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Excluded Minors for Boolean Polymatroids*

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Abstract. New necessary and sufficient conditions for a polymatroid to be Boolean are presented. The excluded minors for the class of Boolean polymatroids are explicitly described.

Let N be a finite set and $h\colon 2^N \to \mathbb{Z}$ be an integer-valued function on the power set of N. The pair (N, h) is a *polymatroid* if the function h is normalized, $h(\emptyset) = 0$, nondecreasing, $h(I) \leqslant h(J)$ for $I \subseteq J \subseteq N$, and submodular $h(I)+h(J) \geqslant h(I\cup J)+h(I\cap J)$, $I, J \subseteq N$. Here N is the ground set and h the rank function of the polymatroid (N, h).

Definition 1. *A polymatroid (N, h) is Boolean if there exists a finite set M and a mapping $\phi\colon N \to 2^M$ such that $h(I)$ equals the cardinality of $\phi(I) = \bigcup_{i\in I} \phi(i)$.*

Every polymatroid with the ground set N of cardinality $|N| \leqslant 2$ is Boolean. In this note we present an excluded minor characterization of the class of Boolean polymatroids.

A polymatroid (N, h) is a *matroid* if $h(i) \leqslant 1$ for all singletons i of N. An element $i \in N$ is a loop of the polymatroid if $h(i) = 0$. The number $h(N)$ is the rank of the polymatroid. If it happens that $h(N) = 1$ then the set $J = \{i \in N;\, h(i) = 1\}$ is nonempty and uniquely characterizes the (poly)matroid by $h(I) = \min\{|I \cap J|, 1\}$, $I \subseteq N$. These rank-one matroids will be denoted by (N, r_J), $\emptyset \neq J \subseteq N$; the elements of $N - J$ are loops.

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Lemma 2. *For a normalized function $h\colon 2^N \to \mathbb{Z}$ the following statements are equivalent.*

(i) (N,h) *is a Boolean polymatroid.*

(ii) $h = \sum_{\emptyset \neq J \subseteq N} c_J r_J$ *for nonnegative integers* c_J.

(iii) $\mathbb{A}h(K) = \sum_{I \subseteq K} (-1)^{|K-I|} \big[h(N) - h(N-I)\big] \geqslant 0$, *for all* $K \subseteq N$.

Proof. Let (N,h) be a Boolean polymatroid originating from a mapping $\phi\colon N \to 2^M$. We denote $\phi'\colon M \to 2^N$ the mapping given by $\phi'(\ell) = \{j \in N;\ \ell \in \phi(j)\}$. Let c_J be the number of the elements $\ell \in M$ such that $\phi'(\ell) = J$, $\emptyset \neq J \subseteq N$. We claim that $h = \sum_{\emptyset \neq J \subseteq N} c_J r_J$. In fact, evaluating the sum in a set $I \subseteq N$ we get $\sum_{J\colon J \cap I \neq \emptyset} c_J$ which is equal to the number of $\ell \in M$ satisfying $\phi'(\ell) \cap I \neq \emptyset$. But, this is exactly $|\phi(I)| = h(I)$.

If $h = \sum_{\emptyset \neq J \subseteq N} c_J r_J$ where c_J are nonnegative integer numbers then h is a sum of the rank functions of polymatroids and (N,h) must be a polymatroid. Let the set M be given as $\{\ell_t^J;\ \emptyset \neq J \subseteq N, 1 \leqslant t \leqslant c_J\}$, taking the elements ℓ_t^J pairwise distinct, and let $\phi(i)$ equal $\{\ell_t^J;\ i \in J \subseteq N, 1 \leqslant t \leqslant c_J\} \subseteq M$, $i \in N$. Then the union $\phi(I)$ is $\{\ell_t^J;\ I \cap J \neq \emptyset, 1 \leqslant t \leqslant c_J\}$, $I \in N$, and has $\sum_{J\colon J \cap I \neq \emptyset} c_J = h(I)$ elements. The polymatroid (N,h) is therefore Boolean.

It is a matter of an easy calculation to show that $\mathbb{A}r_J = \delta_J$, $\emptyset \neq J \subseteq N$, where $\delta_J(K) = 1$ for $J = K$ and $\delta_J(K) = 0$ for $J \neq K$. Hence, $\{r_J;\ \emptyset \neq J \subseteq N\}$, and obviously $\{\delta_J;\ \emptyset \neq J \subseteq N\}$, is a base of the linear space of all real-valued normalized functions h. The equivalence of *(ii)* and *(iii)* follows. ∎

The *restriction* of a polymatroid (N,h) by a set $I \subseteq N$ is the polymatroid $(N-I, h_{\setminus I})$ where $h_{\setminus I}(J) = h(J)$, $J \subseteq N - I$, and its *contraction* by the set I is the polymatroid $(N-I, h_{/I})$ where $h_{/I}(J) = h(J \cup I) - h(I)$, $J \subseteq N - I$. A *minor* of a polymatroid is a contraction of some of its restrictions, or equivalently, a restriction of some of its contractions. A minor of a polymatroid is proper if its ground set is a proper subset of the ground set of the polymatroid. It is obvious that minors of (Boolean) polymatroids are (Boolean) polymatroids.

If a class of polymatroids contains all minors of its polymatroids then it is called minor-closed. A polymatroid being out of a minor-closed class but having all proper minors within the class is *excluded minor* for the class. A natural problem is to find all excluded minors for the class.

Let $n \geqslant 3$ be the cardinality of a set N. We denote by $q_N = -r_N + \sum_{i \in N} r_{N-i}$ so that $q_N(i) = n-2$, $i \in N$, and $q_N(I) = n-1$, $I \subseteq N$, $|I| \geqslant 2$. Obviously, (N, q_N) is a polymatroid. By Lemma 2 *(iii)*, $K = N$, no polymatroid

$$\Big(N, c_N q_N + \sum_{\emptyset \neq J \subsetneq N} c_J\, r_J\Big)$$

where all c_J are nonnegative integers and $c_N \geqslant 1$ is Boolean. The main result of this note identifies (all isomorphic copies of) these polymatroids as the very excluded minors of the class of Boolean polymatroids.

Theorem 3. *For a polymatroid (N,h) with the ground set of cardinality $n \geqslant 3$ the following three assertions are equivalent.*

(i) *All proper minors of (N,h) are Boolean polymatroids.*
(ii) *(N,h) is Boolean or $h = c_N q_N + \sum_{\emptyset \neq J \subsetneq N} c_J r_J$ for integers $c_J \geqslant 0$, $c_N \geqslant 1$.*
(iii) *$\mathbb{A}h(K) \geqslant 0$, $K \subsetneq N$, and $\mathbb{A}h(N) + \mathbb{A}h(N-i) \geqslant 0$, $i \in N$.* ‡

Proof. We saw in the previous proof that every normalized function h can be uniquely written as the linear combination $\sum_{\emptyset \neq J \subseteq N} \mathbb{A}h(J)\, r_J$. Then, after an easy reflection on the rectrictons and contractions of the matroids (N, r_J), we see

$$h_{\setminus i} = \sum_{\emptyset \neq J \subseteq N-i} \big[\mathbb{A}h(J) + \mathbb{A}h(i \cup J)\big]\, r_J \quad \text{and} \quad h_{/i} = \sum_{\emptyset \neq J \subseteq N-i} \mathbb{A}h(J)\, r_J$$

for $i \in N$. All proper minors of (N,h) are Boolean if and only if $(N, h_{\setminus i})$ and $(N, h_{/i})$ are Boolean for $i \in N$ and this is equivalent to $\mathbb{A}h(J) \geqslant 0$, $J \subsetneq N$, and $\mathbb{A}h(J) + \mathbb{A}h(i \cup J) \geqslant 0$, $i \in N$, $j \subseteq N - I$ by Lemma 2 *(iii)*. We arrive at the equivalence of *(i)* and *(iii)*.

The implication *(ii)* $\Rightarrow$ *(iii)* is trivial.

Let *(iii)* be satisfied for a polymatroid (N,h). If $\mathbb{A}h(N) \geqslant 0$ then, owing to Lemma 2 *(iii)*, the polymatroid is Boolean. If $\mathbb{A}h(N) = -c_N < 0$ we set $f = h - c_N q_N$ and observe that $\mathbb{A}f(K) = \mathbb{A}h(K) \geqslant 0$ if $|K| < n-1$, $\mathbb{A}f(N-i) = \mathbb{A}h(N-i) - c_N \geqslant 0$ for $i \in N$, and $\mathbb{A}f(N) = 0$. Therefore, again by Lemma 2 *(iii)*, (N,f) is a Boolean polymatroid and $f = \sum_{\emptyset \neq J \subsetneq N} c_J r_J$ by the same lemma *(ii)*. ∎

Where $k \geqslant 1$ is an integer, a polymatroid (N,h) is k-polymatroid once $h(i) \leqslant k$, $i \in N$. Minors of k-polymatroids are k-polymatroids.

Example 4. Boolean 1-polymatroids, in fact matroids, are free, $h(I) = |I|$, up to loops and parallel elements ($i, j \in N$ are parallel if $h(i) = h(j) = h(i \cup j)$). There is one excluded polymatroid for this class namely the uniform matroid $(\{1,2,3\}, q_{123})$ of the rank 2 on a three-element set. The rank functions of the excluded polymatroids for the class of Boolean 2-polymatroids are q_{123}, $q_{123} + r_1$, $q_{123} + r_1 + r_2$, $q_{123} + r_1 + r_2 + r_3$, $q_{123} + r_{12}$, $q_{123} + r_{12} + r_3$, $2q_{123}$ and q_{1234}, cf. [3] p. 475. It is not difficult to find that there are 45 excluded minors for the class of Boolean 3-polymatroids. It is clear that the number of excluded minors for the class of Boolean k-polymatroids is finite and grows with k at least exponentially. Note that for these excluded minors $|N| \leqslant h(i) + 2 \leqslant k + 2$, $i \in N$, cf. Theorem 3.1. of [3].

Proposition 5. *Ranks of the excluded minors (N,h), $|N| = n$, of Boolean k-polymatroids are bounded by*

$$h(N) \leqslant \left\lceil \tfrac{k(k+6)}{4} + 1 \right\rceil .$$

The equality takes place if and only if

‡Somebody might prefer to write $N \setminus \{i\}$ instead of ours $N - i$.

$$\begin{array}{lll} k = 2n-3 \geqslant 3 & and & h = q_N + (n-1)\sum_{i\in N} r_i, \\ k = 2n-2 \geqslant 4 & and & h = q_N + n\sum_{i\in N} r_i, \\ k = 2n-4 \geqslant 4 & and & h = q_N + (n-2)\sum_{i\in N} r_i. \end{array}$$

Proof. If h has the form as in Theorem 3 *(ii)* then $h(i) = d(n-2) + \sum_{J\,:\,i\in J} c_J$ where $d = c_N \geqslant 1$ and

$$h(N) = d(n-1) + \sum_{\emptyset \neq J \subsetneq N} c_J \leqslant d(n-1) + \sum_{\emptyset \neq J \subsetneq N} c_J|J| = d(n-1) + \sum_{i\in N}\sum_{J\,:\,i\in J} c_J \ .$$

Hence $h(N)$ is bounded from above by

$$d(n-1) + \sum_{i\in N}\big(h(i) - d(n-2)\big) \leqslant d(n-1) + n\big[k - d(n-2)\big] = -dn^2 + (k+3d)n - d \ .$$

Here we can substitute $d = 1$. The upper bound $-n^2+(k+3)n-1$ is at most $(k+3)^2/4-1$ for k odd and $(k+2)(k+4)/4-1$ for k even. It is now easy to trace out all polymatroids attaining it. ∎

Remarks. 1. The notion of Boolean polymatroid, Definition 1, was introduced in [1] under the label 'covering hypermatroid'. We follow the terminology of [3] and [4]. In the paper [1] a necessary and sufficient condition for a polymatroid to be Boolean was formulated in the form of 4^n inequalities. In our Lemma 2 *(iii)* it is only 2^n-1 inequalities; the inequality indexed by $K = \emptyset$ is trivial.

2. In [3], Theorem 2.5., a necessary and sufficient condition for a polymatroid to be the excluded Boolean minor was found by means of inequalities indexed by all minors of the polymatroid. Theorem 3 was inspired by the considerations about the cone H_{n-1} in Corollary 3 of [2]. Our Proposition 5 repairs Proposition 3.5. of [3] which is erroneous. It also disproves Conjecture 3.6. of the same paper.

3. It may be advantageous to imagine instead of ϕ in Definition 1 the bipartite graph $(M, N, \mathcal{I})$ where $(\ell, i) \in \mathcal{I} \subseteq M \times N$ if and only if $\ell \in \phi(i)$, or to speak about a hypergraph with the vertex set M and the (hyper)edge set N. A natural assumption is $\phi(N) = M$ excluding isolated vertices of the hypergraph. Then it is easy to see from the proof of Lemma 2 that two hypergraphs give rise to the same Boolean polymatroid if and only if they are isomorphic. In symbols, if ϕ and ϕ' provide the same Boolean polymatroid on a set N then $\phi'(i) = \{\tau(\ell);\ \ell \in \phi(i)\}$, $i \in N$, for some bijection $\tau\colon \phi(N) \to \phi'(N)$. Therefore, Boolean polymatroids are cryptographic versions of hypergraphs.

References

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