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EXTREME CONVEX SET FUNCTIONS WITH MANY NONNEGATIVE DIFFERENCES [1]

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Abstract. Where N is a finite set of the cardinality n and $\mathcal{P}$ the family of all its subsets, we study real functions on $\mathcal{P}$ having nonnegative differences of orders $n-2$, $n-1$ and n. Nonnegative differences of zeroth order, first order, and second order may be interpreted as nonnegativity, nonincreasingness and convexity, respectively. If all differences up to order n of a function are nonnegative, the set function is called completely monotone in analogy to the continuous case. We present a discrete Bernstein-type theorem for these functions with Möbius inversion in the place of Laplace one. Numbers of all extreme functions with nonnegative differences up to the orders n, $n-1$ and $n-2$, which is the most sophisticated case, and their Möbius transforms are found. As an example, we write out all extreme nonnegative nondecreasing and semimodular functions to the set N with four elements.

1. Introduction

Let N be a finite basic set of the cardinality $n = |N|$ and $\mathcal{P}_r^s(N)$, $0 \le r, s \le n$, be the family of all the subsets K of N with cardinalities bounded by the integers r and s; $r \le |K| \le s$. For the sake of notational simplicity we frequently omit the reference to N in this notation and shorten also $\mathcal{P}_r^r$ to $\mathcal{P}_r$ and $\mathcal{P}_0^n$ to $\mathcal{P}$.

The algebra $R^{\mathcal{P}(N)}$ of all real functions f defined on $\mathcal{P}(N)$ admits a discrete differential calculus, see [1]. The difference (Δ) and the sum (σ) operators used in this paper are defined as follows

$$\begin{aligned} \Delta_K^L f &= \sum_{K \subset I \subset L} (-1)^{|I-K|} f(I), \\ \sigma_K^L f &= \sum_{K \subset I \subset L} f(I), \qquad\qquad K \subset L \subset N. \end{aligned}$$

Accordingly, the differences $\Delta_K^K f = f(K)$, $K \subset N$, are of zeroth order, the differences $\Delta_K^{iK} f = f(K) - f(iK)$, $K \subset N$, $i \in N - K$, of first order, the differences $\Delta_K^{ijK} f = f(K) - f(iK) - f(jK) + f(ijK)$, $K \subset N$, $ij \subset N - K$, $i \ne j$, of second order, etc. Note that no distinction is made between elements and singletons on N and that the sign of union is omitted. The difference $\Delta_K^L - \Delta_{iK}^{iL}$, $K \subset L \subset N$, $i \in N - L$, of two differences of the same order $|L-K|$ equals to the difference Δ_K^{iL} of the subsequent order $|L-K|+1$ and analogically for the sums ($\sigma_K^L + \sigma_{iK}^{iL} = \sigma_K^{iL}$).

Our main attention will be devoted to these (convex) cones

$$H_r = \{f \in R^{\mathcal{P}};\ \Delta_K^L f \ge 0,\ K \subset L \subset N,\ |L-K| \le r\}, \qquad\qquad 0 \le r \le n,$$

and especially to their extreme functions and rays (for the related notions see [4]). For example, functions from H_0 are nonnegative, functions from $H_1 \subset H_0$ are locally nonincreasing (and hence also globally nonincreasing: $f(K) \ge f(L)$, $K \subset L \subset N$), functions from $H_2 \subset H_1$ are, in addition, locally convex (and hence also, by induction, globally convex: $f(K) + f(L) \le f(K \cap L) + f(K \cup L)$, $K, L \subset N$), etc.

The cones H_r, $0 \le r \le n$, are pointed and since they are defined by a finite number of inequalities with integer coefficients, they have a finite number of extreme rays all of which contain an integer-valued

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(extreme) function. The cone H_0 is spanned trivially by the functions δ_K, $K \subset N$, where $\delta_K(I)$ equals one if $I = K$ and zero otherwise. It is also easy to prove that the extreme functions of H_1 have the form $u \sum_{K \in \mathcal{M}} \delta_K$, where u is a positive real number and $\mathcal{M} \subset \mathcal{P}(N)$ is any nonempty hereditary family of subsets of N, that is $K \in \mathcal{M}$ if $K \subset L$ for some $L \in \mathcal{M}$.

In [6] the extreme integer-valued functions of the cone

$$\{f \in R^{\mathcal{P}};\ \ f(\emptyset) + f(N) - f \in H_2\}$$

of all nonnegative nondecreasing and semimodular ($f(K) + f(L) \geq f(K \cap L) + f(K \cup L)$, $K, L \subset N$; some authors use also the term submodular) functions were characterized by means of expansions and their connectivity. Examples of these functions are the rank functions of matroids; they are extreme in the above cone if and only if the matroid is connected after deleting all its loops. For other results on semimodular functions we refer the reader to [3], [2] and [8]. Another linear transformation of H_2 gives the cone

$$\{g \in R^{\mathcal{P}};\ g(I) = f(N - I) - f(N),\ I \subset N,\ f \in H_2\}$$

consisting of the normalized ($g(\emptyset) = 0$) nonnegative and convex set functions. Its extreme functions, which are interesting from the game theory point of view, were described in [9] to have special canonical decompositions.

The aim of this paper is to examine the cones H_n, H_{n-1}, H_{n-2}. We shall present an analogy between the cone H_n and a classical continuous result (Section 2). Let us remind that an infinitely differentiable real function f defined on the open interval $(0, \infty)$ is called completely monotone if $(-1)^r f^{(r)} \geq 0$, $r = 0, 1, 2, \ldots$, i.e. if it is nonnegative, nonincreasing, convex, etc., and that the extreme completely monotone bounded functions are exponentials ($f(x) = ue^{-\alpha x}$, $u > 0$, $0 \leq \alpha < \infty$), see [7], [12]. Bernstein's theorem claims that every completely monotone bounded function is the Laplace transform ($f(x) = \int_0^\infty e^{-\alpha x} d\mu(\alpha)$) of a uniquely determined finite Borel measure on the one-point compactification of the interval $(0, \infty)$. In the discrete case presented here the extreme functions of the cone H_n are $\sum_{I \subset K} \delta_I$, $K \subset N$, and the role of Laplace transform is played, as can be expected, by Möbius inversion.

Though the cone H_{n-1} has only one extreme ray more than H_n, the case of H_{n-2} is technically much more complicated and occupies the greater part of the paper. We present the number of extreme rays of H_{n-2} and the complete list of their Möbius transforms. At the end (Section 5) we show as an example all extreme nonnegative nondecreasing and semimodular four-element-set functions. These functions were our main motivation in connection with an entropic approach to the probabilistic representations of conditional independence structures, as in [5].

2. A discrete Bernstein-type theorem

Instead of working with the cones H_r for r close to n it is advantageous to relate them to these cones given by the sum operators

$$C_r = \{f \in R^{\mathcal{P}};\ \sigma_K^L f \geq 0,\ K \subset L \in \mathcal{P}_r^n(N)\}, \quad 0 \leq r \leq n.$$

Lemma 1. *The Möbius transform $T : R^{\mathcal{P}} \to R^{\mathcal{P}}$ in the form $Tf(K) = \Delta_K^N f$, $K \subset N$, is a bijection between H_r and C_{n-r}, $0 \leq r \leq n$.*

Proof. Let us recall [1] that the inverse Möbius transform is given by $T^{-1}f(K) = \sigma_K^N f$, $K \subset N$. Our claim follows from the identity $\Delta_K^L T^{-1} = \sigma_K^{K(N-L)}$, $K \subset L \subset N$, which can be verified in this way

$$\begin{aligned}\Delta_K^L T^{-1} f &= \sum_{K \subset I \subset L} (-1)^{|I-K|} \sigma_I^N f = \sum_{K \subset I \subset L} (-1)^{|I-K|} \sum_{I \subset J} f(J) = \sum_{K \subset J} f(J) \sum_{K \subset I \subset J \cap L} (-1)^{|I-K|} = \\ &= \sum_{K \subset J \subset K(N-L)} f(J) = \sigma_K^{K(N-L)} f, \qquad\qquad f \in R^{\mathcal{P}(N)}.\end{aligned}$$

□

Proposition 1. *The functions* $\delta_K, K \in \mathcal{P}^n_{r+1}(N)$, *together with the cone*

$$D_r = \{f \in R^{\mathcal{P}}; \sigma^L_K f \geq 0, K \subset L \in \mathcal{P}^r_r(N) \text{ and } f(J) = 0,\ |J| > r\}$$

span C_r, $0 \leq r \leq n$, *and, moreover, every function* $f \in C_r$ *can be uniquely decomposed into* $g + \sum u_K \delta_K$ *where* $g \in D_r$, u_K *are nonnegative real numbers and the sumation is extended over* $\mathcal{P}^n_{r+1}(N)$.

Proof. First we observe that if $f \in C_r$ then $g = f - \sum f(K)\delta_K$ (here again $|K| > r$) belongs to D_r and thus f can be expressed uniquely in the desired form with $u_K = f(K) = \sigma^K_K f \geq 0,\ K \in \mathcal{P}^n_{r+1}(N)$.

The very nontrivial part to be proved is that D_r is a subcone of C_r, i.e. $\sigma^L_K f \geq 0$ as soon as $f \in D_r$ and $K \subset L \in \mathcal{P}^n_r(N)$. We shall proceed by induction on $s = |L - K|$.

If $s = 0$ we have $\sigma^L_K f = f(L)$ what is for $f \in D_r$ nonnegative if $|L| = r$ and zero if $|L| > r$. Now let our claim is valid for all $f \in D_r$ and $K \subset L \in \mathcal{P}^n_r(N)$ such that $0 \leq |L - K| \leq s < n$ and consider $|L - K| = s + 1 \geq 1$. If $|L| = r$ then $\sigma^L_K f \geq 0$ by the definition of D_r. If $|L| > r$ then $|L - i| \geq r$ for any fixed $i \in L - K$ and by the induction argument $\sigma^{L-i}_K f \geq 0$ and $\sigma^L_{iK} f \geq 0$. Adding these two inequalities we conclude $\sigma^L_K f \geq 0$. □

Corollary 1. *An extreme function of the cone* C_r, $0 \leq r \leq n$, *is either extreme in* D_r *or a positive multiple of the function* δ_K, $K \in \mathcal{P}^n_{r+1}(N)$.

On account of the previous assertions we see that the task of finding extreme rays of the cones H_n, H_{n-1} and H_{n-2} is reduced to that one of finding extreme rays of D_0, D_1 and D_2, respectively. The first two cases can be solved swifty.

Corollary 2. *The extreme completely monotone set functions* $h_K = T^{-1}\delta_K = \sum_{I \subset K} \delta_I$, $K \subset N$, *form a base of* $R^{\mathcal{P}(N)}$ *and span the cone* H_n. *Hence, to every completely monotone set function* f *there exists a unique finite measure* μ *defined on the algebra of subsets of* $\mathcal{P}(N)$ *such that*

$$f(L) = {}_{\mathcal{P}(N)}\!\int h_K(L) d\mu(K) = \sum_{K \subset N} h_K(L)\mu(\{K\}), \qquad L \subset N.$$

Corollary 3. *The cone* H_{n-1}, $n \geq 2$, has $2^n + 1$ *extreme rays which are specified by the functions* h_K, $K \subset N$, *and by the function* $(n-1)\delta_\emptyset + \sum_{i \in N} \delta_i$ *(if* $n = 1$ *then* H_{n-1} *has two extreme rays).*

Proof. If we decompose the cone

$$D_1 = \{f \in R^{\mathcal{P}(N)};\ f(i) \geq 0,\ f(i) + f(\emptyset) \geq 0,\ i \in N, \text{ and } f(K) = 0,\ |K| \geq 2\}$$

into the union of two subcones

$$D_1 = \{f \in D_1;\ f(\emptyset) \geq 0\} \cup \{f \in D_1;\ f(\emptyset) \leq 0\}$$

we see that the functions δ_K, $K \in \mathcal{P}^1_0(N)$, span the first one and that any function f from the second one takes the form

$$f = |f(\emptyset)|\varepsilon_\emptyset + \sum_{i \in N} (f(i) + f(\emptyset))\ \delta_i\ ,$$

where $\varepsilon_\emptyset = -\delta_\emptyset + \sum_{i \in N} \delta_i \in D_1$. It remains to mention that $T^{-1}\varepsilon_\emptyset = (n-1)\delta_\emptyset + \sum_{i \in N} \delta_i$. □

3. The cone $\mathbf{D_2^+}$

In this and the following section we analyze the pointed cone ($n \geq 2$)

$$D_2 = \{f \in R^{\mathcal{P}(N)};\ \ f(ij) \geq 0,\ \ f(ij) + f(i) \geq 0,\ \ f(ij) + f(i) + f(j) + f(\emptyset) \geq 0,$$
$$i \in N,\ j \in N - i,\ \text{and}\ f(K) = 0,\ |K| \geq 3\}$$

and namely its two closed subcones D_2^+ and D_2^- defined according to the sign of the value $f(\emptyset)$ as before.

The attentive reader will find that the lemma below is not used in its full strength in the sequel. Nevertheless, we believe it is interesting on its own right in this context.

Lemma 2. *If $n \geq 3$ then the cone $D_2^+ \cap D_2^- = \{f \in D_2;\ f(\emptyset) = 0\}$ has $\frac{1}{2}n(n+3)$ extreme rays corresponding to the functions δ_K, $K \in \mathcal{P}_1^2$, and $\varepsilon_i = -\delta_i + \sum_{j \in N-i} \delta_{ij}$, $i \in N$ (for $n = 2$ it has only 4 extreme rays).*

Proof. If f is any element of D_2 and $f(\emptyset) = 0$ we set $I = \{i \in N;\ f(i) < 0\}$ and write

$$\begin{aligned} f &= \sum_{i \in N-I} f(i)\,\delta_i + \sum_{i \in I} f(i) \left(-\varepsilon_i + \sum_{j \in N-i} \delta_{ij} \right) + \sum_{K \in \mathcal{P}_2(N)} f(K)\,\delta_K = \\ &= \sum_{i \in N-I} f(i)\,\delta_i + \sum_{i \in I} |f(i)|\,\varepsilon_i + \sum_{i \in I} \sum_{j \in N-I} \left(\sigma_i^{ij} f\right)\,\delta_{ij} + \sum_{K \in \mathcal{P}_2(I)} \left(\sigma_\emptyset^K f\right)\,\delta_K + \sum_{K \in \mathcal{P}_2(N-I)} f(K)\,\delta_K\ , \end{aligned}$$

what shows that the distinguished functions span $D_2^+ \cap D_2^-$. It is also easy to see that for $n \geq 3$ neither a function δ_K, $K \in \mathcal{P}_1^2(N)$, nor a function ε_i, $i \in N$, is a conical combination (that is a linear combination with nonnegative coefficients) of the remaining ones. For $n = 2$ and $N = \{i, j\}$ note that $\delta_i + \varepsilon_i = \delta_{ij}$. $\square$

Proposition 2. *The functions δ_K, $K \in \mathcal{P}_0^2$, ε_i, $i \in N$, $\varphi = \sum_{|K| \leq 2} (-1)^{|K|} \delta_K$ and $\varphi_I = \delta_\emptyset - \sum_{i \in I} \delta_i + \sum \delta_K$ (the last sum is extended over all two-element subsets K of N not contained in $N - I$; see Fig.1), $I \in \mathcal{P}_2^{n-2}$, span the cone D_2^+.*

Proof. Let us introduce the notation

$$f \succeq g \quad \Leftrightarrow \quad (\,\forall K \subset L \in \mathcal{P}_2:\ \ \sigma_K^L f \geq \sigma_K^L g\,), \qquad\qquad f, g \in R^{\mathcal{P}},$$

for the ordering according to the values of the considered sum operators. Then a function f belongs to D_2^+ if and only if its support is included in $\mathcal{P}_0^2$, $f \succeq 0$ and $f(\emptyset) \geq 0$. Our endeavour is to express every $f \in D_2^+$ as a conical combination of the listed functions (note that they trivially belong to this cone).

Let $f \in D_2^+$. If for some $L \in \mathcal{P}_2$ the number $a_L(f) = \min_{K \subset L}\{\sigma_K^L f\}$ is positive then we can write $f = a_L(f)\delta_L + g$ where $f \succeq g \succeq 0$, $f(\emptyset) = g(\emptyset) \geq 0$ (that means $g \in D_2^+$) and $a_L(g) = 0$. Having proved that g is the desired conical combination we obtain the same assertion for f too. So, there is no loss of generality if we would suppose from the very beginning, and we shall do it, that $a_L(f) = 0$. Repeating this idea finitely many times we can pass to the functions $f \in D_2^+$ with the property

(1) $$\forall L \in \mathcal{P}_2:\ \ a_L(f) = \min_{K \subset L}\{\sigma_K^L f\} = 0$$

and, moreover (working with δ_i, ε_i and φ instead of δ_L, respectively), with the properties

(2) $$\forall i \in N:\ \ b_i(f) = \min_{j \in N-i}\{\sigma_i^{ij} f, \sigma_\emptyset^{ij} f\} = 0,$$

(3) $$\forall i \in N:\ \ c_i(f) = \min_{j \in N-i}\{\sigma_{ij}^{ij} f, \sigma_j^{ij} f\} = 0,$$

(4) $$d(f) = \min_{L\in\mathcal{P}_2}\{f(\emptyset), f(L)\} = 0.$$

We derive now four consequences of these properties. Let us start with

(5) $$\forall i \in N: \quad f(i) \le 0.$$

In fact, if $f(i) > 0$ for some $i \in N$ then by (2) there exists $j \in N - i$ such that $\sigma^{ij}_{\emptyset} f = 0$ (as we know that $\sigma^{ij}_i f \ge f(i) > 0,\ j \in N$) and then $\sigma^{ij}_j f = f(ij) + f(j) = -f(i) - f(\emptyset) < 0$, a contradiction to the fact $f \in D^+_2$. Further, let us consider the set $I = I(f) = \{i \in N;\ f(i) < 0\}$ and notice that if $f(\emptyset) > 0$ then on account of (4) $f(ij) = 0$ for some $i \in N$ and $j \in N - i$ and then the cardinality of I is at most $n-2$ as $f(i) = f(j) = 0$ (combine (5) with the nonnegativity of $\sigma^{ij}_i f$ and $\sigma^{ij}_j f$). Applying (1) we have immediately

(6) $$\forall i \in I \quad \forall j \in N - I: \quad f(i) + f(ij) = 0\ ,$$

(7) $$\forall L \in \mathcal{P}_2(N - I): \quad f(L) = 0$$

and due to (3)

(8) $$\forall i \in I: \quad f(i) + f(\emptyset) \ge 0;$$

indeed, note that for $i \in I$ we have $f(ij)(f(j) + f(ij)) = 0$ choosing properly $j \in N - i$ and since $f(ij) > 0$ (otherwise $\sigma^{ij}_i f < 0$) we arrive at $f(i) + f(\emptyset) = \sigma^{ij}_{\emptyset} f \ge 0$.

We proceed by induction on $s = |I(f)|,\ 0 \le s \le n$, to prove that every function $f \in D^+_2$ fulfilling (1)–(4) (and then also (5)–(8)) is a conical combination of $\delta_\emptyset$ and $\varphi_J,\ J \in \mathcal{P}^{n-2}_2$.

If $I(f) = \emptyset$ then $f = f(\emptyset)\delta_\emptyset$ by (7) and if $I(f) = i$ is a singleton then $f = -f(i)\varepsilon_i + f(\emptyset)\delta_\emptyset$ by (6) and (7). A look at (3) brings $0 = c_i(f) = -f(i)$ and thus $I(f)$ is never a singleton in this situation. We point out that if $f(\emptyset) = 0$ then by (5) and (8) $I(f) = \emptyset$ and in the following induction step $f(\emptyset) > 0$ can be supposed. Then it is enough to proceed only until $s \le n - 2$.

Let $f \in D^+_2$ enjoy (1)–(4), $f(\emptyset) > 0$ and $2 \le s = |I(f)| \le n-2$. Knowing from (6) that $f(ij) \ge -f(i) > 0$ for $i \in I$ and $j \in N - i$ we can write

$$0 < e = \min_{i\in I,\ j\in N-i}\ \{f(ij)\}$$

and again due to (6) and (8) we get $f(\emptyset) \ge f(ij),\ i \in I,\ j \in N - I$, and thus $e \le f(\emptyset)$. As the reader expects the key of the induction step is the decomposition $f = e\varphi_I + g$. Namely, it is easy to see that $0 \le \sigma^L_K g \le \sigma^L_K f,\ \emptyset \ne K \subset L \in \mathcal{P}_2$, and further that for any $i, j \in N$ distinct

$$\begin{aligned} \sigma^{ij}_{\emptyset} g &= \sigma^{ij}_{\emptyset} f \ge 0, && ij \subset I,\\ &= \sigma^{ij}_{\emptyset} f - e = f(ij) + f(\emptyset) - e = f(\emptyset) - e \ge 0, && ij \subset N - I,\\ &= \sigma^{ij}_{\emptyset} f - e = f(ij) + f(i) + f(\emptyset) - e = f(\emptyset) - e \ge 0, && i \in I,\ j \in N - I \end{aligned}$$

(in the last line we used (6) and in the last but one (7)). We have obtained $f \succeq g \succeq 0$ and $g(\emptyset) = f(\emptyset) - e \ge 0$ what implies that the function g belongs to D^+_2 and enjoys (1)–(4).

The most important point is that the inclusion $I(g) \subset I(f)$ is strict. In fact, $g(ij) = 0$ for properly chosen $i \in I(f),\ j \in N - i$ and then $g(i) = 0$ as $0 \le \sigma^{ij}_i g = g(i)$ and $g(i) \le 0$ by (5). By the induction assumption we can express the function g as a conical combination of $\delta_\emptyset$ and $\varphi_J,\ J \in \mathcal{P}^{n-2}_2$, and then we can manage it with the function f too. $\square$

Lemma 3. *The cone D_2^+, $n \geq 3$, has $2^n + \frac{1}{2}n(n-1)$ extreme rays (for $n = 2$ six extreme rays).*

Proof. We have to verify that all functions listed in Proposition 2 are extreme. Trivially the extreme functions of $D_2^+ \cap D_2^-$ are also extreme in D_2^+. Extremity of the function $\delta_\emptyset$ is transparent too.

Let us assume the function φ in the form

$$\varphi = \sum_{|K|\leq 2} u_K \delta_K + \sum_{i \in N} v_i \varepsilon_i + \sum_{2 \leq |I| \leq n-2} w_I \varphi_I$$

where all numbers u_K, v_i and w_I are nonnegative. From $\sigma_\emptyset^L \varphi = 0$, $L \in \mathcal{P}_2$, we deduce that all considered u_K and w_I are equal to zero. As the equality $\varphi = \sum_{i \in N} v_i \varepsilon_i$ is not valid (substitute $\emptyset$ for the argument) we conclude that φ is extreme.

Let $I \in \mathcal{P}_2^{n-2}$ and assume the combination

$$\varphi_I = t\varphi + \sum_{|K|\leq 2} u_K \delta_K + \sum_{i \in N} v_i \varepsilon_i + \sum_{2 \leq |J| \leq n-2} w_J \varphi_J$$

with nonnegative t, u_K, v_i and w_J. The observations $a_L(\varphi_I) = 0$, $L \in \mathcal{P}_2$, $b_i(\varphi_I) = 0$, $i \in N$, $c_i(\varphi_I) = 0$, $i \in N$, and $d(\varphi_I) = 0$ imply

$$\varphi_I = u_\emptyset \delta_\emptyset + \sum_{2 \leq |J| \leq n-2} w_J \varphi_J.$$

Since $\varphi_I(L) = 0$ for all $L \in \mathcal{P}_2(N - I)$ we obtain $w_J = 0$ if $J \not\subset I$. If $j \in J \subset I$ and $i \in I - J$ then $\sigma_\emptyset^{ij} \varphi_I = 0$ gives $w_J = 0$ and thus $\varphi_I = u_\emptyset \delta_\emptyset + w_I \varphi_I$. Finally, the only nonzero coefficient of our combination is $w_I = 1$ what means that the function φ_I is extreme. □

4. The cone $\mathbf{D_2^-}$

In this section we will perform tedious investigation of the subcone D_2^- of the cone D_2 specified by the inequality $f(\emptyset) \leq 0$.

Proposition 3. *The cone D_2^- is spanned beside the functions δ_K, $K \in \mathcal{P}_1^2$, and ε_i, $i \in N$, by the following functions (see Fig.2 and Fig.3)*

$$\begin{aligned}
\gamma &= -2\delta_\emptyset + \sum_{i \in N} \delta_i, & \\
\beta_I &= -\delta_\emptyset + \sum_{i \in I} \delta_i + \sum_{K \in \mathcal{P}_2(N-I)} \delta_K, & I \in \mathcal{P}_1^{n-1}, \\
\alpha_{J,K} &= -2\delta_\emptyset + \sum_{j \in J} \delta_j + 2\sum_{k \in K} \delta_k + \sum_{j \in J}\sum_{i \in N-JK} \delta_{ij} + 2 \sum_{L \in \mathcal{P}_2(N-JK)} \delta_L, & (J,K) \in \mathcal{Q}(N),
\end{aligned}$$

where $\mathcal{Q}(N) = \{(J,K) \in \mathcal{P}(N)^2;\ J \in \mathcal{P}_3^{n-2},\ J \cap K = \emptyset,\ \emptyset \neq K \neq N - J\}$.

Proof. A trivial reasoning supported by a look at Fig.2 and Fig.3 assures that all mentioned functions belong to D_2^- and we are going to express any function $f \in D_2^-$ as a conical combination of them. The first part of this demonstration is similar to the foregoing one. Loosing no generality we can assume that our function $f \in D_2^-$ satisfies the properties (1), (2) and (3) and in addition (by an analogical consideration for the function γ instead of $\delta_\emptyset$)

(4) $$d(f) = \min_{i \in N, j \in N-i}\{-\tfrac{1}{2}f(\emptyset), \sigma_i^{ij} f\} = 0.$$

First, we derive some consequences of these four properties. We start with

$$\forall i \in N : \quad f(i) \geq 0. \tag{5}$$

In fact, if $f(i) < 0$ for some $i \in N$ then by (3) there exists $j \in N - i$ such that $f(ij) = 0$ or $f(j) + f(ij) = 0$ having contradictions to the facts $\sigma_i^{ij} f \geq 0$ and $\sigma_\emptyset^{ij} f = f(i) + f(\emptyset) \geq 0$, correspondingly. We introduce the set $I = I(f) = \{i \in N;\ f(i) > 0\}$ and notice that if $f(\emptyset) < 0$ then $I \neq \emptyset$ (if $I = \emptyset$ then using (3) $f(L) = 0$ for some $L \in \mathcal{P}_2$ and then $0 \leq \sigma_\emptyset^L f = f(\emptyset) \leq 0$) and $I \neq N$ by (4). Applying (1) and (2) we have

$$\forall i \in I \quad \forall j \in N - I : \quad f(ij) = -f(\emptyset) - f(i) \geq 0 \tag{6}$$

(if $f(ij) > 0$ it is trivial; if $f(ij) = 0$ then (1) implies $\sigma_\emptyset^{ij} f = f(\emptyset) + f(i) \geq 0$ and the strict inequality here would get $\sigma_i^{ik} f\ \sigma_\emptyset^{ik} f > \sigma_{ik}^{ik} f\ \sigma_k^{ik} f \geq 0$, $k \in N - i$, i.e. $b_i(f) > 0$) and directly from (1) we obtain

$$\forall L \in \mathcal{P}_2(N - I) : \quad f(L) + f(\emptyset) = 0. \tag{7}$$

We shall need this additional notation

$$K = K(f) = \{i \in I;\ f(i) > -\tfrac{1}{2} f(\emptyset)\}\ , \qquad J = J(f) = \{i \in I;\ f(i) = -\tfrac{1}{2} f(\emptyset)\}$$

and explain

$$f(\emptyset) < 0 \ \Rightarrow\ \exists i \in K : \quad f(i) = -f(\emptyset). \tag{8}$$

In fact, if $f(\emptyset) < 0$ we can choose $j \in N - I$ arbitrarily and then $i \in N - j$ by (3) to get $f(ij) = 0$; having in mind (7) we know that $i \in I$ and conclude $f(i) = -f(\emptyset)$ on account of (6). Applying again (1) we obtain easily

$$\forall i \in I - JK\ \forall j \in I - iJK : \quad f(ij) = -f(\emptyset) - f(i) - f(j), \tag{9}$$

$$\forall i \in I - JK\ \forall j \in J : \quad f(ij) = -\tfrac{1}{2} f(\emptyset) - f(i), \tag{10}$$

$$\forall i \in JK\ \forall j \in JK - i : \quad f(ij) = 0 \tag{11}$$

closing our preliminary considerations.

We start to prove that any function $f \in D_2^-$ fulfilling (1)–(4) (and then (5)–(11)) is a conical combination of the functions of the type β and α. We employ induction on $r(f) = |\mathcal{A}(f)| + |\mathcal{B}(f)|$ where

$$\mathcal{A}(f) = \{L \in \mathcal{P}_2;\ \sigma_L^L f > 0\} \quad \text{and} \quad \mathcal{B}(f) = \{L \in \mathcal{P}_2;\ \sigma_\emptyset^L f > 0\}.$$

If $r(f) = 0$ and $f(\emptyset) < 0$ then $f = f(\emptyset)\delta_\emptyset + \sum_{i \in N} f(i)\delta_i$ and $f(i) + f(j) + f(\emptyset) = 0$, $\ i, j \in N$ distinct. Looking at (8) we find that $n = 2$, $I(f)$ is a singleton and $f = -f(\emptyset)\beta_I$. We point out that once $f(\emptyset) = 0$ the properties (5) and (6) force f to be equal identically to zero; below we assume only negative $f(\emptyset)$.

Let $f \in D_2^-$ fulfil (1)–(4), $f(\emptyset) < 0$ and $r(f) > 0$. If $I(f) = i$ is a singleton then (6)–(8) imply $f = -f(\emptyset)\beta_i$ and we finished, hence $|I(f)| \geq 2$. We distinguish the following two cases.

Case 1. $|J(f)| \leq 2$

Now we can partition the set $I(f)$ into L, $|L| \geq 2$, and $I - L$ (possibly empty) ensuring the strict inequalities

$$\begin{array}{ll} \forall i \in L \ \ \forall j \in L - i : & f(i) + f(j) + f(\emptyset) > 0 \\ \forall i \in I - L \ \ \forall j \in I - iL : & f(i) + f(j) + f(\emptyset) < 0. \end{array}$$

In fact, if $J \neq \emptyset$ we choose fixed $j \in J$ and set $L = jK$ ($|L| \geq 2$ due to (8)). If the set J is empty and $I = K$ we set $L = K$ ($|L| \geq 2, I - L = \emptyset$). If $J = \emptyset$ and $I \neq K$ we look for an element $i \in I - K$ such that $f(i) + f(k) + f(\emptyset) > 0$ for all $k \in K$ and if such element exists, in the case $|K| = 1$ it really does by (8), we chose a fixed one and set $L = iK$; otherwise $L = K$ (always $|L| \geq 2$).

Owing to the choice of L the number

$$e_L = \min\{|f(i) + f(j) + f(\emptyset)|;\ i, j \in N \text{ distinct},\ ij \subset L \text{ or } ij \subset I - L\}$$

is positive. Since (5) and (6) bring $0 \leq f(i) \leq -f(\emptyset)$, $i \in N$, the number e_L is bounded from above by $-f(\emptyset)$. The reader surely expects that we are going to combine $f = e_L \beta_L + g$ and to verify $g \in D_2^-$.

Firstly, using (1) we see immediately

$$\begin{aligned} \sigma_\emptyset^{ij} g &= \sigma_\emptyset^{ij} f - e_L = f(i) + f(j) + f(\emptyset) - e_L \geq 0, && i, j \in L \text{ distinct}, \\ &= \sigma_\emptyset^{ij} f \geq 0, && \text{otherwise}. \end{aligned}$$

Secondly,

$$\begin{aligned} g(i) &= f(i) - e_L \geq f(i) + f(j) + f(\emptyset) - e_L \geq 0, && i \in L, \\ &= f(i) \geq 0, && i \in N - L, \end{aligned}$$

where we took $j \in L - i$ and applied (6). And thirdly, for $i, j \in N$ distinct we employ (1) and (7) to get

$$\begin{aligned} g(ij) &= f(ij) - e_L & &= -f(\emptyset) - e_L \geq 0, && i, j \in N - I \text{ distinct}, \\ & & &= -f(\emptyset) - f(i) - e_L, && i \in I - L,\ j \in N - I, \\ & & &= |f(i) + f(j) + f(\emptyset)| - e_L \geq 0, && i, j \in I - L \text{ distinct}, \\ &= f(ij) \geq 0, & & && \text{otherwise}. \end{aligned}$$

The missing inequality needs an additional explanation. Where $i, k \in I - L$ are distinct and $j \in N - I$ then

$$g(ij) = -f(\emptyset) - f(i) - e_L > -f(\emptyset) - f(i) - f(k) - e_L \geq 0$$

but the argument fails if $I - L$ is a singleton. Yet, in this case we write $e_L = f(k) + f(l) + f(\emptyset)$, $f(k) \leq f(l)$, for distinct $k, l \in L$ and since $f(i) + f(k) + f(\emptyset) \leq 0$ (note that if $J \neq \emptyset$ then $f(k) = -\frac{1}{2} f(\emptyset)$ and if $J = \emptyset$ then this inequality holds due to our specific choice of L) we obtain by (6)

$$g(ij) = -f(\emptyset) - f(i) - e_L = -f(\emptyset) - f(i) - f(k) - f(l) - f(\emptyset) \geq -f(l) - f(\emptyset) \geq 0.$$

We can summarize. We wrote the function f as the sum of $e_L \beta_L$ and g and found $0 \preceq g \preceq f$ and $g(\emptyset) = f(\emptyset) + e_L \leq 0$. The function g belongs to D_2^- and enjoys (1)–(4). Moreover, $r(g) < r(f)$ (see $g(ij)$, $i, j \in I - L$, and $\sigma_\emptyset^{ij} g$, $i, j \in L$). Thus the function f is expressible in the desired form by an appeal to the induction principle.

Case 2. $|J(f)| \geq 3$

The number

$$e_{J,K} = \min_{i \in I - J}\{|\tfrac{1}{2} f(\emptyset) + f(i)|\}$$

is positive and $e_{J,K} \leq -\frac{1}{2} f(\emptyset)$ (see (5) and (6)). In the expression $f = e_{J,K} \alpha_{J,K} + g$ the function g will turn out to be in D_2^-.

Firstly, for $i, j \in N$ distinct we use (11) and (5) to get

$$\begin{aligned} \sigma_\emptyset^{ij} g &= \sigma_\emptyset^{ij} f - 2e_{J,K} = f(i) + f(j) + f(\emptyset) - 2e_{J,K} \geq 0, && i, j \in K \text{ distinct}, \\ &= \sigma_\emptyset^{ij} f - e_{J,K} = \tfrac{1}{2} f(\emptyset) + f(j) - e_{J,K} \geq 0, && i \in J,\ j \in K, \\ &= \sigma_\emptyset^{ij} f \geq 0, && \text{otherwise}. \end{aligned}$$

Further, due to (6) and (5) we see

$$\begin{aligned} g(i) &= f(i) - 2e_{J,K} \geq f(i) - 2(f(i) + \tfrac{1}{2} f(\emptyset)) = -f(i) - f(\emptyset) \geq 0, && i \in K, \\ &= f(i) - e_{J,K} = -\tfrac{1}{2} f(\emptyset) - e_{J,K} \geq 0, && i \in J, \\ &= f(i) \geq 0, && i \in N - JK. \end{aligned}$$

And finally the most tedious case is to verify ($i,j \in N$ distinct)

$$\begin{array}{llll} g(ij) \;= f(ij) - 2e_{J,K} & = -f(\emptyset) - 2e_{J,K} \geq 0, & i,j \in N-I \text{ distinct}, & \text{(see (7))} \\ & = -f(\emptyset) - f(j) - 2e_{J,K} \geq f(j) > 0, & i \in N-I, j \in I-JK, & \text{(see (6))} \\ & = -f(\emptyset) - f(i) - f(j) - 2e_{J,K} \geq 0, & i,j \in I-JK \text{ distinct}, & \text{(see (9))} \\ \;= f(ij) - e_{J,K} & = -\frac{1}{2}f(\emptyset) - e_{J,K} \geq 0, & i \in N-I, j \in J, & \text{(see (6))} \\ & = |\frac{1}{2}f(\emptyset) + f(i)| - e_{J,K} \geq 0, & i \in I-JK, j \in J, & \text{(see (10))} \\ \;= f(ij) \geq 0, & & \text{otherwise}, & \end{array}$$

what together with the previous estimation enables to conclude $0 \preceq g \preceq f$. Hence, the function $g \in D_2^-$ (note that $g(\emptyset) = f(\emptyset) + 2e_{J,K} \leq 0$) fulfils (1)–(4) and, in addition, $r(g) < r(f)$ (see $\sigma_\emptyset^{ij} g$, $i \in J$, $j \in K$, and $g(ij)$, $i \in I-JK$, $j \in J$). The induction argument shows that f can be expressed as the desired conical combination. □

Lemma 4. *All functions mentioned in Proposition 3 are extreme in D_2^-, $n \geq 3$, i.e. this cone has*

$$3^n - 2^{n-3}(n^2+3n+16) + \tfrac{1}{2}(3n^2+5n+4)$$

extreme rays (for $n=2$ six extreme rays).

Proof. On account of Lemma 2 it remains to prove that the functions of the types γ, β and α are extreme. First we observe that the functionals a_L, b_i, c_i and d map them into zero with one exception $d(\gamma) = 1$. Hence we can consider only conical combinations not containing δ_K, $K \in \mathcal{P}_1^2$, ε_i, $i \in N$, and γ.

Let us suppose γ in the form (we omit the ranges of summation)

$$\gamma = \sum u_I \beta_I + \sum v_{J,K} \alpha_{J,K}$$

where all coefficients are nonnegative. Since $\gamma(L) = 0$, $L \in \mathcal{P}_2$, all $v_{J,K}$ are zero and also $u_I = 0$, $I \in \mathcal{P}_1^{n-2}$. Then $\gamma = \sum_{i \in N} u_{N-i} \beta_{N-i}$ what gives $\gamma(\emptyset) = -2 = -\sum_{i \in N} u_{N-i}$ and $\gamma(j) = 1 = \sum_{i \in N-j} u_{N-i}$, $j \in N$, and then $u_{N-i} = 1$, $i \in N$, and $n = 2$. For two-element groundset $N = \{i,j\}$ the cone D_2^- has six extreme functions: β_i, β_j, δ_i, δ_j, ε_i and ε_j. Otherwise the function γ is extreme.

Let us suppose β_I, $I \in \mathcal{P}_1^{n-1}$, in the form

$$\beta_I = \sum u_L \beta_L + \sum v_{J,K} \alpha_{J,K}.$$

As all involved functions are nonnegative on singletons we deduce $u_L = 0$, $\emptyset \neq L \not\subset I$, and $v_{J,K} = 0$, $\emptyset \neq JK \not\subset I$. But if $i \in I-L$, $L \subset I$ and $j \in N-I$ then $\beta_I(ij) = 0$, $\beta_L(ij) = 1$ and thus u_L is nonzero only if $L = I$. Similarly for $i \in J \subset I$ and $j \in N-I$ we get $\beta_I(ij) = 0$, $\alpha_{J,K}(ij) = 1$ what implies $v_{J,K} = 0$, $(J,K) \in \mathcal{Q}(N)$; the function β_I is thus extreme.

Let us write

$$\alpha_{J,K} = \sum u_I \beta_I + \sum v_{L,M} \alpha_{L,M}, \qquad (J,K) \in \mathcal{Q}(N),$$

and consider analogically as above only $I \subset JK$ and $LM \subset JK$. If $i,j \in J$ distinct then $\sigma_\emptyset^{ij} \alpha_{J,K} = 0$ what implies $u_I = 0$ as soon as $|I \cap J| > 1$. But also if $i,j \in J-I$ distinct then $\alpha_{J,K}(ij) = 0 \neq 1 = \beta_I(ij)$ and thus all coefficients u_I, $I \in \mathcal{P}_1^{n-1}$, are equal to zero (note that $|J| \geq 3$). Further, we observe for $j \in N-JK$ that $\alpha_{J,K}(ij)$ equals zero or not if $i \in K$ or $i \in N-jK$, respectively, what gives $v_{L,M} = 0$, $M \not\supset K$. Moreover, if $i \in J \cap M$ and $j \in L \subset J$ (note that also $|L| \geq 3$) then $\sigma_\emptyset^{ij} \alpha_{J,K} = 0 \neq 1 = \sigma_\emptyset^{ij} \alpha_{L,M}$ what leads to $v_{L,M} = 0$ if $M \neq K$. But now if $i \in J-L$ and $j \in L \subset J$ then $\alpha_{J,K}(ij) = 0$ whereas $\alpha_{L,K}(ij) = 1$ and we find that the only positive coefficient is $v_{J,K} = 1$. The function $\alpha_{J,K}$ is extreme. □

5. Main results

Theorem. *The cones C_2 and H_{n-2}, $n \geq 3$, have both*

$$3^n - 2^{n-3}(n^2+3n) + n^2 + 1$$

extreme rays. The extreme functions of C_2 are in our notation of seven types and the table below reports on them and on their numbers.

$\alpha_{J,K}$	$(J,K) \in \mathcal{Q}(N)$	$3^n - 2^{n-3}(n^2+3n+24) + (n^2+n+3)$
β_I	$I \in \mathcal{P}_1^{n-1}(N)$	$2^n - 2$
γ		1
δ_I	$I \in \mathcal{P}(N)$	2^n
ε_i	$i \in N$	n
φ		1
φ_I	$I \in \mathcal{P}_2^{n-2}(N)$	$2^n - 2 - 2n$

Proof. By Lemma 1 it suffices to deal with the cone C_2 only. Proposition 1 shows that the extreme functions of C_2 consist of δ_I, $I \in \mathcal{P}_3^n$, and of the extreme functions of D_2. But D_2 is spanned by the tabulated functions due to Propositions 2 and 3. Lemma 3 and 4 imply that all functions $\alpha_{J,K}$, β_I, γ, $\delta_\emptyset$, φ and φ_I are extreme in D_2. The last fact to prove is that the functions δ_L, $L \in \mathcal{P}_1^2$, and ε_i, $i \in N$, are extreme in D_2 (we remind that $n \geq 3$).

If δ_L, $L \in \mathcal{P}_2$, is a conical combination of all tabulated functions spanning D_2 then this combination does not contain terms with φ_I, φ, ε_i, δ_K for $K \in \mathcal{P}_2(N)$ and $K \neq L$, β_I for $I \in \mathcal{P}_1^{n-3}$ and $\alpha_{J,K}$ for $(J,K) \in \mathcal{Q}(N)$, as these functions are positive on doubletons different from L. However, from the remaining functions $\delta_\emptyset$, δ_i, γ, and β_I, $I \in \mathcal{P}_{n-2}^{n-1}$, no one does not occur too as they are all nonnegative on singletons and thus δ_L is extreme.

Similarly a conical combination of the above type equal to δ_i, $i \in N$, does not contain φ_I, φ, ε_i, δ_K for $K \in \mathcal{P}_2(N)$, β_I for $I \in \mathcal{P}_1^{n-2}$ and $\alpha_{J,K}$ as they are all nonzero on at least one doubleton; the remaining functions being nonnegative on singletons can combine into δ_i only in the trivial way. Hence δ_i is extreme.

The same simple arguments can be aplied again to ε_i, $i \in N$, and we do not write them explicitely. Thus, the table presents the complete list of all extreme functions of C_2. $\square$

Example. Let $n = 4$, $N = \{i,j,k,l\}$. The cone of all nonnegative nondecreasing and semimodular set functions can be obtained from $H_2 = H_{n-2}$ by the self inverse linear transformation (involution) $Sf = f(\emptyset) + f(N) - f$, $f \in R^{\mathcal{P}}$. Due to Theorem it has 42 extreme functions. They can be found applying the composed transform ST^{-1} to the tabulated functions. In this way we obtain

– the constant function identically equal to one (from δ_N),
– the rank functions of the following matroids:
  – the matroids with the rank 1 (from δ_I, $I \in \mathcal{P}_0^3(N)$),
  – the uniform matroid $U_{3,4}$ (from φ),
  – the uniform matroid $U_{2,4}$ (from γ),
  – the matroids with one loop and a submatroid $U_{2,3}$ (from ε_i, $i \in N$),
  – the matroids with a submatroid $U_{2,3}$ and two parallel elements (from β_I, $I \in \mathcal{P}_2$),
– the functions g_i, $i \in N$ (from β_{N-i}, $i \in N$), where

$$\begin{aligned} g_i(L) &= \min\{2, |L|\}, & L &\neq i, \\ &= 2, & L &= i, \end{aligned}$$

– the functions h_i, $i \in N$ (from β_i, $i \in N$), where

$$\begin{aligned} h_i(L) &= \min\{3, |L|+1\}, & i &\in L, \\ &= |L|, & i &\notin L, \end{aligned}$$

– the functions f_{ij}, $ij \in \mathcal{P}_2$ (from φ_I, $I \in \mathcal{P}_2$), where

$$\begin{array}{lll} f_{ij}(L) & = \min\{4, 2|L|\}, & L \notin \{ik, jk, il, jl, kl\}, \\ & = 3, & \text{otherwise.} \end{array}$$

(Let us note that the free expansions of the functions of the last three types are $U_{2,5}$, $U_{3,5}$ and Vamos cube, respectively.) We know in addition that the extreme rays of the subcone given by $\sum_{I \subset N} (-1)^{|I|} f(I) \geq 0$ or by the opposite inequality or even by the equality are among the listed 42 ones. Their numbers are (see Lemmas 1, 2, 3 and 4) 34, 27 and 19, respectively.

The extreme rays of the cone of all convex set functions on the four-element set N that are equal to zero on $\mathcal{P}_0^1$ were listed in [11]. They (37 in total) can be found easily from this example. An extensive study of this cone by different methods is in [10].

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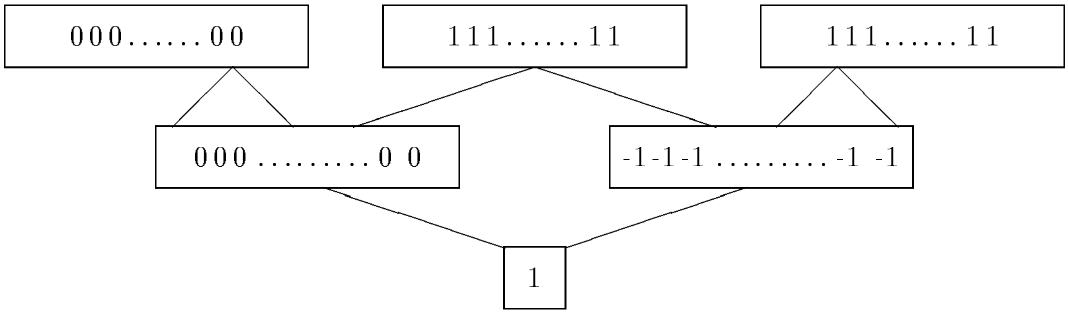

Fig.1 A sketch of the function φ_I, $I \in \mathcal{P}_2^{n-2}$.

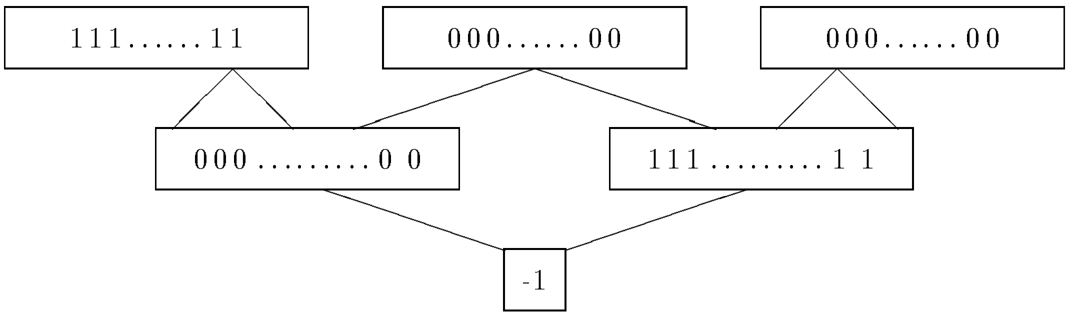

Fig.2 A sketch of the function β_I, $I \in \mathcal{P}_1^{n-1}$.

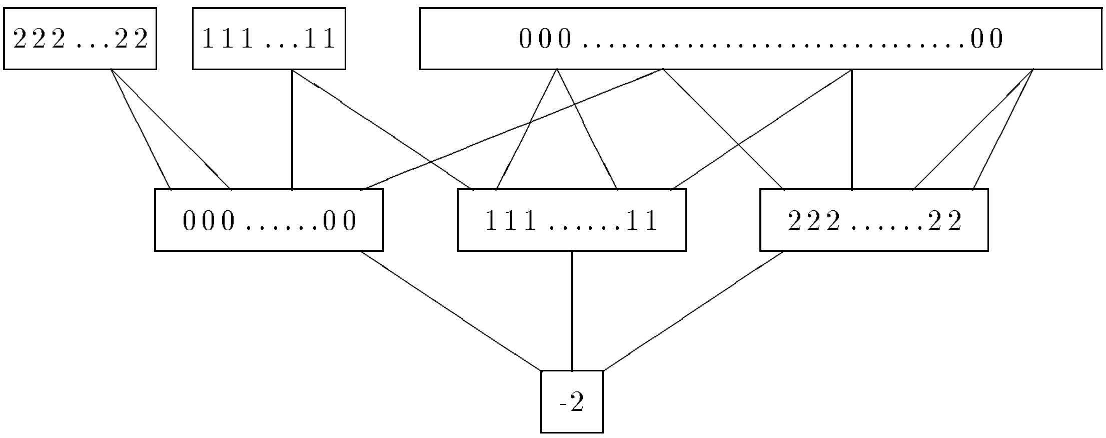

Fig.3 A sketch of the function $\alpha_{J,K}$, $J \in \mathcal{P}_3^{n-2}$, $J \cap K = \emptyset$, $\emptyset \neq K \neq N - J$.