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Combining m-dependence with Markovness

František Matúš

Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
and
Laboratory of Intelligent Systems
University of Economics, Prague

ABSTRACT.[1] — Generally, no stationary sequence of random variables which is Markov of order n but not of order $n-1$ and m-dependent but not $(m-1)$-dependent exists if the state space of the sequence has small cardinality. We show that to ensure the existence for the Markov sequences of order $n=1$ the number of attainable states must be at least $m+2$ and that this bound is tight. Given a small state space such a sequence exists only for special n and m. On a two-element state space the smallest possible n and m are shown to be 3 and 2, respectively. This results from our parametric description of all binary m-dependent sequences, $m \geq 0$, that are Markov of order 3.[2]

RÉSUMÉ.[3] — Si l'espace d'états n'est pas suffisamment riche on ne peut pas construire, pour n et m quelconques, une suite aléatoire stationnaire de Markov d'ordre n et pas d'ordre $n-1$ qui est dans le même temps m-dépendante et pas $(m-1)$-dépendante. Nous montrons que pour les chaînes de Markov, $n=1$, l'espace d'états doit avoir au moins $m+1$ éléments et que ce nombre ne peut pas être amélioré. Pour les suites binaires les plus petits n et m admissibles sont 3 et 2, respectivement. C'est une conséquence de notre description paramétrique de toutes les suites binaires stationnaires m-dépendantes, $m \geq 0$, de Markov d'ordre 3.

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1. INTRODUCTION

Let $\xi = (\xi_i;\ i \geqslant 1)$ be a strictly stationary sequence of random variables taking values in a finite state space S; speaking about a sequence we will always assume these properties. The sequence ξ is *Markov of order* $n \geqslant 0$ if $(\xi_i;\ 1 \leqslant i \leqslant k)$ is conditionally independent of $(\xi_i;\ i \geqslant k+n+1)$ given $(\xi_i;\ k+1 \leqslant i \leqslant k+n)$ for all $k \geqslant 1$. The sequence ξ is *dependent of order* $m \geqslant 0$ if $(\xi_i;\ 1 \leqslant i \leqslant k)$ is unconditionally independent of $(\xi_i;\ i \geqslant k+m+1)$, $k \geqslant 1$. For simplicity, we shorten the expressions "Markov of order n" and "dependent of order m" to n-Markov and m-dependent, correspondingly.

The aim of this note is to examine how these two properties interfere under restrictions on the cardinality of the state space. A more precise formulation will use the following notion of an index of a sequence. Let n_ξ be the smallest nonnegative integer n such that a sequence ξ is n-Markov, let m_ξ be the smallest $m \geqslant 0$ such that ξ is m-dependent and let d_ξ be the cardinality of the set of states which are attained with positive probabilities. Thus, we have $n_\xi \geqslant 0$, $m_\xi \geqslant 0$ and $d_\xi \geqslant 1$ with $n_\xi = 0$ if and only if $m_\xi = 0$. This expresses the sequence ξ is i.i.d. If ξ is Markov of no order $n \geqslant 0$ it is reasonable to write $n_\xi = \infty$ and similarly with the dependence and m_ξ; we shall, however, not deal with these cases at all. The triple $\langle n_\xi, m_\xi, d_\xi \rangle$ will be called *index* of ξ.

A natural question asks which triple can be equal to the index of a sequence ξ. In other words, given a triple of integers $\langle n, m, d \rangle$ does there exist a (stationary) sequence ξ such that $n_\xi = n$, $m_\xi = m$ and $d_\xi = d$, i.e. in the nontrivial case $n > 0$ and $m > 0$, such that it is n-Markov and not $(n-1)$-Markov, m-dependent and not $(m-1)$-dependent and takes exactly d states with positive probabilities?

We present answers only if $n = 1$ and partially if $d = 2$ here. In the second section devoted to the usual Markov chains (1-Markovness) we prove that a triple $\langle 1, m, d \rangle$ is the index of a sequence if and only if $1 \leqslant m \leqslant d-2$. Then we turn our attention entirely to the binary sequences, $S = \{0,1\}$, and during a technical preparation in the third section we reveal that every $(n, m, 2)$-sequence, this is an abbreviation for n-Markov, m-dependent and two-element state space, is i.i.d. provided $n \leqslant 2$ or $m \leqslant 1$. For some years I conjectured this be valid for any n and m nonnegative. It is, however, not the case. We will see in Section 4 that all $(3, m, 2)$-sequences are 3-dependent and therefore the only two candidates for indices of 3-Markov binary sequences are $\langle 3, 2, 2 \rangle$ and $\langle 3, 3, 2 \rangle$. Both these triples are really indices and, moreover, we provide a complete characterization of the distributions of all $(3, 2, 2)$-sequences in the fifth section and all $(3, 3, 2)$-sequences in the sixth section, respectively.

Though Markov chains is an old topic, Markov chains with 1-dependence appeared for the first time in [1] and then in [2], [6] where the focus was on the structure of block-factors. Notes on binary sequences of this type are in [11] and [12]. Our question is akin to the problems around probabilistic conditional independence structures [8]; [4] and [5] settle an unconditional case for sequences of random variables. The lattest review of the field is in [7]. It is also worthwhile to mention the paper [9] where Markovness was combined with m-independence. That means any m variables of ξ are mutually independent.

2. MARKOV SEQUENCES OF FIRST ORDER

It is not unexpected that a solution of our problem for 1-Markov sequences will be based on an analysis of transition matrices. Let us remind that a sequence ξ with the state space $S=\{1,2,\dots,d\}$, $d=d_\xi$, is 1-Markov if and only if the probability of every event $\xi_1=s_1\cdots\xi_{k+1}=s_{k+1}$, denoted by $[s_1\dots s_{k+1}]$, is equal to $[s_1]\,p_{s_1s_2}\cdots p_{s_ks_{k+1}}$, $k\geqslant 1$, where $[s]>0$ is the probability of $\xi_1=s$ and $p_{s,t}$ is the conditional probability of $\xi_2=t$ given $\xi_1=s$, $s,t\in S$. The (s,t)-entry of the k-th power of the transition matrix $\boldsymbol{P}=(p_{s,t};\ 1\leqslant s,\ t\leqslant d)$ contains the conditional probability of $\xi_{k+1}=t$ given $\xi_1=s$, $k\geqslant 1$.

If the sequence ξ is, moreover, m-dependent, $m\geqslant 0$, then ξ_1 is independent of ξ_{m+2} and $\boldsymbol{P}^{m+1}=\boldsymbol{Q}$ where the matrix $\boldsymbol{Q}$ has constant columns, t-th one containing the probability $[t]$, $t\in S$. It is not difficult to see that, on contrary, this matrix equality implies that ξ is m-dependent. In fact, it implies ξ_k is independent of ξ_{k+m+1} what together with the conditional independence of ξ_k and $(\xi_i;\ i\geqslant k+m+2)$ given ξ_{k+m+1} yield ξ_k is independent of $(\xi_i;\ i\geqslant k+m+1)$. Repeating the same reasoning once again we obtain the desired m-dependence.

Lemma 1. *If a Markov sequence of first order with d-element state space, $d\geqslant 2$, is m-dependent, $m\geqslant 0$, then it is $(d-2)$-dependent.*

This assertion is nontrivial for $m>d-2$ as it provides reduction of the order of dependence due to “small” state space.

Proof. Knowing that $\boldsymbol{P}^{m+1}=\boldsymbol{Q}$ we deduce that the spectra of both matrices are equal to $\{0,1\}$. Since $\boldsymbol{P}$ is primitive (all entries of some of its powers are positive) the number 1 is an eigenvalue of $\boldsymbol{P}$ with algebraic multiplicity one, see [10]. We can write $\boldsymbol{P}=\boldsymbol{T}\,\boldsymbol{W}\,\boldsymbol{T}^{-1}$ where $\boldsymbol{T}$ is a regular matrix and $\boldsymbol{W}$ is the Jordan canonical form of $\boldsymbol{P}$, see [3]. The matrix $\boldsymbol{W}$ is block-diagonal. One of the blocks consists of the single eigenvalue 1 and the remaining blocks have zeros on their diagonals and ones on their superdiagonals. The k-th power of such a block of size $b\times b$, $b\geqslant 2$, is a zero matrix once $k\geqslant b$. Thus, we can conclude that each matrix $\boldsymbol{W}^k$, $k\geqslant d-1$, has only one nonzero entry; obviously it is the eigenvalue 1. Now, if $m+1\geqslant d-1$ then $\boldsymbol{W}^{m+1}=\boldsymbol{W}^{d-1}$ and consequently $\boldsymbol{P}^{d-1}=\boldsymbol{P}^{m+1}=\boldsymbol{Q}$ what means that the examined sequence is $(d-2)$-dependent. ∎

Corollary 1. *Every (n,m,d)-sequence, $d\geqslant 2$, is dependent of order (d^n-n-1).*

Proof. If ξ fulfils the assumptions, $n>0$, we consider the sequence $\eta=(\eta_i;\ i\geqslant 1)$ of the random variables $\eta_i=(\xi_i,\dots,\xi_{i+n-1})$. This sequence is obviously 1-Markov, $(n+m-1)$-dependent and $d_\eta\leqslant d_\xi^n\leqslant d^n$. By Lemma 1 it is $(d_\eta-2)$-dependent what implies that ξ is dependent of order $(d^n-2-(n-1))$. ∎

Proposition 1. *A triple of integers $\langle 1,m,d\rangle$ is the index of a sequence if and only if $1\leqslant m\leqslant d-2$.*

Proof. The necessity of the presented condition is a consequence of the previous lemma and the sufficiency will be approved below by a construction of the desired sequences.

Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ be an arbitrary orthonormal base of the Euclidean space $\mathbb{R}^d$ such that $\boldsymbol{x}_1$ has all coordinates equal to $d^{-1/2}$. These vectors are taken as rows and their transpositions, columns, are obtained by using the superindex T. For example, $\boldsymbol{Q} = \boldsymbol{x}_1^T \boldsymbol{x}_1$ is a doubly stochastic matrix. If we set $\boldsymbol{U}_k = \sum_{j=2}^{k} \boldsymbol{x}_j^T \boldsymbol{x}_{j+1}$ for $1 \leqslant k \leqslant d-1$ then the powers of these matrices are $\boldsymbol{U}_k^\ell = \sum_{j=2}^{k+1-\ell} \boldsymbol{x}_j^T \boldsymbol{x}_{j+\ell}$, $1 \leqslant \ell \leqslant k$. This fact can be obtained by a simple induction argument. Note that $\boldsymbol{U}_k^k$, $1 \leqslant k \leqslant d-1$, are zero matrices and that for $\ell < k$ the matrix $\boldsymbol{U}_k^\ell$ is nonzero owing to $\boldsymbol{x}_2 \boldsymbol{U}_k^\ell = \boldsymbol{x}_{2+\ell}$. In addition, $\boldsymbol{Q}\, \boldsymbol{U}_k^\ell = \boldsymbol{U}_k^\ell\, \boldsymbol{Q}$ are zero matrices for $1 \leqslant \ell \leqslant k \leqslant d-1$, too.

Now, let us have a triple $\langle 1, m, d \rangle$ and $1 \leqslant m \leqslant d-2$. The 1-Markov sequence ξ on a d-element state space with the transition matrix $\boldsymbol{P} = \boldsymbol{Q} + \varepsilon\, \boldsymbol{U}_{m+1}$, $\varepsilon \neq 0$ sufficiently small, and the uniform initial distribution is stationary because $\boldsymbol{x}_1\, \boldsymbol{P} = \boldsymbol{x}_1$. In addition, $\boldsymbol{P}^k = \boldsymbol{Q} + \varepsilon^k\, \boldsymbol{U}_{m+1}^k$, $1 \leqslant k \leqslant m+1$, what enables to conclude that ξ is m-dependent but not $(m-1)$-dependent, i.e. $n_\xi = 1$, $m_\xi = m$ and $d_\xi = d$. ∎

3. BINARY SEQUENCES: PRELIMINARIES

From now on we fix the state space as $S = \{0, 1\}$. States $s_k, \dots, s_\ell$ from S, $k \leqslant \ell$, will be concatenated into words and the word $s_k\, s_{k+1} \cdots s_\ell$ will be shortened to s_k^ℓ. The symbol S^k denotes the set of all words made of letters from S which have the length k, $k \geqslant 0$, e.g. $s_1^k \in S^k$ and $s_1^0 \in S^0$ is the empty word.

A sequence ξ is n-Markov, $n \geqslant 0$, if and only if for all $k \geqslant 0$ and $s_1^{n+k} \in S^{n+k}$ the probability $[s_1^{n+k}]$ of the event $\xi_1 = s_1 \cdots \xi_{n+k} = s_{n+k}$ can be factorized as follows

$$[s_1^{n+k}] = [s_1^n] \prod_{j=1}^{k} (s_j^{j+n}) \,.$$

In this formula the numbers (s_j^{j+n}), conditional probabilities, are defined by the equalities $[s_1^{n+1}] = [s_1^n]\, (s_1^{1+n})$. If $[s_1^n] = 0$ the choice of (s_1^{1+n}) is arbitrary and will not affect our next computations.

An n-Markov sequence is m-dependent, $m \geqslant 0$, if and only if for all s_1^n, $s_{n+m+1}^{n+m+n} \in S^n$ the following equality

$$\begin{aligned} 0 &= \square_{n,m}(s_1^n,\, s_{n+m+1}^{n+m+n}) = \\ &= [s_1^n] \left([s_{n+m+1}^{n+m+n}] - \sum_{s_{n+1}^{n+m} \in S^m} \prod_{j=1}^{n+m} (s_j^{j+n}) \right) \end{aligned}$$

takes place. That means $(\xi_{k+i};\, 1 \leqslant i \leqslant n)$ is independent of $(\xi_{k+n+m+i};\, 1 \leqslant i \leqslant n)$, $k \geqslant 0$, and this fact implies m-dependence in a similar way as was done above with $n = 1$. We shall also need the symbol $\square_{n,m}^*(s_1^n, s_{n+m+1}^{n+m+n})$ denoting the difference in parentheses. Sometimes an argument in $\square_{n,m}(s_1^n, \cdot)$ is omitted to work with a function on S^n.

An n-Markov sequence, $n \geqslant 1$, is $(n-1)$-Markov if and only if

$$0 = \triangle_{n-1}(s_2^n) = [0\, s_2^n\, 0]\, [1\, s_2^n\, 1] - [0\, s_2^n\, 1]\, [1\, s_2^n\, 0]$$

for all $s_2^n \in S^{n-1}$. The equalities express that the variable ξ_k is independent of ξ_{k+n} given $(\xi_{k+i};\ 1 \leqslant i \leqslant n-1)$. We shall also need the symbol $\triangle^*_{n-1}(s_2^n)$ denoting the above difference with the brackets replaced by parentheses. Thus, $\triangle_{n-1}(s_2^n) = [0\, s_2^n]\, [1\, s_2^n]\, \triangle^*_{n-1}(s_2^n)$.

Beside the foregoing basic observations we want to summarize and label some other useful facts concerning $(n,m,2)$-sequences, $n,\ m \geqslant 1$, $(k \geqslant 0)$

1. $\Box_{n,m-1}(s_2^n\, 0, \cdot) + \Box_{n,m-1}(s_2^n\, 1, \cdot) = 0$
2. $[0\, s_2^n]\, \Box_{n,k}(1\, s_2^n, \cdot) = [1\, s_2^n]\, \Box_{n,k}(0\, s_2^n, \cdot)$ if $\triangle_{n-1}(s_2^n) = 0$
3. $\Box^*_{n,m-1}(s_2^n\, 0, \cdot) = \Box^*_{n,m-1}(s_2^n\, 1, \cdot) = 0$ if $\triangle_{n-1}(s_2^n) \neq 0$
4. $\Box_{n,m-1}(\cdot, 0\, s_2^n) = \Box_{n,m-1}(\cdot, 1\, s_2^n) = 0$ if $\triangle^*_{n-1}(s_2^n) \neq 0$.

Some comments are in order. The expression in 1. equals the sum of $\Box_{n,m}(0\, s_2^n, \cdot)$ and $\Box_{n,m}(1\, s_2^n, \cdot)$. The validity of 2. is clear if $[0\, s_2^n]$ or $[1\, s_2^n]$ is zero; if they are both positive we use $(0\, s_2^{n+1}) = (1\, s_2^{n+1})$. To see 3. we write

$$0 = \Box_{n,m}(s_1^n, \cdot) = \textstyle\sum_{t \in S}\, [s_1^n\, t]\ \Box^*_{n,m-1}(s_2^n\, t, \cdot)$$

and match these equalities into pairs corresponding to $0\, s_2^n$ and $1\, s_2^n$; the assumption $\triangle_{n-1}(s_2^n) \neq 0$ means that the determinant of two equations in a pair is nonzero. Finally, the validity of 4. is obtained similarly from

$$0 = \Box_{n,m}(\cdot, s_{n+m+1}^{n+m+n}) = \textstyle\sum_{t \in S}\, (s_{n+m}^{2n+m})\ \Box_{n,m-1}(\cdot, t\, s_{n+m+1}^{2n+m-1})\ .$$

Lemma 2. *If ξ is a $(n,m,2)$-sequence where $n \leqslant 2$ or $m \leqslant 1$ then ξ is i.i.d.*

Proof. Using Corollary 1 we can restrict ourselves to $m = 1$ and $n \geqslant 2$. We shall demonstrate by contradiction that $\triangle_{n-1}$ is identically zero and then apply the induction argument.

Let $\triangle_{n-1}(t_2^n) \neq 0$ for some $t_2^n \in S^{n-1}$. By fact 3. we have

$$[s_{n+2}^{2n+1}] = \textstyle\prod_{j=2}^{n+1}\, (s_j^{j+n})$$

as soon as the word s_2^{2n+1} begins with t_2^n. We multiply both sides by $[s_{n+1}^{2n}]$ and sum over s_{2n+1} what gives

$$[s_{n+1}^{2n}]\, [s_{n+2}^{2n}] = [s_{n+1}^{2n}]\, \textstyle\prod_{j=2}^{n}\, (s_j^{j+n})\ .$$

The conclusion is $[s_{n+1}^{2n+1}]\, [s_{n+2}^{2n}] = [s_{n+1}^{2n}]\, [s_{n+2}^{2n+1}]$ for all $s_{n+1}^{2n+1} \in S^{n+1}$ contradicting the assumption $\triangle_{n-1}(t_2^n) \neq 0$. ∎

4. BINARY 3-MARKOV SEQUENCES

From Corollary 1 we know that every $(3,m,2)$-sequence is 4-dependent. We will see in a moment that it is even 3-dependent. By Lemma 2 the only nontrivial m's to be examined are then 2 and 3. The aim of this section is to prove some auxiliary results about these two cases.

Lemma 3. *For every* $(3,m,2)$*-sequence* $\triangle_2(01)=0$ *or* $\triangle_2(10)=0$.

Proof. Let us suppose that both $\triangle_2(01)$ and $\triangle_2(10)$ are nonzero. By fact 3. we deduce

$$0=\square^*_{3,m-1}(01t,\cdot)=\square^*_{3,m-1}(10t,\cdot)\,,\quad t\in S.$$

If $\triangle_2(00)=0$ then by fact 2.

$$0=[000]\square_{3,m-1}(100,\cdot)=[100]\square_{3,m-1}(000,\cdot)$$

and since $[100]\neq 0$ (otherwise $\triangle_2(10)=0$) we employ 1. to obtain

$$0=\square_{3,m-1}(00t,\cdot)\,,\quad t\in S.$$

If $\triangle_2(00)\neq 0$ we have this equality immediately by 3. The same reasoning applies symmetrically to $\triangle_2(11)$. Thus, we see that the sequence is $(m-1)$-dependent. By induction, it is i.i.d., a contradiction. ∎

Lemma 4. *A* $(3,m,2)$*-sequence is i.i.d. if and only if both numbers* $\triangle_2(01)$ *and* $\triangle_2(10)$ *are equal to zero.*

Proof. One implication is trivial. If $\triangle_2(01)=\triangle_2(10)=0$ and both $\triangle_2(00)$ and $\triangle_2(11)$ are nonzero we recall 3., 2. and 1. and, similarly as in the proof above, keep lowering of the order of dependence. Hence, by symmetry, let $\triangle_2(00)=0$. Then $\triangle_2(11)=0$ would imply the sequence is 2-Markov and by Lemma 2 also i.i.d.

From $\triangle_2(11)\neq 0$ we deduce $\square_{3,m-1}(s1t,\cdot)=0$ for $s,t\in S$ using 3., 2. and 1. as usually. Further, 1. and 2. enable to write the following four linear equations ($s\in S$)

$$\begin{aligned}&\square_{3,m-1}(s00,\cdot)+\square_{3,m-1}(s01,\cdot)=0\,,\\&[00s]\square_{3,m-1}(10s,\cdot)-[10s]\square_{3,m-1}(00s,\cdot)=0\,.\end{aligned}$$

If the determinant $\triangle_1(0)$ of the system of equations is nonzero then the order of dependence of the sequence decreases. Analogically as soon as $[00]=0$. Let $\triangle_1(0)=0$ and $[00]\neq 0$. Since $[110]\neq 0$ we know that $[s0t]=[s0][0t]/[0]>0$ for $s,t\in S$ and thus

$$(110s)\,(10st)=(10s)\,(0st)=(0s)\,(00st)=(000s)\,(00st)\,.$$

This equality implies $0=\square^*_{3,m-1}(110,\cdot)=\square^*_{3,m-1}(000,\cdot)$ and then $0=\square_{3,m-1}(s0t,\cdot)$ for all $s,t\in S$, having decrease of the order of dependence, too. ∎

Lemma 5. *In every* $(3,m,2)$*-sequence* $\triangle_2(00)=0$ *or* $\triangle_2(11)=0$.

Proof. By Lemma 3, Lemma 4 and symmetry we can assume $\triangle_2(10)\neq 0$ and $\triangle_2(01)=0$. We start from the opposite $\triangle_2(00)\triangle_2(11)\neq 0$ aiming at a contradiction. Note that $[s_1^3]$ is positive for $s_1^3\in S$. Since $\square_{3,4}(\cdot,\cdot)=0$ by Corollary 1, one has $\square_{3,3}(s0t,\cdot)=0$ and $\square_{3,2}(00s,\cdot)=0$ by means of 3., $s,\,t\in S$. Owing to 4. $\square_{3,1}(00s,t1u)=0$ and $\square_{3,0}(00s,t11)=0$ for any $s,\,t,\,u\in S$. The choice $st=00$ in the latter equality gives $[01]=(0000)(0001)$. Then we substitute $st=01$ and $st=11$ and find $(0001)=(1111)$.

On the other hand, from $\square_{3,2}(00s,\cdot)=0$ and $\triangle_2(00)\neq 0$ we have also $\square_{3,1}(00s,t00)=0$ again by 4., $s,\,t\in S$. This provides for $st=01$

$$[100]=(0000)(0001)(010)(0100)+(0001)(011)(0110)(1100)$$

where the left product equals [0100]. Thus, [1100] is equal to the right product and then

$$[110] = (0001)(011)(0110)\,.$$

Let us multiply both sides by (0000) whence $(0000)[011] = [0110]$. The contradiction sounds $(0110) = (0000) = (1110)$. ∎

Corollary 2. *$\langle 3,4,2\rangle$ is not an index.*

Proof. Let a sequence ξ has the index $\langle 3,4,2\rangle$. Then index of the sequence of triples $\eta = ((\xi_i, \xi_{i+1}, \xi_{i+2});\ i \geqslant 1)$ is $\langle 1,6,d\rangle$ whence $d = 8$ by Proposition 1. From the proof of Lemma 1 we know that the transition matrix of η has the rank 7. But, due to Lemma 3 and Lemma 5 this matrix has at least two pairs of equal rows, a contradiction. ∎

Lemma 6. *Let $\triangle_2(10) \neq 0$ and $\triangle_2(01) = 0$ in a $(3,m,2)$-sequence ξ. Then this sequence is 2-dependent if and only if both numbers $\triangle_2(00)$ and $\triangle_2(11)$ are equal to zero.*

Proof. Let us observe that $[s_1^3] > 0$ for $[s_1^3] \in S - \{000, 111\}$ and then the 1-Markov sequence η constructed by grouping triples of consequent variables as above has the index $\langle 1,m,d\rangle$ where $6 \le d \le 8$. The transition matrix of η has the two rows indexed 001 and 101 identical. If $\triangle_2(ss) = 0$ and $[sss] > 0$ then also the two rows indexed by $0ss$ and $1ss$ coincide, $s \in S$. Hence, the matrix has rank at most 5. Then η is 4-dependent and ξ must be 2-dependent.

On the other hand, let ξ be 2-dependent. If $\triangle_2(00) \neq 0$ then by the fact 3. we have $\square_{3,1}(s0t,\cdot) = 0$ for $s,t \in S$ and then by 3. again $\square^*_{3,0}(00t,\cdot) = 0$ for $t \in S$. We can argue as in the proof of Lemma 2 to arrive at a contradiction and thus necessarily $\triangle_2(00) = 0$.

If $\triangle_2(11) \neq 0$ then by 3. we get $\square^*_{3,1}(11t,\cdot) = 0$, $t \in S$, and by the fact 4.

$$\square^*_{3,0}(11t,\ s_5 1 s_7) = 0\,, \quad t, s_5, s_7 \in S\,.$$

The choice $ts_5 = 01$ leads to

$$[11s] = (1101)\,(1011)\,(011s)\,, \quad s \in S\,,$$

and then (add the above equations) to $[11s] = [11]\,(011s)$. Hence $\triangle_2(11)$ equals zero, a contradiction. ∎

Under the assumptions of Lemma 6, the sequence ξ is 3-dependent and not 2-dependent if and only if exactly one of the numbers $\triangle_2(00)$ and $\triangle_2(11)$ is equal to zero. This follows from Corollary 1, Corollary 2, Lemma 5 and Lemma 6.

5. $(3,2,2)$-SEQUENCES

In this section we will describe parametrically all binary 2-dependent sequences that are Markov of order 3.

Proposition 2. *Let ξ be a binary 3-Markov sequence such that*

$$\triangle_2(10) \neq 0 = \triangle_2(00) = \triangle_2(01) = \triangle_2(11) = 0\,.$$

Then ξ is 2-dependent if and only if $[0st1] = [0s][t1]$, $s,\, t \in S$.

Proof. If ξ is a $(3,2,2)$-sequence with $\triangle_2$'s as above then by 3. and 4. we derive

$$\Box^*_{3,1}(10s,\cdot) = \Box^*_{3,0}(10s,\, t10) = 0\,, \quad s,t \in S\,.$$

Since $[s_1^3]$ are positive if s_1^3 is different from 000 and 111 we obtain $[t10] = (0st)\,(0st1)\,(st10)$ and for $st \neq 11$ immediately the desired equalities. But,

$$\textstyle\sum_{s,\,t\in S} [0st1] - [0s][t1] = 0$$

by 2-dependence and we have also $[0111] = [01][11]$.

In the opposite direction, we deduce first from $[0ss1] = [0s][s1]$ that the probabilities $[000]$ and $[111]$ are positive, too. Then

$$[s_5 1 s_7 s_8] = (s_2 0 s_4 s_5)\,(0 s_4 s_5 1)\,(s_4 s_5 1 s_7)\,(s_5 1 s_7 s_8)$$

because $(s_2 0 s_4 s_5) = (0 s_4 s_5)$, $(s_5 1 s_7 s_8) = (1 s_7 s_8)$ and $[0 s_4 s_5 1] = [0 s_4][s_5 1]$. We multiply the above equation by $(s_1 s_2 0 s_4)$, sum over s_4 and s_5 and arrive at $\Box^*_{3,2}(s_1^2 0,\, 1 s_7^8) = 0$.

Further, we are going to verify the equality $\Box^*_{3,2}(s_1^2 1,\, 1 s_7^8) = 0$ for $s_1^2,\, s_7^8 \in S^2$, which is equivalent to

$$[1 s_7^8] - \sum_{s_5 \in S} \frac{[s_5 1 s_7 s_8]}{[s_2 1]\,[s_5 1]} \sum_{s_4 \in S} [s_2 1 s_4 s_5]\,(1 s_4 s_5 1) = 0\,.$$

This will be clear if we show that

$$\nabla(s_2, s_5) = [s_2 1]\,[s_5 1] - \textstyle\sum_{s_4 \in S} [s_2 1 s_4 s_5]\,(1 s_4 s_5 1) = 0$$

for every s_2 and s_5 from S. But,

$$[01]^{-1}\nabla(0,1) = [11] - [01]\,(011) - [11]\,(111) = 0$$

is straightforward and

$$\nabla(1,0) = [11]\,[01] - [1100]\,(001) - (111)\,[1101] = 0$$

owing to $(001) = (111)$ which is a consequence of

$$[000][11] - [011][00] = [001]^{-1}\big([0001][11][00] - [0011][00][01]\big) = 0\,.$$

The fact

$$\textstyle\sum_{s_2 \in S} \nabla(s_2, s_5) = [1][s_5 1] - \sum_{s_4 \in S} [s_4 s_5 1] - [0 s_4 s_5 1] = 0$$

implies that ∇ is identically zero, indeed.

At this moment we know $\Box^*_{3,2}(\cdot, 1 s_7^8) = 0$ for $s_7^8 \in S^2$. The multiplication by (s_0^3) and the summation over $s_3 \in S$ will give $\Box^*_{3,3}(\cdot, 1 s_8^9) = 0$. Now, we sum over s_9 and compare the result with $\Box^*_{3,2}(\cdot, 11 s_8)$. We have $\Box^*_{3,2}(\cdot, 01 s_8) = 0$. Repeating the same trick we arrive at $\Box^*_{3,2}(\cdot, 001) = 0$. But, the sum of $\Box^*_{3,2}(\cdot, s_6^8)$ over $s_6^8 \in S^3$ is zero and thus $\Box^*_{3,2}(\cdot, 000) = 0$. Since $\Box_{3,2}$ is identically zero the sequence ξ is 2-dependent. ∎

Theorem 1. *Let $\alpha,\, \beta \in \mathbb{R}$ satisfy the two inequalities*

$$\pm(4\alpha - 3\beta - \beta^3) \leqslant 1 - \beta^2\,.$$

The binary 3-Markov sequence $\zeta^{\alpha,\beta}$, which has its distribution of first four variables proportional to the function given by the following table is 2-dependent.

0000	$(1-\beta)^2\,(1+\beta^2-2\alpha)$
1000	$(1-\beta^2)\,(1+\beta^2-2\alpha)$
0100	$(1-\beta)^3\,(1+\beta)+4(1-\beta)\,(\beta-\alpha)$
1100	$(1-\beta^2)^2+8\beta(\beta-\alpha)$
0010	$(1+\beta^2-2\alpha)\,(1-\beta^2+2\beta-2\alpha)$
1010	$(1-\beta^2)^2-4(\alpha-\beta)^2$
0110	$(1-\beta)^2\,(1+\beta^2+2\alpha)$
1110	$(1-\beta^2)\,(1+\beta^2+2\alpha)$
0001	$(1-\beta^2)\,(1+\beta^2-2\alpha)$
1001	$(1+\beta)^2\,(1+\beta^2-2\alpha)$
0101	$(1-\beta^2)^2$
1101	$(1+\beta)^3\,(1-\beta)+4(1+\beta)\,(\alpha-\beta)$
0011	$(1+\beta^2)^2-4\alpha^2$
1011	$(1+\beta^2+2\alpha)\,(1-\beta^2+2\alpha-2\beta)$
0111	$(1-\beta^2)\,(1+\beta^2+2\alpha)$
1111	$(1+\beta)^2\,(1+\beta^2+2\alpha)$

Every $(3,2,2)$-sequence ξ is equal in distribution to some $\zeta^{\alpha,\beta}$ or to the sequence obtained from some $\zeta^{\alpha,\beta}$ by interchanging zeros and ones.

Proof. The proportionality factor is $1/16$. The conditions imposed on α and β restrict the parameters to be between -1 and 1, see Figure 1, and guarantee that all entries of the table are nonnegative; notably the critical inequalities are $[0100] \geqslant 0$ and $[1101] \geqslant 0$.

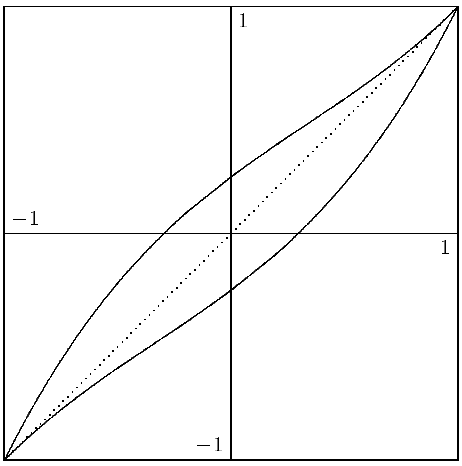

Figure 1. Parameters (α,β) are from the region bordered by two cubic curves.

It is easy to verify that $\sum_{t\in S}[t\,s_1^3]-[s_1^3 t]=0$, $s_1^3\in S_3$, i.e. that every sequence $\zeta^{\alpha,\beta}$ is strictly stationary. To this end we remark that

$$\begin{aligned} 8\,[000] &= (1-\beta)(1+\beta^2-2\alpha) & 8\,[010] &= (1-\beta)(1-\beta^2+2\beta-2\alpha)\\ 8\,[111] &= (1+\beta)(1+\beta^2+2\alpha) & 8\,[101] &= (1+\beta)(1-\beta^2+2\alpha-2\beta)\\ 8\,[100] &= (1+\beta)(1+\beta^2-2\alpha) & 8\,[110] &= (1-\beta)(1+\beta^2+2\alpha) \end{aligned}$$

and $[100]=[001]$, $[110]=[011]$. It is also not difficult to see that $\zeta^{\alpha,\beta}$, $\alpha\neq\beta$, fulfils the assumptions of Proposition 2, especially

$$(0001)=(1001)=\frac{1+\beta}{2}=(0111)=(1111)$$
$$(0011)=(1011)=\frac{1+\beta^2+2\alpha}{2(1+\beta)}$$

and $\triangle_2(10)\neq 0$. Hence, all sequences $\zeta^{\alpha,\beta}$ are 2-dependent. Note that $\zeta^{\alpha,\alpha}$ are i.i.d., $|\alpha|\leqslant 1$ (the dotted segment in Figure 1).

In the opposite direction, let ξ be a (3,2,2)-sequence. By Lemmas 3 and 6 we know that up to switching between zeros and ones $\triangle_2(st)=0$ for $st\neq 10$. If also $\triangle_2(10)=0$ then ξ is i.i.d. by Lemma 4 and thus equal in distribution to some $\zeta^{\alpha,\alpha}$. If $\triangle_2(10)\neq 0$ then $[0st1]=[0s]\,[t1]$ for $s,\,t\in S$ by Proposition 2. Thus, with $st=11$ here, the quadratic equation in $[111]$

$$[111]^2-[11]\,[111]+[11]^2\,[01]=0$$

has nonnegative discriminant equal to $[11]^2\,(1-4[01])$. We set $\beta^2=1-4[01]$ and then we have $2[111]=[11]\,(1+\beta)$. If we take $\alpha=2[1]-1$ we can compute $4[00]=1+\beta^2-2\alpha$ and $4[11]=1+\beta^2+2\alpha$. Using the listed properties of ξ and the stationarity it is easy, but a bit laborious, to compute first the probabilities $[s_1s_2s_3]$ and then to construct the whole table. ∎

Corollary 3. *The triple $\langle 3,2,2\rangle$ is the index of $\zeta^{0.1,0}$.*

Remarks. 1. Let us mention that the sequence $\zeta^{-\alpha,\alpha}$, $\alpha\neq 0$ small, has the same index as $\zeta^{0.1,0}$. In addition, every its two consequent variables are independent. But, no of the sequences $\zeta^{\alpha,\beta}$, $\alpha\neq\beta$, is 2-independent.

2. From the topological point of view the class of $(3,2,2)$-sequences with the week topology is homeomorphic to two closed circles (disks) pasted together along its diameters; the common diameter corresponds to the i.i.d. sequences and switching between circles to switching between 0 and 1.

3. Reversing time in a $(3,2,2)$-sequence indexed by integer numbers one obtains again a $(3,2,2)$-sequence. In our parametrization this corresponds to the transition $0\leftrightarrow 1$ and $(\alpha,\beta)\leftrightarrow(-\alpha,-\beta)$ simultaneously.

6. $(3,3,2)$-SEQUENCES

In this section we will describe parametrically all binary 3-dependent sequences that are Markov of order 3. Since all of them which are 2-dependent were investigated in the previous section we concentrate on the non-2-dependent ones. This will close the description of all $(3,m,2)$-sequences.

Proposition 3. *Let ξ be a binary 3-Markov sequence satisfying $\triangle_2(1s) \neq 0 = \triangle_2(0s)$ for $s \in S$. Then ξ is 3-dependent if and only if $[111] = (001)^3$, $[0s01] = [0s][01]$, $s \in S$, and $(11st) = (00t)$, $s, t \in S$.*

Proof. Obviously, $[s_1^3] > 0$ for $s_1^3 \neq 000$. If ξ is 3-dependent then by 3. $\square_{3,2}(1st,\cdot) = 0$ and by 4. $\square^*_{3,1}(1st,u1v) = 0$ and $\square^*_{3,0}(1st,u11) = 0$, $s,t,u,v \in S$. For $su = 00$ the last equation gives $[000] > 0$ and $[011] = (0t0)(0t01)(011)$, i.e. $[0t][01] = [0t01]$; as a useful consequence we have $[01] = (000)(001)$. The choice $su = 01$ provides $[01] = (11t0)(1t01)$ what reads as $(000) = (1100)$ if $t = 0$ and as $(000) = (1110)$ if $t = 1$. And finaly, $stu = 111$ leads to $[111] = (1111)^3 = (001)^3$.

For the reverse implication we will first demonstrate $\square^*_{3,0}(1st,u11) = 0$. For $u = 0$ this is equivalent to $[01] = (1st0)(st01)$ which certainly holds if $s = 0$; if $s = 1$ we have $[01] = (000)(001)$. For $u = 1$ we want to see that $[111] = (1st1)(st11)(t111)$ or, rewritten, $[t11] = (1st1)(st11)(111t)$. If $t = 0$ this means $[011] = (001)(011)(000)$. If $t = 1$ this amounts $[s11] = (1s11)(111s)(1111)$ which is fulfilled for $s = 1$ as $(1111) = (001)$ and which holds also for $s = 0$ having $[011] = (011)(000)(001)$.

The next step is to show $\square^*_{3,1}(1st,u1v) = 0$. For $u = 1$ this can be obtained from $\square^*_{3,0}(1st,u'11) = 0$ by multiplication with $(u'11v)$ and summation over u'. For $u = 0$ we want to verify

$$[01] = \textstyle\sum_{v\in S} (1stv)(stv0)(tv01)$$

which is clear for $t = 0$: the product equals $(1s0v)[01]$. If $t = 1$ then we rewrite it into

$$(000) = \textstyle\sum_{v\in S} (1s1v)(s1v0)$$

being satisfied for $s = 1$. For $s = 0$ we have

$$(000)[01] = [0100] + [0110] = [01] - [0101] - [0111] = [01] - [01]^2 - (1110)[111]\,,$$

which can be casted into $(001)[01] = [01]^2 + (000)(001)^3$ and $(001) = [01] + (001)^2$.

Knowing that $\square^*_{3,1}(1st,u1v) = 0$ we can obtain $\square^*_{3,2}(1st,1vu) = 0$ and $\square^*_{3,2}(s1t,u1v) = 0$. These two equalities imply $\sum_{t\in S} \square^*_{3,2}(11s,00t) = 0$ where both summands must equal zero due to $(001)\square^*_{3,2}(\cdot,000) = (000)\square^*_{3,2}(\cdot,001)$ (cf. the fact 2.). Hence, $\square^*_{3,2}(11s,\cdot) = 0$ and then $\square^*_{3,3}(11s,\cdot) = 0$ and $\square^*_{3,3}(s11,\cdot) = 0$ arguing as usually. But owing to the equality $(110t)(10tu) = (s00t)(00tu)$ we have $\square^*_{3,2}(110,\cdot) = \square^*_{3,2}(s00,\cdot)$ whence $\square^*_{3,3}(s00,\cdot) = 0$ and $\square^*_{3,4}(s00,\cdot) = 0$. We can write now

$$0 = \square_{3,4}(000,\cdot) + \square_{3,4}(100,\cdot) = \square_{3,3}(000,\cdot) + \square_{3,3}(001,\cdot)$$

and deduce $\square^*_{3,3}(s01,\cdot) = 0$. Thence $\square_{3,3}(\cdot,\cdot) = 0$ and ξ is 3-dependent. ∎

Theorem 2. *Let α, $\beta \in \mathbb{R}$ satisfy the two inequalities*

$$-(1-\beta^2)^2 \le 8\alpha - 8\beta \le (1-\beta)^3\,(1+\beta)\,.$$

The binary 3-Markov sequence $\theta^{\alpha,\beta}$ which has its distribution of first four variables proportional to the function given by the following table is 3-dependent.

0000	$(1-\beta)^2\,(1+\beta^2-2\alpha)$
1000	$(1-\beta^2)\,(1+\beta^2-2\alpha)$
0100	$(1-\beta)^3\,(1+\beta)+8(\beta-\alpha)$
1100	$(1-\beta^2)^2+4(1-\beta)\,(\alpha-\beta)$
0010	$(1+\beta^2-2\alpha)\,(1+3\beta-\beta^2+\beta^3-4\alpha)/(1-\beta)$
1010	$(1-\beta^2+2\alpha-2\beta)\,(1+3\beta-\beta^2+\beta^3-4\alpha)/(1-\beta)$
0110	$(1-\beta^2)^2+8(\alpha-\beta)$
1110	$(1-\beta^2)\,(1+\beta)^2$
0001	$(1-\beta^2)\,(1+\beta^2-2\alpha)$
1001	$(1+\beta)^2\,(1+\beta^2-2\alpha)$
0101	$(1-\beta^2)^2$
1101	$(1+\beta)^3\,(1-\beta)+4(1+\beta)\,(\alpha-\beta)$
0011	$(1+\beta^2-2\alpha)\,(1-3\beta-\beta^2-\beta^3+4\alpha)/(1-\beta)$
1011	$(1-\beta^2+2\alpha-2\beta)\,(1-3\beta-\beta^2-\beta^3+4\alpha)/(1-\beta)$
0111	$(1-\beta^2)\,(1+\beta)^2$
1111	$(1+\beta)^4$

Every $(3,3,2)$-sequence ξ which is not 2-dependent equals in distribution to some $\theta^{\alpha,\beta}$, $\alpha \neq \beta$, up to the switching of zeros and ones or up to the time reversal.

Proof. The proportionality factor is $1/16$. The conditions imposed on α and β restrict the parameters to be strictly between -1 and 1, see Figure 2, and guarantee that all entries of the table are nonnegative (the critical inequalities are $[0100] \geqslant 0$ and $[0110] \geqslant 0$). By continuity, $\theta^{1,1}$ will be a constant sequence. The dashed curves in Figure 2 remind the restrictions from Theorem 1 which can be here interpreted as $[010] \ge 0$ and $[011] \ge 0$.

It is easy to verify that the sequences $\theta^{\alpha,\beta}$ are strictly stationary. For this purpose we write down

$$\begin{aligned}
8\,[000] &= (1-\beta)(1+\beta^2-2\alpha) & \qquad 8\,[111] &= (1+\beta)^3\\
8\,[100] &= (1+\beta)(1+\beta^2-2\alpha) & 8\,[010] &= (1+3\beta-\beta^2+\beta^3-4\alpha)\\
8\,[101] &= (1+\beta)(1-\beta^2+2\alpha-2\beta) & 8\,[110] &= (1-3\beta-\beta^2-\beta^3+4\alpha)\,.
\end{aligned}$$

From

$$(0001) = (1001) = \frac{1+\beta}{2} = (1101) = (1111)$$

$$(0011) = (1011) = \frac{1-3\beta-\beta^2-\beta^3+4\alpha}{2(1-\beta^2)}$$

we see that the sequences $\theta^{\alpha,\beta}$ satisfy the assumptions of Proposition 3 and are thus 3-dependent.

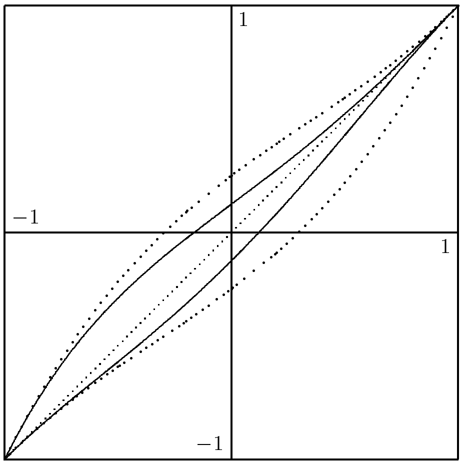

Figure 2. Parameters (α, β) are from the region bordered by two biquadratic curves.

Let ξ be a (3,3,2)-sequence which is not 2-dependent. Switching zeros and ones and reversing time, if necessary, we know that $\triangle_2(0s) = 0$ and $\triangle_2(1s) \neq 0$, $s \in S$; see the lemmas of Section 4. By Proposition 3, $[000][001] = [0001][00] = [00]^2[01]$, which can be casted into a quadratic equation in $[000]$ similarly as in the proof of Theorem 1. The equation has nonnegative discriminant equal to $[11]^2\,(1 - 4[01])$. We set $\beta^2 = 1 - 4[01]$, $\alpha = 2[1] - 1$ and then we can easily compute $[000]$, $[001]$ and $[101]$. Using $[111] = (011)^3$ we obtain $[111]$, $[110]$ and $[010]$. From $(11st) = (00t)$ and $(s0tu) = (0tu)$ the whole table can be computed with a little effort. ∎

Corollary 4. *The triple $\langle 3, 3, 2\rangle$ is the index of $\theta^{0,0.1}$.*

Remark. From the topological point of view the union of the classes of $(3, m, 2)$-sequences over $m \geq 0$ finite, with the week topology, is homeomorphic to six closed disks pasted together along its diameters. The common diameter corresponds to the i.i.d. sequences, two disks to the $(3, 2, 2)$-sequences and the remaining four disks without their common diameter to the $(3, 3, 2)$-sequences that are not 2-dependent.

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References

[1] Aaronson J., Gilat D. and Keane M. S. (1992) On the structure of 1-dependent Markov shifts, *J. Theor. Probab.* **5**, 545–561.

[2] Burton R. M., Goulet M. and Meester R. W. J. (1993) On 1-dependent processes and k-block factors, *The Annals of Probability* **21**, 2157–2168.

[3] Horn R. A. and Johnson C. R. (1986) *Matrix Analysis*, Cambridge University Press, Cambridge.

[4] Matúš F. (1988) Independence and Radon projections on compact groups, PhD. theses (in Slovak), Institute of Information Theory and Automation, Prague.

[5] Matúš F. (1994) Stochastic independence, algebraic independence and abstract connectedness, *Theor. Comp. Science* **134**, 455-471.

[6] Matúš F. (1996) On two-block-factor sequences and one-dependence, *Proc. Amer. Math. Soc.* **14**, 1237–1242.

[7] Matúš F. (1997) Conditional independence structures examined via minors, *Annals of Mathematics and AI* **21**, 99–128.

[8] Pearl J. (1988) *Probabilistic Reasoning in Intelligent Systems*, Morgan Kaufman, San Meteo, California.

[9] Rosenblatt M. and Slepian D. (1962) Nth order Markov chains with every N variables independent, *J. Soc. Indust. Appl. Math.* **10**, 537–549.

[10] Seneta E. (1973) *Non-Negative Matrices*, G. Allen & Unwin Ltd., London.

[11] Valk V. de (1988) One-dependent processes, PhD. theses, Delft Univ. Technology, Delft Univ. Press.

[12] Valk V. de (1994) One-dependent processes: two-block-factors and non-two-block-factors, CWI Tract 85, Amsterdam

František Matúš
Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Pod vodárenskou věží 4
182 08 Prague, Czech Republic
email: matus@utia.cas.cz