like in Theorem 3. For example, $\boldsymbol{H}_N^{\nabla}$ has $35 + 9 \cdot 6 = 89$ extreme rays, namely the extreme rays of $\boldsymbol{H}_N^{\square}$ and the rays generated by the functions obtained from (15) through permutations.

Lemma 7. *The polymatroids* $f_{ik} + r_1^k$, $f_{ik} + r_1^{ij}$, $f_{ik} + r_1^{il}$ *and the polymatroids* (15) *are selfadhesive at* kl.

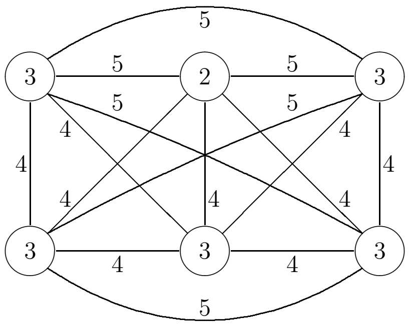
Proof. By symmetries, it suffices to prove that each of the seven polymatroids

$$\begin{array}{cccc} f_{13} + r_1^{3} & f_{13} + r_1^{12} & f_{13} + r_1^{14} & \\ f_{12} + r_1^{1} + r_1^{2} & f_{12} + r_1^{1} + r_1^{13} & f_{12} + r_1^{13} + r_1^{23} & f_{12} + r_1^{13} + r_1^{24} \end{array}$$

with the ground set $N = 1234$ is selfadhesive at 34. For the second one, $f_{13} + r_1^{12}$, this was proved in Example 1. For each of the remaining six polymatroids and π-copies with $M = 3456$ and $\pi(1) = 5$, $\pi(2) = 6$ an adhesive extension is constructed below. The union $N \cup M$ is organized visually as $\begin{smallmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{smallmatrix}$ and each extension f on $N \cup M$ is given by a labeled complete graph that specifies the values of f on singletons and two-element sets, similarly to Example 1, and by a table specifying the remaining values of f.

In each of the six cases below, it is necessary to verify that the extension f is a polymatroid, thus $\Delta_{ij|K}\, f \geqslant 0$ holds for $(ij|K) \in \mathcal{E}_{N \cup M}$, and that the extension is adhesive. The latter is obvious from the tables checked against the edge label for $f(34)$. To verify the inequalities observe first that in all cases $f(I) = f(N \cup M)$ for I that has the cardinality at least four and is different from N and M. This implies that $\Delta_{i|(N\cup M)\setminus i}\, f$ equals zero. Henceforth, only different i, j are considered. For K of the cardinality four $\Delta_{ij|K} f$ vanishes due to the above observation unless K equals N or M in which cases it equals one, by checking the tables. For K of the cardinality three the observation implies that $f(ijK)$ equals $f(iK)$ or $f(jK)$, because $\{iK, jK\}$ is different from $\{N, M\}$, and thus $\Delta_{ij|K}\, f \geqslant 0$ follows from $f(iK) \geqslant f(K)$ for iK different from N and M, which is obvious from the tables. The inequalities $\Delta_{ij|\emptyset}\, f \geqslant 0$ are equivalent to the assertion that every edge label does not exceed sum of the labels of its adjacent vertices, which verifies visually in graphs. The proof is completed by verifying $\Delta_{ij|K}\, f \geqslant 0$ for K of cardinality one or two which is done below case-by-case.

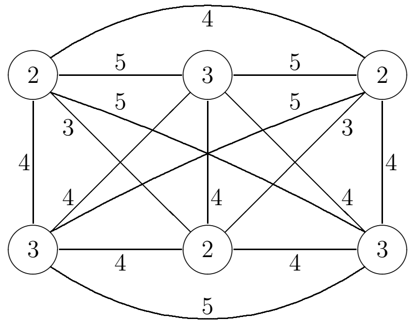
The extension of $f_{13} + r_1^{3}$.

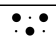
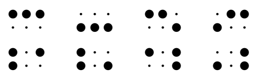


$\lvert I\rvert$	$f(I)$	I
3	5	$I \subseteq N$ or $I \subseteq M$ or $4 \in I$
	6	otherwise
4,5,6	5	$I = N$ or $I = M$
	6	otherwise

The inequality $\Delta_{ij|k}\, f \geqslant 0$ obviously holds if $f(ik) + f(jk) \geqslant 9$ or if $f(k) = 2$. Otherwise, ik and jk are edges labeled by 4 and k is a vertex labeled by 3. This implies $f(ijk) = 5$ and, in turn, $\Delta_{ij|k}\, f = 0$. To prove nontrivial cases of $\Delta_{ij|kl}\, f \geqslant 0$, it suffices to consider $f(kl) = 5$ and $ijkl$ different from N and M. Then, at least one of the numbers $f(ikl)$ and $f(jkl)$ equals 6, and hence $f(ikl) + f(jkl) \geqslant 11 \geqslant f(ijkl) + f(kl)$.

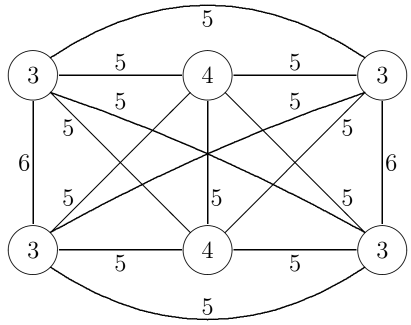
The extension of $f_{13} + r_1^{14}$.



$\lvert I\rvert$	$f(I)$	I
3	4	
	5	otherwise
	6	
4,5,6	5	$I = N$ or $I = M$
	6	otherwise

To prove $\Delta_{ij|k}\, f \geqslant 0$, consider first $f(ik) = 3$. This implies $f(k) = 2$. If $f(jk) = 5$ then obviously $f(ik) + f(jk) = 8$ and $8 \geqslant f(ijk) + f(k)$. If $f(jk) \leqslant 4$ then $\Delta_{ij|k}\, f \geqslant 0$ follows from an inspection of the graph and table. Otherwise, $f(ik)$ and $f(jk)$ are greater than three and $\Delta_{ij|k}\, f \geqslant 0$ provided $f(k) = 2$. If $f(k) = 3$ then $f(ik) + f(jk) \geqslant 9$ unless ijk is a subset of N or a subset of M or $ij = 26$ and $k = 4$. In all cases, $\Delta_{ij|k}\, f \geqslant 0$. To verify $\Delta_{ij|kl}\, f \geqslant 0$, observe $f(ikl) + f(jkl) \geqslant 9$. If $f(kl) = 5$ then $f(ikl) + f(jkl) \geqslant 11$ unless $ijkl$ coincides with N or M. If $f(kl) = 4$ then $f(ikl) + f(jkl) \geqslant 10 \geqslant f(ijkl) + f(kl)$.

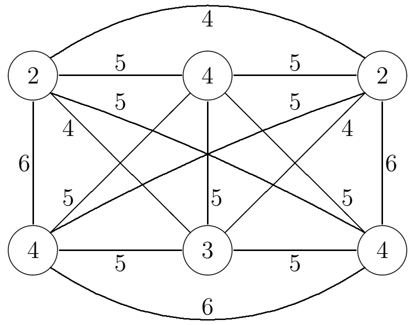
The extension of $f_{12} + r_1^{1} + r_1^{2}$.

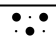
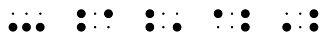


$\lvert I\rvert$	$f(I)$	I
3	6	otherwise
	7	
4,5,6	6	$I = N$ or $I = M$
	7	otherwise

To verify $\Delta_{ij|k}\, f \geqslant 0$, observe $f(ik) + f(jk) \geqslant 10$. However, $10 \geqslant f(ijk) + f(k)$ if $f(ijk) = 6$ or if $f(ijk) = 7$ since in this case $f(i) = 3$. To verify $\Delta_{ij|kl}\, f \geqslant 0$, observe $f(ikl) + f(jkl) \geqslant 12$. However, $12 \geqslant f(ijkl) + f(kl)$ holds when $f(kl) = 5$. Otherwise, if $kl = 12$ or $kl = 56$, the equality $f(ikl) + f(jkl) = 12$ implies $f(ijkl) = 6$.

The extension of $f_{12} + r_1^{1} + r_1^{13}$.

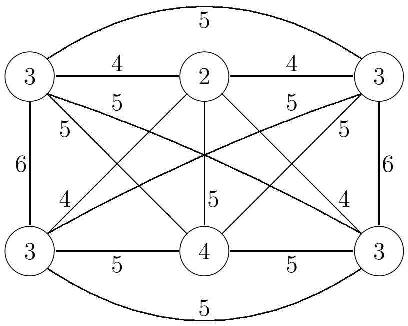


$\lvert I\rvert$	$f(I)$	I
3	5	
	6	otherwise
	7	
4,5,6	6	$I = N$ or $I = M$
	7	otherwise

The nonnegativity of $\Delta_{ij|k}\, f$ is obvious if $f(k) = 2$ or $f(ik) = f(jk) = 4$. Otherwise, if $f(ik) + f(jk) = 9$ then $k = 4$, and thus $f(k) = 3$ and $f(ijk) = 6$. It suffices to verify the cases with $f(ijk) = 7$ since otherwise $f(ik) + f(jk) \geqslant 10$ and $10 \geqslant f(ijk) + f(k)$. To prove $\Delta_{ij|kl}\, f \geqslant 0$, if $f(ikl)$ or $f(jkl)$ equals 5 then $f(ikl) + f(jkl) \geqslant 11$ and $f(kl) = 4$.

Hence, $11 \geqslant f(ijkl) + f(kl)$. Otherwise, $f(ikl) + f(jkl) \geqslant 12$ and $\Delta_{ij|kl}\, f \geqslant 0$ holds when $f(kl) < 6$. In the case $f(kl) = 6$ it follows $f(ikl) + f(jkl) \geqslant 13$ and obviously $13 \geqslant f(ijkl) + f(kl)$ up to the trivial situation when $ijkl$ equals N or M.

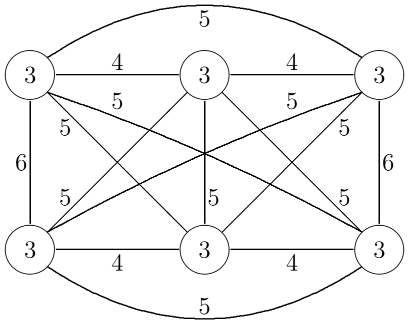
The extension of $f_{12} + r_1^{13} + r_1^{23}$.



$\lvert I\rvert$	$f(I)$	I
3	6	otherwise
	7	$\begin{smallmatrix}\bullet&\cdot&\bullet\\ \bullet&\cdot&\cdot\end{smallmatrix}$ $\begin{smallmatrix}\bullet&\cdot&\cdot\\ \bullet&\cdot&\bullet\end{smallmatrix}$ $\begin{smallmatrix}\bullet&\cdot&\bullet\\ \cdot&\cdot&\bullet\end{smallmatrix}$ $\begin{smallmatrix}\cdot&\cdot&\bullet\\ \bullet&\cdot&\bullet\end{smallmatrix}$
4,5,6	6	$I = N$ or $I = M$
	7	otherwise

The nonnegativity of $\Delta_{ij|k} f$ is verified first for $f(ijk) = 7$, then for $f(ik) = f(jk) = 4$ (i.e. $k = 3$), and then for $f(k) = 4$ (i.e. $k = 4$); otherwise $f(ik) + f(jk) \geqslant 9 \geqslant f(ijk) + f(k)$. The nonnegativity of $\Delta_{ij|kl} f$ is obvious when $f(kl) \leqslant 5$, if $f(kl) = 6$ then $f(ikl)+f(jkl) \geqslant 13$ and $13 \geqslant f(ijkl) + f(kl)$ unless $ijkl$ equals N or M.

The extension of $f_{12} + r_1^{13} + r_1^{24}$.



$\lvert I\rvert$	$f(I)$	I
3	5	$\begin{smallmatrix}\bullet&\bullet&\bullet\\ \cdot&\cdot&\cdot\end{smallmatrix}$ $\begin{smallmatrix}\cdot&\cdot&\cdot\\ \bullet&\bullet&\bullet\end{smallmatrix}$
	6	otherwise
	7	$\begin{smallmatrix}\bullet&\cdot&\bullet\\ \cdot&\bullet&\cdot\end{smallmatrix}$ $\begin{smallmatrix}\cdot&\bullet&\cdot\\ \bullet&\cdot&\bullet\end{smallmatrix}$ $\begin{smallmatrix}\bullet&\cdot&\bullet\\ \bullet&\cdot&\cdot\end{smallmatrix}$ $\begin{smallmatrix}\bullet&\cdot&\cdot\\ \bullet&\cdot&\bullet\end{smallmatrix}$ $\begin{smallmatrix}\bullet&\cdot&\bullet\\ \cdot&\cdot&\bullet\end{smallmatrix}$ $\begin{smallmatrix}\cdot&\cdot&\bullet\\ \bullet&\cdot&\bullet\end{smallmatrix}$
4,5,6	6	$I = N$ or $I = M$
	7	otherwise

The inequality $\Delta_{ij|k}\, f \geqslant 0$ holds if $f(ik) + f(jk) \geqslant 10$ or if both $f(ik)$ and $f(jk)$ are equal to 4 which implies $f(ijk) = 5$. Otherwise, $f(ik)+f(jk) = 9$ and, using symmetries, $ik = 13$. In these cases, $9 \geqslant f(ijk)+f(k)$. To prove $\Delta_{ij|kl} f \geqslant 0$, observe $f(ikl)+f(jkl) \geqslant 11$, and thus the case $f(kl) = 4$ is straightforward. If $f(kl) = 5$ then by symmetries $kl \in \{14, 15, 16\}$, and thus $f(ikl) + f(jkl) \geqslant 12$. The same inequality holds if $f(kl) = 6$, with equality for $ijkl$ equal to N or M. $\square$

Proof of Theorem 3. If a polymatroid $(ijkl, g)$ is selfadhesive at kl then $\nabla_{i,j}\, g \geqslant 0$ follows from the argument given prior to the formulation of Theorem 3. By symmetry, $\nabla_{j,i}\, g \geqslant 0$. To prove $\nabla_{k,i}\, g \geqslant 0$, consider an adhesive extension f of g and its π-copy with $M = kl\hat{\imath}\hat{\jmath}$, $\pi(i) = \hat{\imath}$ and $\pi(j) = \hat{\jmath}$. Since the restrictions of f to $ijkl$ and $kl\hat{\jmath}$ are adhesive Corollary 5 implies

$$\Delta^*_{i,j,k,l,\hat{\jmath}} f = \;\square_{ik}\, f + \Delta_{\hat{\jmath}k|l}\, f + \Delta_{\hat{\jmath}l|k}\, f + \Delta_{kl|\hat{\jmath}}\, f \geqslant 0\ .$$

This rewrites to $\nabla_{k,i}\, g \geqslant 0$ since f extends g and its π-copy. By symmetries, the scalar products of $\nabla_{k,j}$, $\nabla_{l,i}$, $\nabla_{l,j}$ with g are nonnegative.

If a polymatroid g makes the six scalar products nonnegative then it belongs to the union of $\boldsymbol{H}_N^{\square}$, $\boldsymbol{H}_N^{\square}(kl)$, $\boldsymbol{H}_N^{\nabla}(k,i)$, $\boldsymbol{H}_N^{\nabla}(k,j)$, $\boldsymbol{H}_N^{\nabla}(l,i)$, $\boldsymbol{H}_N^{\nabla}(l,j)$ and $\boldsymbol{H}_N^{\nabla}(i,j) \cap \boldsymbol{H}_N^{\nabla}(j,i)$ due to Remark 7. By Corollary 7, any $g \in \boldsymbol{H}_N^{\square}$ is selfadhesive. If $g \in \boldsymbol{H}_N^{\square}(kl)$ then g is a combination of a polymatroid from $\boldsymbol{H}_N^{\square}$ and f_{kl}. The polymatroid f_{kl} is selfadhesive at kl, using parallel extensions or Corollary 3, and therefore g is adhesive at kl by Corollary 7. If $g \in \boldsymbol{H}_N^{\nabla}(k,i)$ then g is a combination of a polymatroid from $\boldsymbol{H}_N^{\square}$ and $f_{ik} + r_1^k$, $f_{ik} + r_1^{ij}$, $f_{ik} + r_1^{jl}$ by Lemma 5. Then g is selfadhesive at kl because these polymatroids are self-adhesive at kl by Corollary 7 and Lemma 7. With regard to symmetries, it remains to consider $g \in \boldsymbol{H}_N^{\nabla}(i,j) \cap \boldsymbol{H}_N^{\nabla}(j,i)$ in which case the selfadhesivity of g at kl follows from Corollary 7, Lemma 6 and Lemma 7. □

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