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ON TWO–BLOCK–FACTOR SEQUENCES AND ONE–DEPENDENCE

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Abstract. The distributions of two–block–factors $(f(\eta_i, \eta_{i+1});\ i \geq 1)$ arising from i.i.d. sequences $(\eta_i;\ i \geq 1)$ are observed to coincide with the distributions of the superdiagonals $(\zeta_{i,i+1};\ i \geq 1)$ of jointly exchangeable and dissociated arrays $(\zeta_{i,j};\ i, j \geq 1)$. An inequality for superdiagonal probabilities of the arrays is presented. It provides, together with the observation, a simple proof of the fact that a special one–dependent Markov sequence of Aaronson, Gilat and Keane [1] is not a two–block factor.[2]

1. Introduction. Let $\eta = (\eta_i;\ i \geq 1)$ be an i.i.d. sequence with the variables ranging in a measurable space $(U, \mathcal{U})$ and let f be a measurable function defined on the product space $(U^{m+1}, \mathcal{U}^{m+1})$, $m \geq 0$, with its values in a measurable space $(S, \mathcal{S})$. The random sequence $(f(\eta_i, \eta_{i+1}, \ldots, \eta_{i+m});\ i \geq 1)$ is called $(m+1)$*–block–factor* of the sequence η. The problem of determining which random sequence $\xi = (\xi_i;\ i \geq 1)$ is equal in distribution to an $(m+1)$–block–factor has been likely contemplated by many probabilists during last decades.

A sequence ξ having the distribution of an $(m+1)$–block–factor must be, trivially, strictly stationary and m*–dependent* in the sense that $(\xi_i;\ j > i \geq 1)$ is stochastically independent of $(\xi_i; i \geq j + m)$ for every $j > 1$. The fact that these two necessary conditions are not sufficient, or in other words, that the distributions of $(m+1)$–block–factors do not exhaust the distributions of all m–dependent stationaly sequences, $m > 0$, was already mentioned in Ibragimov and Linnik [9] (see the footnote on p. 469). The first published examples of one–dependent stationary

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sequences which are not two–block–factors appeared much later, see [2]. Markov sequences were examined from this point of view in [1] where an example of a five–state one–dependent homogeneous Markov sequence which is not a two–block–factor is presented. Moreover, it was proved that there is no such example with less than five states. More sophisticated examples of one–dependent stationary sequences which are even not m–block–factors for any $m \geq 0$ were introduced recently in [5]. For lists of further references on the topic we refer the reader to [7] and [11].

The aim of the present note is to relate the problem which sequences have the distributions of $(m+1)$–block–factors to known results on representations of partially exchangeable random arrays. This will provide a characterization of $(m+1)$–block–factors in terms of jointly exchangeable and dissociated arrays. Since the generalization for $m > 1$ is straightforward we confine, for simplicity, to the case $m = 1$. If S is a Polish space then as a consequence of the characterization we derive that the distributions of two–block–factors form a close set under week topology. This fact combined with the example from [2] implies that they are not dense in the set of one–dependent stationary distributions provided S has at least two elements.

Further we employ the standpoint of arrays to simplify and strengthen the results of Aaronson, Gilat and Keane [1] on the five–state Markov sequence. By transparent graphical manipulations we prove an inequality concerning the superdiagonals of arrays. This inequality enables to modify slightly their example to obtain a five–state one–dependent Markov sequence which is not a two–block–factor and has positive probabilities of all cylinders.

2. Block–factors in random arrays. An array $\zeta = (\zeta_{i,j};\ i,j \geq 1)$ of random variables with values in $(S,\mathcal{S})$ is called *jointly* (or also weakly) *exchangeable*, see [10] and [6], if its distribution coincides with the distribution of every array $(\zeta_{\pi(i),\pi(j)};\ i,j \geq 1)$. Here both indices are permuted simultaneously by any permutation π which moves only finite number of positive integers. An array ζ is *dissociated* if $(\zeta_{i,j};\ k > i \geq 1\,,\ k > j \geq 1)$ is independent of $(\zeta_{i,j};\ i \geq k\,,\ j \geq k)$, $k > 1$.

Proposition. *Let $(S,\mathcal{S})$ be a Borel space. A sequence $\xi = (\xi_i;\ i \geq 1)$ is equal in distribution to a two–block–factor if and only if there exists a jointly exchangeable and dissociated array $\zeta = (\zeta_{i,j};\ i,j \geq 1)$ with the superdiagonal $(\zeta_{i,i+1};\ i \geq 1)$ equal in distribution to ξ.*

Proof. If ξ has the distribution of a two–block–factor, in symbols

$$\mathcal{D}\ (\xi) = \mathcal{D}\ (f(\eta_i, \eta_{i+1});\ i \geq 1)\,,$$

then it has the distribution of the superdiagonal of the array $(f(\eta_i, \eta_j);\ i,j \geq 1)$ which is trivially jointly exchangeable and dissociated.

In the opposite direction, let ξ be equal in distribution to the superdiagonal of such an array ζ. Then there are real random variables $(\beta_i;\ i \geq 1)$ and $(\gamma_{i,j}; j > i \geq 1)$ which are altogether mutually independent and distributed uniformly on the unit interval $[0,1]$ and there exists a measurable function g on $[0,1]^3$ with values in S such that

$$\mathcal{D}\ (\zeta_{i,j};\ j > i \geq 1)\ =\ \mathcal{D}\ (g(\beta_i, \beta_j, \gamma_{i,j});\ j > i \geq 1)\,.$$

This claim is a simple consequence of Theorem 3.1 from [10] (cf. also [3], [4] and [8]). If we take now $U = [0,1]^2$ and $\eta_i = (\beta_i, \gamma_{i,i+1})$ then $\eta = (\eta_i; i \geq 1)$ is an i.i.d. sequence and $\mathcal{D}\ (\xi) = \mathcal{D}\ (f(\eta_i, \eta_{i+1});\ i \geq 1)$ for the measurable function f given by $f((x_1, x_2), (y_1, y_2)) = g(x_1, y_1, x_2)$. ∎

Consequence. *Let $(S, \mathcal{S})$ be a Polish space with its Borel σ–algebra of subsets. The family $\mathcal{P}_{2bf}(S)$ of all distributions of two–block–factors ranging in S is closed in the usual weak topology.*

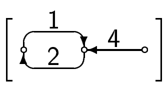
Proof. It is a matter of an elementary Polish space calculus to verify that the mapping

$$\lambda : \mathcal{D}\ (\zeta_{i,j};\ j > i \geq 1) \to \mathcal{D}\ (\zeta_{i,i+1};\ i \geq 1)$$

is continuous w.r.t. the weak topologies on the distributions of arrays and sequences. If S is compact then the family $\mathcal{P}_{jed}(S)$ of the distributions $\mathcal{D}\ (\zeta_{i,j};\ j > i \geq 1)$ for ζ jointly exchangeable and dissociated is compact and then by the proposition $\mathcal{P}_{2bf}(S) = \lambda(\mathcal{P}_{jed}(S))$ is compact, too. To see that $\mathcal{P}_{2bf}(S)$ is closed for noncompact S we can identify S with a (G_δ and dense) subspace of a compact Polish space T. Now, if $Q_n \in \mathcal{P}_{2bf}(S)$ converges to the distribution Q of a sequence with values in S then also $Q_n \to Q$ when considered over the space T. But for some $P_n \in \mathcal{P}_{jed}(S)$ we have $\lambda(P_n) = Q_n$ and passing to a subsequence, if necessary, we can assume that $P_n \in \mathcal{P}_{jed}(T)$ converges to a distribution $P \in \mathcal{P}_{jed}(T)$. As $\lambda(P) = Q$ the triangular array with the distribution P ranges, in fact, in S. Namely, all its variables are equidistributed. Thus $P \in \mathcal{P}_{jed}(S)$ and then $Q \in \mathcal{P}_{2bf}(S)$. ∎

Knowing from [2] that $\mathcal{P}_{2bf}(S)$ is a proper subset of the family $\mathcal{P}_{1ds}(S)$ of all distributions of one–dependent stationary sequences ranging in S one can conclude immediately that $\mathcal{P}_{2bf}(S)$ is not dense in $\mathcal{P}_{1ds}(S)$ for nontrivial S.

3. Inequality for superdiagonals. Let ζ be a fixed jointly exchangeable and dissociated array with values in the space $S = \{1, 2, 3, 4, 5\}$; we will have in mind mainly the "trivial" arrays $(f(\eta_i, \eta_j);\ i, j \geq 1)$. For our purposes it is advantageous to denote the probabilities of events like $(\zeta_{i,j} = 1, \zeta_{j,i} = 2, \zeta_{k,j} = 4)$, i, j, k distinct, by means of oriented graphs and brackets. For example, this event is represented by the 1–graph with three vertices i, j and k and with three edges $(i,j), (j,i)$ and (k,j) labeled by the elements 1, 2 and 4 of S, respectively. Its probability will be written as



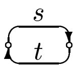
The notation of vertices by i, j and k can be omitted here due to the joint exchangeability. Further, we shall abbreviate these two kinds of probabilities concerning the superdiagonal

$$\left[\circ\xrightarrow{1}\circ\right] = [1] = P(\zeta_{i,i+1} = 1) \text{ and } \left[\circ\xrightarrow{1}\circ\xrightarrow{3}\circ\right] = [13] = P(\zeta_{i,i+1} = 1, \zeta_{i+1,i+2} = 3)\ ;$$

for the "trivial" arrays the probabilities like [1] and [13] concern two–block–factors.

We shall need the following auxiliary result.

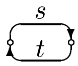
Lemma.

[]$^2 \leq [st]\,, \qquad\qquad s,t \in \mathcal{S}\,.$

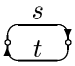
Proof. From Theorem 3.1 of [10] we deduce that the array ζ, without its diagonal, can be represented in the form

$$(\zeta_{i,j};\ i,j \geq 1,\ i \neq j) \ = \ (g(\beta_i, \beta_j, \gamma_{i,j});\ i,j \geq 1,\ i \neq j)$$

with β_i a $\gamma_{i,j}$ as before but $\gamma_{i,j} = \gamma_{j,i}$. The equality is taken almost sure. Let $i,j,k \geq 1$ be distinct and $\alpha^s_{i,j}$, $s \in S$, denote the indicator of the event $(\zeta_{i,j} = s)$. Then by using Cauchy inequality twice we get

[] $= \mathsf{E}(\mathsf{E}(\alpha^s_{i,j}\alpha^t_{j,i}|\beta_j)) \leq \mathsf{E}(\mathsf{E}(\alpha^s_{i,j}|\beta_j)^{1/2}\mathsf{E}(\alpha^t_{j,i}|\beta_j)^{1/2}) \leq \mathsf{E}(\mathsf{E}(\alpha^s_{i,j}|\beta_j)\mathsf{E}(\alpha^t_{j,i}|\beta_j))^{1/2}.$

Since $\mathsf{E}(\alpha^t_{j,i}|\beta_j) = \mathsf{E}(\alpha^t_{j,k}|\beta_j)$ a.s. and $\alpha^s_{i,j}$ is conditionally independent of $\alpha^t_{j,k}$ given β_j (note that $\alpha^s_{i,j}$ is a function of β_i, β_j and $\gamma_{i,j}$) we can conclude

[]$^2 \leq \mathsf{E}(\mathsf{E}(\alpha^s_{i,j}\ \alpha^t_{j,k}|\beta_j)) = [st]\,.$ ∎

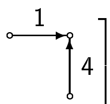
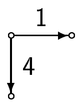
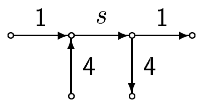
The following result can be formulated equivalently for the two–block–factors as well.

Theorem. *The inequality*

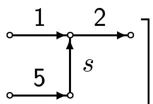
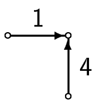
$$[13] + [24]^{1/2} + [32]^{1/2} + [41] + [51] + [52] + [55] \ \geq \ \frac{1}{4}\ [5]^2[11][12][23]$$

is valid for every jointly exchangeable and dissociated array ζ.

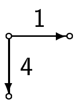
Proof. We start with the calculation

[] [] $= \sum_{s\in\mathcal{S}}$ [] $\leq [41] + [24] + [13] + [41] + [51]$

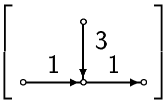
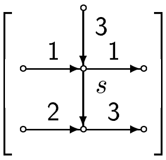
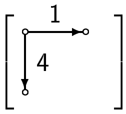
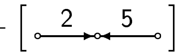
using the dissociation property. The whole proof consists exclusively of tricks of this kind. The next one is

$[5][12] = \sum_{s\in S}$ [] $\leq [51] + [52] + [32] +$ [] $+ [55]$

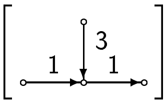
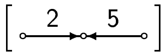
which, together with the foregoing inequality, gives

$[5][12]$ [] $\leq 2([41] + [51]) + [13] + [24] + [32] + [52] + [55]\,.$

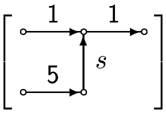
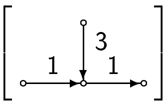
Further the inequality

$[23]$  $= \sum_{s\in S}$  $\leq [13] + [32] + [13] +$  $+$ 

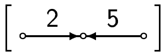
provides the estimation

$[5][12][23]$  $\leq 3[13]+2([32]+[41]+[51])+[24]+[52]+[55]+$  .

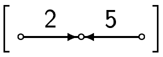
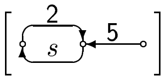
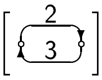
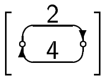
Similarly from

$[5][11] = \sum_{s\in S}$  $\leq [51] + [52] +$  $+ [41] + [55]$

we obtain

$[5]^2[11][12][23] \leq 3([13] + [41] + [51]) + 2([32] + [52] + [55]) + [24] +$  .

Finally, the casting

 $= \sum_{s\in S}$  $\leq [51] + [52] +$  $+$  $+ [55]$

together with the Lemma lead to the assertion of the Theorem. ∎

The inequality of our Theorem improves Theorem 4 of [1] which claims

$$[13] + [24] + [32] + [41] + [51] + [52] + [55] = 0 \quad \Rightarrow \quad [11][12][14][23][53][54] = 0\,,$$

for probabilities in the two–block–factors of i.i.d. sequences.

Consequence. *The homogeneous Markov sequence on S with the uniform initial distribution and with the transition matrix*

$$\frac{1-\varepsilon}{10}\begin{pmatrix} 4 & 2 & 0 & 1 & 3 \\ 2 & 4 & 1 & 0 & 3 \\ 4 & 0 & 1 & 3 & 2 \\ 0 & 4 & 3 & 1 & 2 \\ 0 & 0 & 5 & 5 & 0 \end{pmatrix} + \frac{\varepsilon}{5}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad 0 \leq \varepsilon \leq 1\,,$$

is not a two–block–factor of an i.i.d. sequence if ε is close to zero. Trivially, it is one–dependent (the square of the transition matrix has constant entries) and all cylinders have positive probabilities for positive ε. ∎

The Markov sequence with $\varepsilon = 0$ was found in [1]. Knowing that this sequence is not a two–block–factor of an i.i.d. sequence the Consequence follows also from the closedness of $\mathcal{P}_{2bf}(S)$.

Let us mention that in the perturbed sequences of Burton, Goulet and Meester [5] (p. 2163) cylinders have positive probabilities, too.

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