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On finite sequences conditionally uniform given minima and maxima

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Abstract. Conditioning in a finite sequence $X^{(N)} = (X_1, X_2, \ldots, X_N)$ of real random variables by $\max X^{(N)}$ and by $\max X^{(N)}$ together with $\min X^{(N)}$ are considered. If $X^{(N)}$ is conditionally uniform in a very general sense with respect to a reference Borel measure ν then a shorter subsequence $X^{(n)} = (X_1, X_2, \ldots, X_n)$, $1 \leqslant n < N$, can be well approximated, in the variation distance, by a mixture of n-powers of restrictions of ν. These finite de Finetti type results are then used to obtain integral representations of infinite sequences which have all their finite subsequences conditionally uniform.

1. Introduction

In an exchangeable sequence $X_1, X_2, \ldots$ of random variables the distribution does not change when the variables are permuted arbitrarily. More complex notions of symmetry, invariance or sufficiency requirements are termed as partial exchangeability, [1, 8, 9]. The distributions of sequences under a symmetry assumption form usually a convex set and then the extreme distributions and mixture integral representations are of primary interest. Results on the mixture representations are known as de Finetti-type theorems, [3, 14, 16]. For finite sequences with a symmetry, subsequences need not be often represented but can be well approximated by mixtures. Starting from [4, 7] one speaks about finite de Finetti type theorems. By limiting procedures the infinite de Finetti-type theorems follow from the finite ones.

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In Statistics, natural kinds of symmetries of random sequences are specified by means of sufficient statistics and kernels, see [15, 5, 10]. Though there are very general results in this direction each particular case requires additional work to identify explicitely the form of mixtures and mixing probabilities. The majority of the results in this field characterize models with sufficient statistics of the form $\sum_{i=1}^{n}\Psi(X_i)$. Extremal distributions are then products of measures in a regular exponential family. Extensions of these results in the context of linear models appeared in [6].

In this work we present finite and infinite versions of de Finetti-type theorems for sequences of real random variables which are conditionally uniform given the maxima, or both, the minima and the maxima. The adjective 'uniform' is specified in a very general sense by a reference Borel measure. This approach admits a unification of discrete and continuous situations and a comprehensive treatment and generalization of cases examined before. First, aggregates of product measures modelling the conditional uniformity are examined in Section 2. Using the aggregates we obtain easily the corresponding finite de Finetti-type theorems, when conditioned on the maxima in Section 3 and when conditioned on minima and maxima in Section 5. The infinite versions are worked out in Section 4 and Section 6, respectively.

2. Aggregates of product measures

Let $(A,\mathcal{A},\lambda)$ be a measure space, $0<\lambda(A)<\infty$ and n a positive integer number. An element a of A is fixed assuming that the singleton $\{a\}$ is $\mathcal{A}$-measurable. The Dirac measure sitting in a is denoted by δ_a and the restriction of λ to a set $B\in\mathcal{A}$ by $\lambda_{|B}$, that means $\lambda_{|B}(C)=\lambda(B\cap C)$ for $C\in\mathcal{A}$.

In the first part of this section we examine the following measure on product space A^n

$$\alpha_n=\sum_{\emptyset\neq I\subset\hat{n}}\lambda(\{a\})^{|I|-1}\,\delta_a^I\times\lambda_{|A-\{a\}}^{\hat{n}-I}$$

where $\hat{n}=\{1,2,\dots,n\}$, $|I|$ is the cardinality of I and $0^0=1$ by convention. Note that the summands are mutually singular product measures. Let us observe that α_n measure of $(A-\{a\})^n$ is zero and α_n restricted to $A^k\times\{a\}\times A^{n-1-k}$ is equal to $\lambda^k\times\delta_a\times\lambda^{n-1-k}$, $0\leqslant k\leqslant n-1$. Roughly speaking, α_n is constructed by pasting the latter n measures together. We set $t_n=\alpha_n(A^n)$.

Example 1. Let $A=\{0,1,2,\dots a\}$ where a is a nonnegative integer number, let $\mathcal{A}$ be the power set of A and λ be the counting measure on A. Then α_n is the counting measure on the set of all n-tuples $(a_1,\dots,a_n)\in A^n$ having at least one coordinate equal to a, i.e. satisfying $\max_{1\leqslant j\leqslant n}a_j=a$.

Example 2. Let $(A,\mathcal{A})$ be an interval $[0,a]$ of real line $\mathbb{R}$ (a is any positive real number) endowed with the σ-algebra of its Borel subsets and let λ be the Lebesque measure on this interval. Then α_n is simply the sum of n product measures and its support is again the set of all n-tuples $(a_1,\dots,a_n)\in A^n$ with the maximum equal to a.

Having a measure on product space A^{n+k}, $k\geqslant 1$, its marginal measure on A^n will be denoted by means of the superindex (n). The bar over a nonzero finite measure will normalize it to a probability measure. Our first result estimates the variation distance between a marginal of the normalization of α_{n+k} and a power of normalized λ.

Lemma 1. *For every $n\geqslant 1$ and $k\geqslant 1$ the total variation $\|\,\overline{\alpha}_{n+k}^{(n)}-\overline{\lambda}^n\,\|$ is less or equal to $2n/(n+k)$ with the equality if and only if $\lambda(\{a\})=0$.*

Proof. Let $r=\lambda(\{a\})$ and $s=\lambda(A-\{a\})$, then obviously $t_n=[(r+s)^n-s^n]r^{-1}$ if $r>0$ and $t_n=ns^{n-1}$ otherwise. It is also easy to see that $\alpha_{n+1}^{(n)}=s\,\alpha_n+\lambda^n$.

Using the identity $t_{n+k} = s^k t_n + t_k (r+s)^n$ one can obtain $\alpha^{(n)}_{n+k} = s^k \alpha_n + t_k \lambda^n$, $k \geqslant 1$, by induction. This yields

$$\left\| \overline{\alpha}^{(n)}_{n+k} - \overline{\lambda}^n \right\| \;=\; \frac{s^k t_n}{t_{n+k}} \left\| \overline{\alpha}_n - \overline{\lambda}^n \right\|$$

what is for $r = 0$ equal to the desired bound $2n/(n+k)$. Here, the variation distance of the probability measures $\overline{\alpha}_n$ and $\overline{\lambda}^n$ was two as they are singular. If $r > 0$ the distance is strictly smaller than two. Expanding powers of $r+s$ in the above ratio and comparing corresponding terms one can see that it is bounded from above by $n/(n+k)$. □

Second part of this section is devoted to a similar result for another aggregate of product measures of λ's and δ's. Let us fix now two (different) elements a and b of A both singletons $\{a\}$ and $\{b\}$ being measurable. We set

$$\gamma_n = \sum \lambda(\{a\})^{|I|-1}\, \delta_a^I \;\times\; \lambda(\{b\})^{|J|-1}\, \delta_b^J \;\times\; \lambda^{\hat{n}-(I+J)}_{|A-\{a,b\}}$$

where the summation is extended over all ordered pairs (I, J) of nonempty and disjoint subsets of $\hat{n}$. Analogically as above, the measure γ_n is pasted together from $n(n-1)$ product measures $\lambda^k \times \delta_a \times \lambda^l \times \delta_b \times \lambda^{n-2-k-l}$ and $\lambda^k \times \delta_b \times \lambda^l \times \delta_a \times \lambda^{n-2-k-l}$, $0 \leqslant k+l \leqslant n-2$. We set $\gamma_n(A^n) = v_n$.

Example 3. Let $A = \{a, a+1, \dots, b\}$ where $a \leqslant b$ are integer numbers, let $\mathcal{A}$ be the power set of A and λ be the counting measure on A. Then γ_n is the counting measure on the set of all n-tuples $(a_1, \dots, a_n) \in A^n$ having at least one coordinate equal to a and at least one coordinate equal to b, i.e. satisfying $\min_{1 \leqslant j \leqslant n} a_j = a$ and $\max_{1 \leqslant j \leqslant n} a_j = b$.

Example 4. Let $(A, \mathcal{A})$ be an interval $[a, b]$ of the real line $\mathbb{R}$ ($a < b$ are real numbers) endowed with the σ-algebra of its Borel subsets and λ be the Lebesque measure on this interval. Then α_n is sum of $2n$ product measures and its support is again the set of all n-tuples $(a_1, \dots, a_n) \in A^n$ with the minimum equal to a and the maximum equal to b.

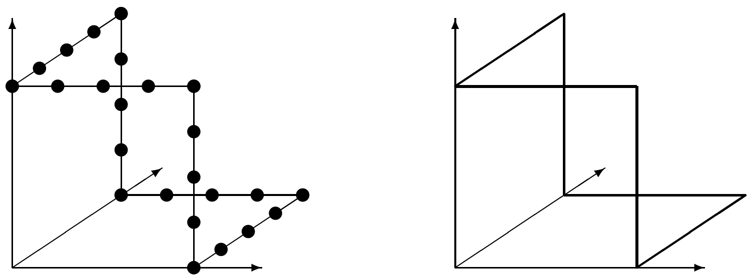

Figure 1. Examples of supports of measures γ_3.

Lemma 2. *Let $\lambda(A - \{a\}) > 0$ and $\lambda(A - \{b\}) > 0$. For every $n \geqslant 1$ and $k \geqslant 1$*

$$\left\| \overline{\gamma}^{(n)}_{n+k} - \overline{\lambda}^n \right\| \leqslant \frac{2n(n+2k-1)}{(n+k)(n+k-1)}$$

the equality being the case if and only if $\lambda(\{a\}) = \lambda(\{b\}) = 0$.

Proof. The number v_n is value of the following polynomial in three variables

$$\frac{1}{r_a r_b} \Big[(r_a + r_b + s)^n - (r_a + s)^n - (r_b + s)^n + s^n \Big]$$

when substituting $r_a = \lambda(\{a\})$, $r_b = \lambda(\{b\})$ and $s = \lambda(A - \{a, b\})$. The positivity assumptions imply $v_n > 0$. It is not difficult to establish $\gamma^{(n)}_{n+1} = s\gamma_n + \alpha_n + \beta_n$ where β_n is constructed analogically as

α_n when a is switched to b. After verification of the polynomial identity

$$v_{n+k} = s^k v_n + t_{k,b}^{\{a\}} t_{n,a} + t_{k,a}^{\{b\}} t_{n,b} + v_k (r_a + r_b + s)^n$$

where $t_{k,b}^{\{a\}} = [(r_b+s)^k - s^k] r_b^{-1}$, $t_{n,a} = [(r_a+r_b+s)^n - (r_b+s)^n] r_a^{-1}$ and $t_{k,a}^{\{b\}}$ and $t_{n,b}$ are defined correspondingly, one can obtain by induction

$$\gamma_{n+k}^{(n)} = s^k \gamma_n + t_{k,b}^{\{a\}} \alpha_n + t_{k,a}^{\{b\}} \beta_n + v_k \lambda^n \ .$$

The desired total variation takes the form

$$\left(1 - \frac{v_k}{v_{n+k}} (r_a + r_b + s)^n\right) \left\| \mu - \overline{\lambda}^n \right\|$$

where μ is a convex combination of $\overline{\alpha}_n, \overline{\beta}_n$ and $\overline{\gamma}_n$. If $r_a = 0$ and $r_b = 0$ then μ and $\overline{\lambda}^n$ are singular, $v_n = n(n-1)s^{n-2}$ and the above expression is equal to the upper bound. In the opposite case, $r_a + r_b > 0$, it suffices to prove that

$$(r_a+r_b+s)^n \frac{(r_a+r_b+s)^k - (r_a+s)^k - (r_b+s)^k + s^k}{(r_a+r_b+s)^{n+k} - (r_a+s)^{n+k} - (r_b+s)^{n+k} - s^{n+k}} \geqslant \frac{k(k-1)}{(n+k)(n+k-1)} \ .$$

Without a loss of generality we can suppose that $r_a + r_b + s = 1$. Let us write $x = r_a + s$, $y = r_b + s$, $m = n + k$ and look at the function

$$g(x,y) = m(m-1)\big[1 - x^k - y^k + (x+y-1)^k\big] - k(k-1)\big[1 - x^m - y^m + (x+y-1)^m\big]$$

that should be nonnegative for $0 \leqslant x \leqslant y \leqslant 1$ and $x + y \geqslant 1$. The expression

$$\frac{1}{mk} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) g(x,y) = (m-1)(x^{k-1} - y^{k-1}) - (k-1)(x^{m-1} - y^{m-1})$$

is, due to $(k-1)(1 - z^{m-1}) \leqslant (m-1)(1 - z^{k-1})$, $0 \leqslant z \leqslant 1$, not positive. Since $g(x,1) = 0$ for $0 \leqslant x \leqslant 1$ the nonnegativity of g is confirmed. □

3. Conditioning by maximum in a finite sequence

Let ν be in this section and the following one a nonzero Borel measure on the real half-line $[0,\infty)$ so that every closed interval $[0,a]$, $a \geqslant 0$, is of finite ν measure. The interval $[0,a]$ with its Borel subsets and the restriction $\nu_{|[0,a]}$ are taken for the measurable space $(A, \mathcal{A}, \lambda)$ and the finite measure $\alpha_{n,a}$ is constructed, $n \geqslant 1$, adding the index a to express explicitly dependence on $a \geqslant 0$. The normalization $\overline{\alpha}_{n,a}$ is a probability measure if $\nu[0,a] > 0$ otherwise it is treated as the zero measure.

Introducing the following definition we had in mind especially the Lebesgue measure or the counting measure on nonnegative integers on the place of ν, see Example 1 and Example 2, correspondingly.

Definition 1. *A finite sequence $X^{(n)} = (X_1, \dots, X_n)$, $n \geqslant 1$, of nonnegative random variables is said to be* ν-uniform given its maximum *if the distribution $P_{X^{(n)}}$ is equal to the mixture*

$$\int_{[0,\infty)} \overline{\alpha}_{n,a} \, P_{\max X^{(n)}}(da) \ .$$

An infinite sequence is ν-uniform given maxima *if every its finite subsequence is ν-uniform given its maximum.*

Equivalently, $\overline{\alpha}_{n,a}$ is assumed to be a conditional distribution of $X^{(n)}$ given $\max X^{(n)} = a$. These sequences are obviously exchangeable.

Every i.i.d. sequence distributed according to $\overline{\nu}_{|[0,a]}, \overline{\nu}_{|[0,a)}$ or $\overline{\nu}$ is ν-uniform/max provided $\nu[0,a] > 0$, $\nu[0,a) > 0$ and ν is finite, respectively. Convex combinations of the distributions of ν-uniform/max sequences are ν-uniform/max.

Proposition 1. *An initial segment $X^{(n)} = (X_1, \ldots, X_n)$ of every finite random sequence $X^{(N)} = (X_1, \ldots, X_N)$, $1 \leqslant n \leqslant N$, that is ν-uniform given its maximum has its distribution approximated by*

$$\left\| P_{X^{(n)}} - \int_{[0,\infty)} \overline{\nu}^n_{|[0,a]} \; P_{\max X^{(N)}}(da) \right\| \;\leqslant\; \frac{2n}{N} \;.$$

Proof. The variation distance does not exceed

$$\int_{[0,\infty)} \left\| \overline{\alpha}^n_{N,a} - \overline{\nu}^n_{|[0,a]} \right\| \; P_{\max X^{(N)}}(da)$$

where values of the integrated function are either zero, as the two measures are zero, or bounded by $2n/N$ owing to Lemma 1. □

4. Conditioning by maxima in infinite sequences

It is straightforward that the measure $\overline{\nu}^\infty_{|[0,a]}$ if $a \geqslant 0$ and $\nu[0,a] > 0$ and the measure $\overline{\nu}^\infty_{|[0,a)}$ if $a > 0$ and $\nu[0,a) > 0$ are distributions of infinite sequences that are ν-uniform given maxima. If ν is a finite Borel measure then also $\overline{\nu}^\infty$ is a distribution of this type. In the assertion below we show that mixtures of all these product measures exhaust distributions of this type. We prefer to work with the family of measures $\nu_{|[0,a]}$, $a \geqslant 0$. A measure $\nu_{|[0,a)}$, $a > 0$, is not in this family if and only if $\nu(\{a\}) > 0$ and $\nu(b,a) > 0$ for every $0 < b < a$. Let us denote by S_ν the set of all these positive numbers a and let, in addition, ∞ belongs to S_ν if and only if ν is finite and $\nu(b,\infty) > 0$ for all $b \geqslant 0$. The set S_ν is at most countable.

Theorem 1. *Let $X = (X_n;\ n \geqslant 1)$ be an infinite sequence of nonnegative random variables that is ν-uniform given maxima. Then there exists a finite measure κ on the real half-line $[0,\infty)$ and nonnegative numbers κ_a for $a \in S_\nu$ such that*

$$P_X \;=\; \int_{[0,\infty)} \overline{\nu}^\infty_{|[0,a]} \; \kappa(da) \;+\; \sum_{a \in S_\nu} \overline{\nu}^\infty_{|[0,a)} \, \kappa_a$$

and $\kappa[0,\infty) + \sum_{a\in S_\nu} \kappa_a = 1$. The measure κ can be chosen to satisfy $\kappa(b,c] = 0$ once $\nu(b,c] = 0$, $0 \leqslant b < c$, and then it is unique with the property. The numbers κ_a are unique as well.

This assertion can be, with some effort, adjusted to the framework of Theorem 1.1 from [8]. We prefer to have a swifty analytical proof identifying explicitly all ingredients of mixtures.

Proof. Uniqueness of the measure κ and of the numbers κ_a, $a \in S_\nu$, follows from uniqueness in the classical de Finetti theorem. In fact, let B_ν be the Borel set $[0,\infty) - \bigcup\{(b,c];\ \nu(b,c] = 0\}$ and let $\phi_\nu(a) = \max\{b \leqslant a;\ b \in B_\nu\}$, $a \geqslant 0$. The above integral can be equivalently written as the integral of the same function over $a \in B_\nu$ with respect to the ϕ_ν image of the measure κ.

Let μ_n denote the distribution of maximum of $X_1, \ldots, X_n$, $n \geqslant 1$. For every $a \geqslant 0$ the sequence $\mu_n[0,a]$, $n \geqslant 1$, does not increase. The function $a \to \lim_{n\to\infty} \mu_n[0,a]$ is nondecreasing and right-continuous and hence equal to $a \to \mu[0,a]$ for a unique Borel measure μ on $[0,\infty)$. Let $\kappa_a = \lim_{k\to\infty} \mu_k[0,a) - \mu[0,a)$, $a > 0$, and $\kappa_\infty = 1 - \mu[0,\infty)$. Note that $0 \leqslant \kappa_a \leqslant \mu(\{a\})$ for any positive a. We set $\kappa = \mu - \sum_{\infty \neq a \in S_\nu} \kappa_a \delta_a$.

If $\nu(b,c]=0$, $0\leqslant b<c$, then also $\mu_n(b,c]=0$ for any $n\geqslant 1$ whence $\mu(b,c]=0$ and $\kappa(b,c]=0$. In fact, by Lemma 1

$$\begin{aligned}\mu_n(b,c] &= P^{(n)}_{X^{(n+k)}}\big([0,c]^n-[0,b]^n\big) = \int_{[0,\infty)} \overline{\alpha}^{(n)}_{n+k,a}\big([0,c]^n-[0,b]^n\big)\,\mu_{n+k}(da)\\ &\leqslant \int_{[0,\infty)} \Big[\big(\overline{\nu}_{|[0,a]}[0,c]\big)^n-\big(\overline{\nu}_{|[0,a]}[0,b]\big)^n\Big]\,\mu_{n+k}(da)+\frac{2n}{n+k}\end{aligned}$$

and since the last integral is zero and k is arbitrary we have indeed $\mu_n(b,c]=0$. As a consequence, $\kappa_a=0$ once $a>0$, $\nu(\{a\})>0$ and $a\notin S_\nu$.

To verify the announced mixture representation of the distribution P_X it suffices to prove that for every $n\geqslant 1$ the distribution $P_{X^{(n)}}$ is equal to mixture of n-th powers of the restricted measures. This is an equality of two distributions that are both ν-uniform given maximum whence if suffices to verify the equality on sets $[0,b]^n$, $b\geqslant 0$. To summarize, we fix $n\geqslant 1$ and $b\geqslant 0$ and want to prove

$$\mu_n[0,b] \;=\; \int_{[0,\infty)} f(a)\,\mu(da) \;+ \sum_{\infty\neq a\in S_\nu} \big(f(a-)-f(a)\big)\,\kappa_a \;+\; f(\infty)\,\kappa_\infty$$

where $f(a)=\big(\overline{\nu}_{|[0,a]}[0,b]\big)^n$ for $a\geqslant 0$, $f(a-)=\big(\overline{\nu}_{|[0,a)}[0,b]\big)^n$ for every $a\in S_\nu-\{\infty\}$ (we remark that $f(a-)$ is really $\lim_{c\nearrow a} f(c)$ in this case) and $f(\infty)=\lim_{c\to\infty} f(c)$ ($f(\infty)=0$ if ν is infinite and $f(\infty)=\big(\overline{\nu}[0,b]\big)^n$ if ν is finite). Note also that $0=f(a-)-f(a)$ if $a\in S_\nu\cap[0,b]$.

The above equality is obviously valid if $\nu[0,b]=0$ because then $\mu_n[0,b]=0$ and the function f is identically equal to zero. From now on we can suppose $\nu[0,b]>0$. By Proposition 1 the integrals $\int_{[0,\infty)} f(a)\,\mu_{n+k}(da)$ converge to $\mu_n[0,b]$ once $k\to\infty$ and thus it suffices to show the convergence

$$I_k=\int_{(b,\infty)} f(a)\,\mu_k(da) \;\to\; \int_{(b,\infty)} f(a)\,\mu(da) \;+ \sum_{a\in S_\nu\cap(b,\infty)} \big(f(a-)-f(a)\big)\,\kappa_a \;+\; f(\infty)\,\kappa_\infty$$

when k approaches ∞. The expression on right will be denoted by K.

Let us observe that the function $f\leqslant 1$ is right-continuous and nonincreasing on interval (b,∞). To prove the above convergence, let $\varepsilon>0$ and $b=b_0<\ldots<b_m=\infty$, $m\geqslant 1$, be chosen to have $f(b_{j-1})\leqslant f(b_j-)+\varepsilon$, $1\leqslant j\leqslant m$. The number I_k belongs to an interval with the endpoints

$$\pm\varepsilon \;+\; \sum_{j=1}^{m} f(b_j-)\,\mu_k(b_{j-1},b_j) \;+\; \sum_{j=1}^{m-1} f(b_j)\mu_k(\{b_j\})\ .$$

Remembering that $\mu_k(c,a)\to\mu(c,a)+\kappa_a$ for $c<a$, $\mu_k(\{a\})\to\mu(\{a\})-\kappa_a$ for $a>0$, and $\mu_k(c,\infty)\to\mu(c,\infty)+\kappa_\infty$, all limit points of the sequence I_k, $k\geqslant 1$, are between

$$\pm\varepsilon \;+\; \sum_{j=1}^{m} f(b_j-)\,\mu(b_{j-1},b_j) \;+\; \sum_{j=1}^{m-1} f(b_j)\mu(\{b_j\})+\big(f(b_j-)-f(b_j)\big)\,\kappa_{b_j} \;+\; f(b_m-)\,\kappa_{b_m}$$

and thus between $\pm 2\varepsilon+K$. Note that $\big(f(a-)-f(a)\big)\,\kappa_a>0$ implies $a\in S_\nu$ and $f(a-)-f(a)\leqslant\varepsilon$ for $a\in S_\nu-\{b_1,\ldots,b_m,\infty\}$. Since ε was arbitrary, I_k converges to K. □

Remarks. 1. If ν is the counting measure on nonnegative integers Theorem 1 reduces to Example 4, p. 910, in Ressel [16], see also Example 5.2, p. 176, in Lauritzen [15].

2. The special case of Theorem 1 for ν being the Lebesque measure on $[0,\infty)$ appeared as Example 2.5, p. 210, in Diaconis and Freedman [8]. Their statement "The necessary and sufficient condition (for a sequence $X_1,X_2,\ldots$, to be a mixture over θ of sequences of independent uniform variables with the range $[0,\theta]$) is that given $M_n=\max(X_1,\ldots,X_n)$, the X_i's are independent and uniform over $[0,M_n]$, for $i=1,\ldots,n$" is not correct. Instead, the X_i's for $i=1,\ldots,n$ and $i\neq j$ should have been independent and uniform over $[0,M_n]$ *given M_n and the maximizer X_j*, for any $j=1,\ldots,n$, cf. Definition 1. This situation was also commented in [2], p. 206.

3. Gnedin [11] published two identical papers on extendibility of finite sequences which are uniform given maxima (and minima), see also [12].

5. Conditioning by minimum and maximum in a finite sequence

Let ν be from now on a fixed nonzero Borel measure on the real line $\mathbb{R}$. A closed interval $[a,b]$ of a positive ν measure, $a<b$, with its Borel subsets and the restriction $\nu_{|[a,b]}$ are taken for $(A,\mathcal{A},\lambda)$ as in the second part of Section 2 and the finite measure $\gamma_{n,a,b}$ is constructed. If $\nu[a,b]=0$, $a<b$, we set $\gamma_{n,a,b}=0$. If $a=b$ we set $\gamma_{n,a,b}=\delta_a^n$ once $\nu(\{a\})>0$ and $\gamma_{n,a,b}=0$ otherwise. Let us denote by $\mathbb{H}$ the closed half-plane of all pairs $(a,b)\in\mathbb{R}^2$ satisfying $a\leqslant b$.

Definition 2. *A finite sequence $X^{(n)}=(X_1,\dots,X_n)$, $n\geqslant 1$, of real random variables is said to be* ν-uniform given its minimum and maximum *if its distribution $P_{X^{(n)}}$ can be written as the mixture*

$$\int_{\mathbb{H}} \overline{\gamma}_{n,a,b}\, P_{(\min X^{(n)},\, \max X^{(n)})}\,(da,db)\ .$$

An infinite sequence is ν-uniform given minima and maxima *if, similarly as in Definition 1, every its finite subsequence is ν-uniform given its minimum and maximum.*

Proposition 2. *An initial segment $X^{(n)}=(X_1,\dots,X_n)$ of every finite random sequence $X^{(N)}=(X_1,\dots,X_N)$, $1\leqslant n<N$, that is ν-uniform given its minimum and maximum has the distribution approximated by*

$$\left\| P_{X^{(n)}} - \int_{\mathbb{H}} \overline{\nu}^{n}_{|[a,b]}\, P_{(\min X^{(N)},\, \max X^{(N)})}\,(da,db) \right\| \leqslant \frac{2n(2N-n-1)}{N(N-1)}\ .$$

Proof. Analogical to the proof to Proposition 1, now using Lemma 2. □

6. Conditioning by minima and maxima in infinite sequences

The measures $\overline{\nu}^{\infty}_{|[a,b]}$ for $a\leqslant b$ and $\overline{\nu}^{\infty}_{|[a,b)}$, $\overline{\nu}^{\infty}_{|(a,b]}$ and $\overline{\nu}^{\infty}_{|(a,b)}$ for $a<b$ are, if nonzero, distributions of infinite sequences that are ν-uniform given minima and maxima. If $\nu_{|[0,+\infty)}$ is a finite Borel measure then also $\overline{\nu}^{\infty}_{|[a,+\infty)}$ and $\overline{\nu}^{\infty}_{|(a,+\infty)}$ for $a\in\mathbb{R}$ are, if nonzero, distributions of this sort and symetrically with $\overline{\nu}^{\infty}_{|(-\infty,b]}$ and $\overline{\nu}^{\infty}_{|(-\infty,b)}$ for $b\in\mathbb{R}$ if $\nu_{|(-\infty,0]}$ is finite. And finally, if even the measure ν is finite then $\overline{\nu}^{\infty}$ is ν-uniform given minima and maxima. In the theorem below we claim that mixtures of these infinite-product probability measures coincide with the distributions of all ν-uniform/min&max sequences.

We need a convenient parametrization of the listed measures. We prefer the closed intervals and then the measures $\overline{\nu}^{\infty}_{|[a,b]}$ are parametrized by $\mathbb{H}$. Further, we write $a\in S^+_\nu$ if $\nu(\{a\})>0$ and $\nu(a,b)>0$ for all $b>a$, in addition, $-\infty\in S^+_\nu$ if $\nu(-\infty,0)$ is finite and $\nu(-\infty,b)$ is positive for all $b\in\mathbb{R}$. Symmetrically, we write $b\in S^-_\nu$ if $\nu(\{b\})>0$ and $\nu(a,b)>0$ for all $a<b$, in addition, $+\infty\in S^-_\nu$ if $\nu(0,+\infty)$ is finite and $\nu(a,+\infty)$ is positive for all $a\in\mathbb{R}$. The sets S^+_ν and S^-_ν are both at most countable.

Theorem 2. *Let $X=(X_n;\, n\geqslant 1)$ be an infinite sequence that is ν-uniform given minima and maxima. Then there exist a finite Borel measure κ on the closed half-plane $\mathbb{H}$, finite Borel measures κ^+_a on $(a,+\infty)$ for $a\in S^+_\nu$, finite Borel measures κ^-_b on $(-\infty,b)$ for $b\in S^-_\nu$ and nonnegative* *numbers $\kappa_{a,b}$ for $a \in S_\nu^+$, $b \in S_\nu^-$ and $a < b$ such that*

$$P_X = \int_{\mathbb{H}} \overline{\nu}^{\infty}_{|[a,b]}\, \kappa(da,db) \;+\; \sum_{a\in S_\nu^+} \int_{(a,+\infty)} \overline{\nu}^{\infty}_{|(a,b]}\, \kappa_a^+(db) \;+$$
$$+ \sum_{b\in S_\nu^-} \int_{(-\infty,b)} \overline{\nu}^{\infty}_{|[a,b)}\, \kappa_b^-(da) + \sum_{a\in S_\nu^+} \sum_{a<b\in S_\nu^-} \overline{\nu}^{\infty}_{|(a,b)}\, \kappa_{a,b} \;.$$

The measure κ can be chosen to satisfy $\kappa\big(\mathbb{R}\times(b,c]\big) = 0$ if $\nu(b,c] = 0$ and $\kappa\big([b,c)\times\mathbb{R}\big) = 0$ if $\nu[b,c) = 0$, $b < c$, and then it is unique with this property. The measures κ_a^+, $a \in S_\nu^+$, can be chosen uniquely to obey $\kappa_a^+[b_1,b_2) = 0$ if $\nu[b_1,b_2) = 0$, $a < b_1 < b_2$, and the measures κ_b^-, $b \in S_\nu^-$, can be chosen uniquely to obey $\kappa_b^-(a_1,a_2] = 0$ if $\nu(a_1,a_2] = 0$, $a_1 < a_2 < b$. The numbers $\kappa_{a,b}$ are unique as well.

Remark. The special case of Theorem 2 with ν being the Lebesque measure on $\mathbb{R}$ was proved in [13]. The same result was treated from the extendibility point of view in [11].

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Appendix: Proof of Theorem 2

Our arguments are sometimes analogical to the ones used in the proof of Theorem 1. The amount of technicalities grows, however, substantially. Where the reasoning is parallel we are concise.

1. Uniqueness follows from the classical de Finetti theorem.

2. Let μ_n be the distribution of $(\min X^{(n)}, \max X^{(n)})$, $n \geqslant 1$. Obviously $\mu_n(\mathbb{R}^2 - \mathbb{H}) = 0$. There exists a unique Borel measure μ on $\mathbb{R}^2$ such that $\mu(\mathbb{R}^2 - \mathbb{H}) = 0$ and $\mu[x,y]^2 = \lim_{n\to\infty} \mu_n[x,y]^2$, $x \leqslant y$. For every $-\infty \leqslant x < +\infty$ there exists a unique Borel measure μ_x^+ on the interval $(x,+\infty)$ such that

$$\mu_x^+(x,y] = \lim_{n\to\infty} \mu_n(x,y]^2 - \mu(x,y]^2 = \lim_{a\searrow x} \lim_{n\to\infty} \mu_n\big((x,a)\times(x,y]\big)\,, \qquad y > x\,.$$

Similarly, for $-\infty < y \leqslant +\infty$ a unique Borel measure μ_y^- on the interval $(-\infty, y)$ is given by the formula

$$\mu_y^-[x,y) = \lim_{n\to\infty} \mu_n[x,y)^2 - \mu[x,y)^2 = \lim_{b\nearrow y}\lim_{n\to\infty} \mu_n\big([x,y)\times(b,y)\big)\ , \qquad x<y\ .$$

It is not difficult to verify

$$\mu\big(\{x\}\times(a,b]\big) - \mu_x^+(a,b] = \lim_{n\to\infty} \mu_n\big(\{x\}\times(a,b]\big) \geqslant 0$$

for any $-\infty < x \leqslant a < b < +\infty$. Hence, denoting by δ_x the probability measure concentrated in the point $x\in\mathbb{R}$ the measure $\mu - \delta_x\times\mu_x^+$ is nonnegative. We see also that μ_x^+ is nonzero only for the countable set of those $x\in\mathbb{R}$ which satisfy $\mu\big(\{x\}\times\mathbb{R}\big) > 0$. Symmetrically, for μ_y^-.

3. For $-\infty \leqslant x < y \leqslant +\infty$ we set

$$\kappa_{x,y} = \lim_{a\searrow x}\lim_{b\nearrow y}\lim_{n\to\infty} \mu_n\big((x,a)\times(b,y)\big)\ .$$

We show in a moment that $\kappa_{x,y}$ is not greater than $\mu_x^+(\{y\})$, y finite, and that it is not greater than $\mu_y^-(\{x\})$, x finite; then, we can define for $-\infty \leqslant x < +\infty$ and $-\infty < y \leqslant +\infty$, respectively, the nonnegative Borel measures

$$\kappa_x^+ = \mu_x^+ - \sum_{y\in S_\nu^-\cap(x,+\infty)} \kappa_{x,y}\,\delta_y \qquad\qquad \kappa_y^- = \mu_y^- - \sum_{x\in S_\nu^+\cap(-\infty,y)} \kappa_{x,y}\,\delta_x\ .$$

To see the inequalities, let us estimate $\kappa_{x,y}$, y finite, from above via triple limit of the expression

$$\mu_n\big((x,a)\times(b,y]\big) = \mu_n\big((x,a)\times(x,y]\big) - \mu_n\big((x,a)\times(x,b]\big) = \mu_n\big((x,a)\times(x,y]\big) - \big(\mu_n(x,b]^2 - \mu_n[a,b]^2\big)$$

which converges to $\mu_x^+(x,y] - \mu_x^+(x,y)$, as desired.

4. Further, the expression

$$\kappa = \mu - \sum_{-\infty<x\in S_\nu^+} \delta_x\times\mu_x^+ - \sum_{+\infty>y\in S_\nu^-} \mu_y^-\times\delta_y + \sum_{-\infty<x\in S_\nu^+}\ \sum_{y\in S_\nu^-\cap(x,+\infty)} \kappa_{x,y}\,\delta_x\times\delta_y\ .$$

consists of four terms each one being a finite measure on $\mathbb{R}^2$. To see that κ is a nonnegative measure we are going to prove first

$$\kappa_{x,y} + \mu\big(\{x\}\times\{y\}\big) \geqslant \mu_x^+(\{y\}) + \mu_y^-(\{x\})$$

for any finite $x<y$. Here, the left-hand side is equal to the triple limit $\lim_{a\searrow x}\lim_{b\nearrow y}\lim_{n\to\infty}$ of

$$\mu_n\big((x,a)\times(b,y)\big) + \mu_n\big([x,a)\times(b,y]\big) \geqslant \mu_n\big((x,a)\times(b,y]\big) + \mu_n\big([x,a)\times(b,y)\big)$$

and the two expressions on the right converge to $\mu_x^+(\{y\})$ and $\mu_y^-(\{x\})$, correspondigly.

The nonnegativity of κ will be clear if we show the nonnegativity of

$$\kappa(\{x\}\times(a,b)) = \mu(\{x\}\times(a,b)) - \mu_x^+(a,b) + \sum_{y\in S_\nu^-\cap(a,b)} \kappa_{x,y} - \mu_y^-(\{x\})$$

for any $-\infty < x \leqslant a < b < +\infty$. This is obvious if the intersection is empty, see 2. above. If the intersection is a finite set the interval (a,b) can be divided into finite number of open subintervals and the argumentation can be done on each one separately and on its endpoints by the previous inequality. If the intersection is an infinite set one can use approximations via its finite subsets.

5. We should remark that

$$1 = \kappa(\mathbb{H}) + \sum_{a\in S_\nu^+} \kappa^+(a,+\infty) + \sum_{b\in S_\nu^-} \kappa^-(-\infty,b) + \sum_{a\in S_\nu^+}\ \sum_{a<b\in S_\nu^-} \kappa_{a,b}$$

to be sure that the mixing is by a probability measure, indeed. The expression on right is equal to $\mu(\mathbb{H}) + \mu_{-\infty}^+(\mathbb{R}) + \mu_{+\infty}^-(\mathbb{R}) + \kappa_{-\infty,+\infty}$ and the latter is equal to the triple limit of

$$\mu_n[a,b]^2 + \mu_n\big((-\infty,a)\times(-\infty,b]\big) + \mu_n\big((a,+\infty)\times(b,+\infty)\big) + \mu_n\big((-\infty,a)\times(b,+\infty)\big) = 1\ .$$

6. We need to write down the limit behaviour of the sequence μ_k for rectangles, segments and points

$$\begin{aligned}
\mu_k\big((x_1,x_0)\times(y_0,y_1)\big) &\to \mu\big((x_1,x_0)\times(y_0,y_1)\big) + \mu^+_{x_1}(y_0,y_1) + \mu^-_{y_1}(x_1,x_0) + \kappa_{x_1,y_1} && -\infty\leqslant x_1<x_0\leqslant y_0<y_1\leqslant+\infty\\
\mu_k\big(\{x\}\times(y_0,y_1)\big) &\to \mu\big(\{x\}\times(y_0,y_1)\big) - \mu^+_{x}(y_0,y_1) + \mu^-_{y_1}(\{x\}) - \kappa_{x,y_1} && -\infty< x\leqslant y_0<y_1\leqslant+\infty\\
\mu_k\big((x_1,x_0)\times\{y\}\big) &\to \mu\big((x_1,x_0)\times\{y\}\big) + \mu^+_{x_1}(\{y\}) - \mu^-_{y}(x_1,x_0) - \kappa_{x_1,y} && -\infty\leqslant x_1<x_0\leqslant y<+\infty\\
\mu_k\big(\{x\}\times\{y\}\big) &\to \mu\big(\{x\}\times\{y\}\big) - \mu^+_{x}(\{y\}) - \mu^-_{y}(\{x\}) + \kappa_{x,y} && -\infty<x<y<+\infty\ .
\end{aligned}$$

For example, to see the first convergence the definition of $\kappa_{x,y}$ can be rewritten as

$$\lim_{n\to\infty}\mu_n(x,y)^2 = \mu(x,y)^2 + \mu^-_y(x,y) + \mu^+_x(x,y) + \kappa_{x,y}$$

and then used to evaluate terms of the decomposition

$$\mu_k\big((x_1,x_0)\times(y_0,y_1)\big) = \mu_k(x_1,y_1)^2 - \mu_k(x_1,y_0]^2 - \mu_k[x_0,y_1)^2 + \mu_k[x_0,y_0]^2\ .$$

7. If $\nu(b,c]=0$, $b<c$, then also $\mu_n\big(\mathbb{R}\times(b,c]\big)=0$ for any $n\geqslant 1$. In fact, by Lemma 2

$$\begin{aligned}
\mu_n\big(\mathbb{R}\times(b,c]\big) &= P^{(n)}_{X(n+k)}\big((-\infty,c]^n-(-\infty,b]^n\big) = \int_{\mathbb{H}} \overline{\gamma}^{(n)}_{n+k,x,y}\big((-\infty,c]^n-(-\infty,b]^n\big)\,\mu_{n+k}(dx,dy)\\
&\leqslant \int_{\mathbb{H}} \Big[\big(\overline{\nu}_{|[x,y]}(-\infty,c]\big)^n - \big(\overline{\nu}_{|[x,y]}(-\infty,b]\big)^n\Big]\,\mu_{n+k}(dx,dy) + \frac{2n(n+2k-1)}{(n+k)(n+k-1)}
\end{aligned}$$

where the integrated function equals identically zero and k may be arbitrarily large. By analogical reasoning we can deduce $\mu_n\big(\mathbb{R}\times\{b\}\big)=0$, $n\geqslant 1$, as soon as $\nu\{b\}=0$.

Hence, if $\nu(b,c]=0$, $b<c$, then $\mu\big(\mathbb{R}\times(b,c]\big)=0$ and $\mu^+_x(b,c]=0$ for any $-\infty\leqslant x<+\infty$. Also, $\mu^-_y=0$ for $y\in(b,c]$ and $\kappa_{x,y}=0$ for $x<y\in(b,c]$. In turn, we see that $\kappa\big(\mathbb{R}\times(b,c]\big)=0$.

The symmetrical assertions, e.g. $\mu_n\big([b,c)\times\mathbb{R}\big)=0$ for any $n\geqslant 1$ provided $\nu[b,c)=0$, will not be listed here but used in the sequel.

Let us remark that if $\nu(\{a\})>0$ but $a\notin S^+_\nu$ then $\mu^+_a=0$ and similarly with μ^-_b. In addition, $\kappa_{a,b}=0$ once $\nu(\{a\})\nu(\{b\})>0$ but $a\notin S^+_\nu$ or $b\notin S^-_\nu$.

8. Where $n\geqslant 1$ and $x\leqslant y$, we will later work with the function

$$f(a,b) = \big(\overline{\nu}_{|[a,b]}[x,y]\big)^n = \begin{cases} \left(\frac{\nu\left([a,b]\cap[x,y]\right)}{\nu[a,b]}\right)^n & \text{if } \nu[a,b]>0\\ 0 & \text{if } \nu[a,b]=0\end{cases}$$

defined for $(a,b)\in\mathbb{H}$. We suppose $\nu[x,y]>0$ otherwise the following statements are trivial. Further let us denote

$$\Delta_a f(b) = f(a+,b)-f(a,b) \qquad\qquad \Delta^b f(a) = f(a,b-)-f(a,b)$$

and

$$\Delta^b_a f = f(a+,b-)-f(a+,b)-f(a,b-)+f(a,b)\ .$$

Obviously, $0\leqslant f\leqslant 1$ and $f(a,b)=0$ if $b<x$ or $a>y$. The expressions $f(a+,b)$, $f(a,b-)$ and $f(a+,b-)$ are defined as one-sided limits of $f(x,y)$ either when $x\searrow a$, $y\nearrow b$ or both, respectively ($f(-\infty,b)$, $f(a,+\infty)$ and $f(-\infty,+\infty)$ will be needed as the corresponding limits of f as well).

First, we examine f on $(-\infty,x]\times[y,+\infty)$. It is obviously left-continuous and nondecreasing in the first variable and right-continuous and nonincreasing in the second one. For $a\leqslant x$ the function $b\to\Delta_a f(b)$ is nonnegative on $[y,+\infty]$ and nonzero if and only if $\nu(\{a\})>0$. It is not difficult to verify that it is nonincreasing as well. Symmetrically, for $b\geqslant y$ the function $a\to\Delta^b f(a)$ is nonnegative on $[-\infty,x]$, nonzero if and only if $\nu(\{b\})>0$ and nondecreasing. Finally, $\Delta^b_a f\geqslant 0$ (for $a\leqslant x$ and $b\geqslant y$) with the strict inequality if and only if $\nu(\{a\})\nu(\{b\})>0$.

Second, we need a little insight to the behaviour of the function f on $(-\infty,z]\times[x,y]$ where $z<x$ is a fixed number such that $\nu(z,x)>0$. In contrast, it is nondecreasing in both variables here. The functions $\Delta_a f(\bullet)$, $a<z$ and $\Delta^b f(\bullet)$, $x\leqslant b\leqslant y$, need not be monotone or nonnegative.

9. To prove the mixture representation equality it suffices to verify that $P_{X^{(n)}}$ is equal to the mixture of n-powers of the restricted measures. To verify this equality one can restrict to the Borel subsets $[x,y]^n$, $x\leqslant y$ exclusively. From now on we fix $n\geqslant 1$ and $x\leqslant y$ and we want to prove

$$\mu_n[x,y]^2=\int_{\mathbb{H}} f\,d\kappa+\sum_{a\in S_\nu^+}\int_{(a,+\infty)} f(a+,\bullet)\,d\kappa_a^+ +\sum_{b\in S_\nu^-}\int_{(-\infty,b)} f(\bullet,b-)\,d\kappa_b^- +\sum_{a\in S_\nu^+}\sum_{a<b\in S_\nu^-} f(a+,b-)\,\kappa_{a,b}.$$

If $\nu[x,y]=0$ then $\mu_n[x,y]^2=0$ and the function f is identical zero on $\mathbb{H}$. The desired equality takes place trivially. Henceforth, let $\nu[x,y]>0$. If $\nu[z,x)=0$ for some $z<x$ then $\mu_n[x,y]^2=\mu_n[z,y]^2$ and $f_{z,y}=f_{x,y}$ so that it suffices to go on with the assumptions $\nu[z,x)>0$ for $z<x$ and $\nu(y,z]>0$ for $z>y$.

For further progress we cannot avoid the substitution of κ_a^+, κ_b^- and $\kappa_{a,b}$ and recasting the desired equality into

$$\begin{aligned}\mu_n[x,y]^2 \;=\; &\int_{\mathbb{H}} f\,d\mu \;+\; \sum_{-\infty<a\in S_\nu^+}\int_{(a,+\infty)} \Delta_a f\,d\mu_a^+ \;+\; \sum_{+\infty>b\in S_\nu^-}\int_{(-\infty,b)} \Delta^b f\,d\mu_b^-\\
&+\int_{\mathbb{R}} f(-\infty,\bullet)\,d\mu_{-\infty}^+ \;+\; \int_{\mathbb{R}} f(\bullet,+\infty)\,d\mu_{+\infty}^- \;+\; \sum_{-\infty<a\in S_\nu^+}\;\sum_{b\in S_\nu^-\cap(a,+\infty)} \Delta_a^b f\,\kappa_{a,b}\\
&+\sum_{-\infty<a\in S_\nu^+}\Delta_a f(+\infty)\,\kappa_{a,+\infty} \;+\; \sum_{+\infty>b\in S_\nu^-}\Delta^b f(-\infty)\,\kappa_{-\infty,b} \;+\; f(-\infty,+\infty)\,\kappa_{-\infty,+\infty}\;.\end{aligned}$$

10. By Proposition 2 we know that $I_k^{\mathbb{H}}=\int_{\mathbb{H}} f\,d\mu_k$ converges to $\mu_n[x,y]^2$ if $k\to\infty$. We partition the integration range $\mathbb{H}$ into $\mathbb{A}+\mathbb{B}^++\mathbb{B}^-+\mathbb{C}$ where $\mathbb{A}=(-\infty,x)\times(y,+\infty)$, $\mathbb{B}^+=[x,y]\times(y,+\infty)$, $\mathbb{B}^-=(-\infty,x)\times[x,y]$ and $\mathbb{C}=[x,y]^2\cap\mathbb{H}$ and will examine $I_k^{\mathbb{A}}$, $I_k^{\mathbb{B}^+}$, $I_k^{\mathbb{B}^-}$ and $I_k^{\mathbb{C}}$ separately.

For $I_k^{\mathbb{C}}$ the task is easy. Let us set $\mathbb{X}_\nu=\mathbb{H}-\bigcup\{[a,b]^2;\nu[a,b]=0\}$ and observe that this is an open set. On $\mathbb{X}_\nu\cap\mathbb{C}$ we have $f=1$ by definition and on $\mathbb{C}-\mathbb{X}_\nu$ on the contrary $f=0$. Then by 7.

$$I_k^{\mathbb{C}}=I_k^{\mathbb{C}\cap\mathbb{X}_\nu}=\mu_k(\mathbb{C}\cap\mathbb{X}_\nu)=\mu_k(\mathbb{C})\to\mu(\mathbb{C})=\mu(\mathbb{C}\cap\mathbb{X}_\nu)=I^{\mathbb{C}\cap\mathbb{X}_\nu}=I^{\mathbb{C}}$$

where the notation is self-explaining, e.g. $I^{\mathbb{C}}=\int_{\mathbb{C}} f\,d\mu$.

11. The limit behaviour of $I_k^{\mathbb{A}}$. Let $\epsilon>0$ be a fixed number. Then there are numbers $m\geqslant 1$, $x=x_0>x_1>\ldots>x_m=-\infty$ and $y=y_0<y_1<\ldots<y_m=+\infty$ such that

$$f(x_{i-1},y_{j-1})\leqslant f(x_i+,y_j-)+\epsilon\qquad f(x_i,y_{j-1})\leqslant f(x_i,y_j-)+\epsilon\qquad f(x_{i-1},y_j)\leqslant f(x_i+,y_j)+\epsilon\;\;.$$

. One can suppose in addition

$$\Delta_{x_i} f(y_{j-1})\leqslant\Delta_{x_i} f(y_j-)+\epsilon\qquad \Delta^{y_j} f(x_{i-1})\leqslant\Delta^{y_j} f(x_i+)+\epsilon\;\;.$$

Then the value $I_k^{\mathbb{A}}$ is between the two numbers

$$\begin{aligned}\pm\,\varepsilon+\sum_{i=1}^{m}\sum_{j=1}^{m} f(x_i+,y_j-)\,\mu_k\big((x_i,x_{i-1})\times(y_{j-1},y_j)\big)+\sum_{i=1}^{m-1}\sum_{j=1}^{m} f(x_i,y_j-)\,\mu_k\big(\{x_i\}\times(y_{j-1},y_j)\big)\\
+\sum_{i=1}^{m}\sum_{j=1}^{m-1} f(x_i+,y_j)\,\mu_k\big((x_i,x_{i-1})\times\{y_j\}\big)+\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} f(x_i,y_j)\,\mu_k\big(\{x_i\}\times\{y_j\}\big)\;\;.\end{aligned}$$

We partitioned $\mathbb{A}$ into open rectangles, open vertical and horizontal segments and points. The function f was estimated on these buiding blocks by its suprema and infima. We employ the fact that the supremum of f is close to its infimum on every block. Let k go to infinity. Using 6. the limit points of $I_k^{\mathbb{A}}$ are bounded by

$$\pm 2\varepsilon + I^{\mathbb{A}} + \sum_{i=1}^{m-1}\left[\sum_{j=1}^{m}\Delta_{x_i} f(y_j-)\,\mu_{x_i}^{+}(y_{j-1},y_j) + \sum_{j=1}^{m-1}\Delta_{x_i} f(y_j)\,\mu_{x_i}^{+}(\{y_j\})\right] + \ldots$$

$$+\left(\sum_{j=1}^{m} f(-\infty,y_j-)\,\mu_{-\infty}^{+}(y_{j-1},y_j) + \sum_{j=1}^{m-1} f(-\infty,y_j)\,\mu_{-\infty}^{+}(\{y_j\})\right) + \ldots$$

$$+\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\Delta_{x_i}^{y_j} f\,\kappa_{x_i,y_j} + \sum_{i=1}^{m-1}\Delta_{x_i} f(+\infty)\,\kappa_{x_i,y_m} + \sum_{j=1}^{m-1}\Delta^{y_j} f(-\infty)\,\kappa_{x_m,y_j} + f(-\infty,+\infty)\,\kappa_{x_m,y_m}$$

where $\ldots$ denotes, here and in what follows, a term symmetrical to the previous one. The upper bound does not exceed

$$2\epsilon + I^{\mathbb{A}} + \sum_{a\in S_\nu^+\cap(-\infty,x)}\int_{(y,+\infty)}\Delta_a f\,d\mu_a^+ + \ldots + \int_{(y,+\infty)} f(-\infty,\bullet)\,d\mu_{-\infty}^+ + \ldots$$

$$+\sum_{a\in S_\nu^+\cap(-\infty,x)}\ \sum_{b\in S_\nu^-\cap(y,+\infty)}\Delta_a^b f\,\kappa_{a,b} + \sum_{a\in S_\nu^+\cap(-\infty,x)}\Delta_a f(+\infty)\,\kappa_{a,+\infty} + \ldots + f(-\infty,+\infty)\,\kappa_{-\infty,+\infty}\ .$$

to be denoted by $K^{\mathbb{A}} + 2\epsilon$. Estimating the limes inferior of $I_k^{\mathbb{A}}$ goes along the same guidelines arriving to its lower bound

$$-2\epsilon + I^{\mathbb{A}} + \sum_{a\in S_\nu^+\cap(-\infty,x)}\int_{(y,+\infty)}\Delta_a f\,d\mu_a^+ + \ldots - \epsilon\sum_{x\in[-\infty,+\infty)}\mu_x^+(\mathbb{R}) - \ldots + \int_{(y,+\infty)} f(-\infty,\bullet)\,d\mu_{-\infty}^+ + \ldots$$

$$+\sum_{a\in S_\nu^+\cap(-\infty,x)}\ \sum_{b\in S_\nu^-\cap(y,+\infty)}\Delta_a^b f\,\kappa_{a,b} + \sum_{a\in S_\nu^+\cap(-\infty,x)}\Delta_a f(+\infty)\,\kappa_{a,+\infty} + \ldots + f(-\infty,+\infty)\,\kappa_{-\infty,+\infty}\,.$$

and finally to $K^{\mathbb{A}} - 4\epsilon$. Since ϵ was arbitrary positive we see that $I_k^{\mathbb{A}}$ converges to $K^{\mathbb{A}}$.

12. The limit behaviour of $I_k^{\mathbb{B}^-}$. We examine first the asymptotics of the integrals $I_k^{\mathbb{B}_z^-}$ where $\mathbb{B}_z^- = (-\infty,z)\times[x,y] \subset \mathbb{B}^-$. Similarly as in 12. $\mathbb{B}_z^-$ will be partitioned into blocks via numbers $z = x_0 > x_1 > \ldots > x_m = -\infty$, $m \geqslant 1$, and $x = y_0 < y_1 < \ldots < y_\ell = y$, $\ell \geqslant 1$, such that the supremum of f on each block is smaller than the infimum of f on the same block plus ϵ. The same can be supposed to be valid for $\Delta_{x_i} f(\bullet)$ and $\Delta^{y_j} f(\bullet)$ on the line segments.

Then the value $I_k^{\mathbb{B}_z^-}$ is between the two numbers

$$\pm\,\epsilon + \sum_{i=1}^{m}\sum_{j=1}^{\ell} f(x_i+,y_j-)\,\mu_k\big((x_i,x_{i-1})\times(y_{j-1},y_j)\big) + \sum_{i=1}^{m-1}\sum_{j=1}^{\ell} f(x_i,y_j-)\,\mu_k\big(\{x_i\}\times(y_{j-1},y_j)\big)$$

$$+\sum_{i=1}^{m}\sum_{j=0}^{\ell} f(x_i+,y_j)\,\mu_k\big((x_i,x_{i-1})\times\{y_j\}\big) + \sum_{i=1}^{m-1}\sum_{j=0}^{\ell} f(x_i,y_j)\,\mu_k\big(\{x_i\}\times\{y_j\}\big)\ .$$

If k goes to infinity then we are still between

$$\pm 2\epsilon + I^{\mathbb{B}_z^-} + \sum_{i=1}^{m-1}\left[\sum_{j=1}^{\ell}\Delta_{x_i}f(y_j-)\,\mu_{x_i}^+(y_{j-1},y_j)+\sum_{j=0}^{\ell}\Delta_{x_i}f(y_j)\,\mu_{x_i}^+(\{y_j\})\right]$$
$$+\left(\sum_{j=1}^{\ell}f(-\infty,y_j-)\,\mu_{-\infty}^+(y_{j-1},y_j)+\sum_{j=0}^{\ell}f(-\infty,y_j)\,\mu_{-\infty}^+(\{y_j\})\right)$$
$$+\sum_{j=0}^{\ell}\left[\sum_{i=1}^{m}\Delta^{y_j}f(x_i+)\,\mu_{y_j}^-(x_i,x_{i-1})+\sum_{i=0}^{m-1}\Delta^{y_j}f(x_i)\,\mu_{y_j}^-(\{x_i\})\right]$$
$$+\sum_{i=1}^{m-1}\sum_{j=0}^{\ell}\Delta_{x_i}^{y_j}f\kappa_{x_i,y_j}+\sum_{j=0}^{\ell}\Delta^{y_j}f(-\infty)\kappa_{-\infty,y_j}$$

Since ϵ was arbitrary, $I_k^{\mathbb{B}_z^-}$ converges to

$$\epsilon + I^{\mathbb{B}_z^-} + \sum_{a\in S_\nu^+\cap(-\infty,z)}\int_{(-\infty,y]}\Delta_a f\,d\mu_a^+ + \sum_{b\in S_\nu^-\cap(-\infty,y]}\int_{(-\infty,b\wedge z)}\Delta^b f\,d\mu_b^-$$
$$+\int_{(-\infty,y]}f(-\infty,\bullet)\,d\mu_{-\infty}^+ + \sum_{a\in S_\nu^+\cap(-\infty,x)}\ \sum_{b\in S_\nu^-\cap(a,y]}\Delta_a^b f\,\kappa_{a,b} + \sum_{b\in S_\nu^-\cap(-\infty,y]}\Delta^b f(-\infty)\,\kappa_{-\infty,b}\ .$$

Moving $z\nearrow x$ we can compute the limit of $I_k^{\mathbb{B}^-}$. It is given by the previous expression with z replaced by x.

13. Combining the limits of $I_k^{\mathbb{C}}$ from 10. $I_k^{\mathbb{A}}$ from 11. and $I_k^{\mathbb{B}^-}$ from 12. together with $I_k^{\mathbb{B}^+}$ (obtained by symmetry) we have the desired equality from 9. $\square$