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Image Representations via a Finite Radon Transform

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Abstract – This paper presents a model of finite Radon transforms composed of Radon projections. The model generalizes to finite groups projections in the classical Radon transform theory. The Radon projector averages a function on a group over cosets of a subgroup. Reconstruction formulae formally similar to the convolved backprojection ones are derived and an iterative reconstruction technique is found to converge after finite number of steps. Applying these results to the group Z_p^2, new computationally favourable image representations have been obtained. A numerical study of the transform coding aspects is attached.

Index Terms – finite Radon transform, Radon projections, Fourier transform, iterative reconstruction, image representations, transform coding, image compression

I. INTRODUCTION

Development of various representations of image data continues to be an area of active research. It has had substantial impact on many image processing and image analysis problems including manipulation, compression, pattern recognition, coding, computer vision, etc. The most common manner of image representation is to use an energy preserving transform. Classical and thoroughly investigated examples include Fourier, cosine, sine, Hadamard, Haar and other unitary transforms [15], [5].

The Radon transform entered the center of interest with its first radioastronomic and tomographic applications [8] and spread quickly into many fields. Recall that the Radon transform (this notion is from integral geometry, cf. [7], [11], [18]) of a real function f defined on Euclidean plane is the function $\hat{f}$, defined on the family of all lines in the plane, having the value $\hat{f}(p)$ equal to the line integral along the line p. It is often advantageous to view the Radon transform as the family $(\hat{f}_q)$ of projections where the projection $\hat{f}_q$ is the restriction of $\hat{f}$ on the set of all lines parallel to the line q containing the origin.

Numerous discretizations of the Radon transforms employed in practice have been devised. For example, tomographic data can also be arranged into a finite subfamily of sampled projections. The utilization of the discretized Radon transforms penetrate the range of image analysis and processing techniques (for an account see [17]). Projection space representations and manipulations of digital images have given rise to new algorithms and have opened new possibilities. A substantial role in this progress has been played by the Parallel Pipeline Projection Engine (P^3E) computer architecture. This is even sometimes compared with the role of FFT algorithm supporting the Fourier transform.

During the last decade, elements of a finite Radon transform theory have appeared mainly in the field of combinatorics ([2], [4], [12], [18]). The fundamental concept, due to Bolker, is as follows. The finite Radon transform of a real function f defined on a finite set S (with respect to a collection $\mathcal{T}$ of its subsets-blocks) is the function $\hat{f}$ on $\mathcal{T}$, the values of which are obtained by summing (we prefer averaging) f over the blocks. It seems quite impossible to say anything about this transform without additional conditions on the collection $\mathcal{T}$, however, when considered on special combinatorial or algebraical structures like designs, matroids, lattices, groups, etc., some results have been attained (for a review see [12]). There seem to be very few resemblences with the classical Radon transform theory.

The main goal of our paper is to introduce and investigate a scheme of finite Radon transforms on groups, to present new results on finite Radon transforms on finite Euclidean planes and to discuss corresponding projection representations of images and their applications. Hence, we are not concerned with the *discretized* Radon transforms but with the *discrete, finite* ones.

Accordingly, the paper is divided into three parts.

First, we investigate Radon transforms viewed as families of Radon projections on finite groups: In Bolker's setting S is taken to be a group and $\mathcal{T}$ a quotient group of S getting the Radon projection $f \to \hat{f}$. The reconstruction from projections is the main theme explored here. Analogies with the classical, analytical situation appear to be much deeper than one can expect. Even an inversion of this Radon transform can be written in the form of filtered backprojection (cf. Theorem 1). Moreover, an iterative reconstruction technique (cf. II. E.) akin to Kacmarz's (see [17]) converges after finite number of steps (equal to the number of the Radon projections, see Theorem 3).

Second, we translate these results into the additive groups Z_p^2 of Euclidean planes. A discrete version of projection slice theorems connecting the finite Radon and finite Fourier transforms is presented. We examine also Radon transforms composed of partial projections, i.e. of the projections in the above sense admitting $\mathcal{T}$ to be an arbitrary subset of a quotient group.

Third, image representations via afore-mentioned finite Radon transforms are suggested and their numerical aspects are discussed. Similarly as with other transforms, maximum information is packed into a small number of samples. In this way, image compression is achieved by the storage of the most informative samples only (this technique, called *transform coding*, is described for other transforms in [1], [5], [10],[16]). In Section IV compression results are visualized, the efficiency of the compression algorithms is discussed and feasibility of real-time implementation through the P^3E architecture approved.

II. FINITE RADON TRANSFORMS ON GROUPS

Let G be a finite group and $\mathbf{A}(G)$ be the linear algebra of all complex functions on G, i.e. $\mathbf{A}(G)$ is a $|G|$-dimensional complex linear space with the convolution of two functions f_1,$f_2 \in \mathbf{A}(G)$ (see [3])

$$f_1 * f_2(x) = |G|^{-1} \sum_{y \in G} f_1(y) f_2(y^{-1}x)$$

as the product. We start with simple notations and technicalities.

A. Radon projections

The Radon projection of a function $f \in \mathbf{A}(G)$ along a normal subgroup H of G is defined (in [14]) to be the function $\Lambda_H f$ on the quotient group G/H given by

$$\Lambda_H f(xH) = |H|^{-1} \sum_{y \in xH} f(y), \quad xH \in G/H;$$

the values of the Radon projection are thus equal to the averages of the function f over the cosets $xH = \{xy;\ y \in H\}$.

We shall frequently use an analogy of the Euclidean plane backprojection. The backprojection of a function $g \in \mathbf{A}(G/H)$ along H is the function $V_H g \in \mathbf{A}(G)$ that is constant and equal to $g(xH)$ on every coset xH.

Considering this scalar product on $\mathbf{A}(G)$

$$\langle f_1, f_2 \rangle = |G|^{-1} \sum_{x \in G} f_1(x) \overline{f_2(x)}, \quad f_1, f_2 \in \mathbf{A}(G),$$

one can easily verify the duality of the operators Λ_H and V_H

$$\langle \Lambda_H f, g \rangle = \langle f, V_H g \rangle, \quad f \in \mathbf{A}(G), g \in \mathbf{A}(G/H).$$

We shall write out a collection of simple assertions on the Radon projections and backprojections below. Proofs are omitted because of their triviality. The $|G/H|$ multiple of the indicator function of H is denoted by λ_H .

Lemma 1:

1. $\Lambda_H V_H g = g, \qquad g \in \mathbf{A}(G/H),$
2. $V_H \Lambda_H f = f * \lambda_H, \qquad f \in \mathbf{A}(G),$
3. $\Lambda_H(f_1 * f_2) = \Lambda_H f_1 * \Lambda_H f_2, \qquad f_1, f_2 \in \mathbf{A}(G),$
4. $V_H(g_1 * g_2) = V_H g_1 * V_H g_1, \qquad g_1, g_2 \in \mathbf{A}(G/H),$
5. $V_H g * f = V_H(g * \Lambda_H f), \qquad f \in \mathbf{A}(G), g \in \mathbf{A}(G/H).$

B. Reconstruction from projections

Let $\mathcal{H} = (H_i)_{i \in N}$ be a nonvoid and finite family of normal subgroups of G (possibly $H_i = H_j$ for $i \neq j$, $i, j \in N$). Instead of Λ_{H_i}, V_{H_i} and λ_{H_i} we prefer to write Λ_i,V_i and λ_i, respectively. Now, our finite Radon transform on finite groups will be defined in accordance with the classical theory as a family of Radon projections.

Definition: The operator $\Lambda : \mathbf{A}(G) \to \mathbf{A}(\mathcal{H}) = \Pi_{i \in N} \mathbf{A}(G|H_i)$ given by $\Lambda f = (\Lambda_i f)_{i \in N}$ is called *finite Radon transform* on G with respect to $\mathcal{H}$.

We remark that the symbol $\mathbf{A}(\mathcal{H})$ denotes the product of the family $(\mathbf{A}(G/H_i))_{i \in N}$ of linear algebras (operations $+$ and $*$ are performed in $\mathbf{A}(\mathcal{H})$ coordinatewisely). All the preliminaries for the precise formulation of our basic problem are now available.

Reconstruction problem: Find the necessary and sufficient conditions for solvability of the equation $\Lambda f = g$ with the given right-hand side $g \in \mathbf{A}(\mathcal{H})$ and find all solutions.

In other words, we should like to solve the system of equations

$$\Lambda_i f = g_i, \quad i \in N,$$

where $g_i \in \mathbf{A}(G|H_i)$, $i \in N$, $g = (g_i)_{i \in N}$. This is, however, equivalent to

$$f * \lambda_i = V_i g_i, \quad i \in N.$$

In fact, if f satisfies the first set of equations then applying V_i and the claim 2. of Lemma 1 we see that it satisfies the second one. Conversely, for f holding the convolution equations the observation

$$g_i = \Lambda_i V_i g_i = \Lambda_i (f * \lambda_i) = \Lambda_i V_i \Lambda_i f = \Lambda_i f$$

(we used consecutively 1., 2. and 1.) yields that f has the prescribed family of projections.

When dealing with convolution equations in algebras like $\mathbf{A}(G)$, the representation theory of groups and algebras is usually the most convenient tool. We shall present here, however, a shorter and simpler algebraic approach without introducing additional, sophisticated notions and facts.

C. Solution

We shall denote by $H_I = \Pi_{i \in I} H_i$, $I \subset N$, the products of subfamilies of $\mathcal{H}$. That means $H_\emptyset = \{e\}$, where e is the identity element of G, $H_{\{i\}} = H_i$ for $i \in N$, $H_{\{i,j\}} = H_i H_j = \{xy \in G;\, x \in H_i,\, y \in H_j\} = H_j H_i$ for $i, j \in N$, etc. It is an elementary piece of group theory that the order of multiplicators in the formula of H_I is not relevant and that every H_I, $I \subset N$, is a normal subgroup of G. To avoid double indexing we write λ_I instead of λ_{H_I}.

The following two expressions (we started to use them in [14] for the first time)

$$\varepsilon = \sum_{\emptyset \neq I \subset N} (-1)^{|I|-1} \lambda_I,$$

$$\omega = \sum_{\emptyset \neq I \subset N} (-1)^{|I|-1} \lambda_I \sum_{s=1}^{|I|} \frac{1}{s}$$

define two functions of $\mathbf{A}(G)$ which play the key role in our problem. The later one can be called *convolving function.*

Lemma 2:

6. $\lambda_I * \lambda_J = \lambda_{I \cup J}, \quad I, J \subset N,$
7. $\varepsilon * \lambda_I = \lambda_I, \quad \emptyset \neq I \subset N,$
8. $\varepsilon * \varepsilon = \varepsilon,$
9. $\sum_{i \in N} \lambda_i * \omega = \varepsilon.$

For the proof of this lemma and the following three theorems see Appendix A. Before stating the main results of this section we shall find an explicit form of the dual operator of Λ. If the scalar product on $\mathbf{A}(\mathcal{H})$ is considered to be the sum of scalar products

$$\langle g^{(1)}, g^{(2)} \rangle = \sum_{i \in N} \langle g_i^{(1)}, g_i^{(2)} \rangle, \quad g^{(s)} = (g_i^{(s)})_{i \in N} \in \mathbf{A}(\mathcal{H}), \quad s = 1, 2,$$

then from

$$\langle \Lambda f, g \rangle = \sum_{i \in N} \langle \Lambda_i f, g_i \rangle = \sum_{i \in N} \langle f, V_i g_i \rangle, \quad f \in \mathbf{A}(G), \; g \in \mathbf{A}(\mathcal{H}),$$

we get immediately the form of the dual $V : \mathbf{A}(\mathcal{H}) \to \mathbf{A}(G)$ of Λ

$$V g = \sum_{i \in N} V_i g_i, \qquad g \in \mathbf{A}(\mathcal{H}).$$

Theorem 1: The equation $\Lambda f = g$, where $g \in \mathbf{A}(\mathcal{H})$ is given, has a solution if and only if $\Lambda(Vg * \omega) = g$ and in this case $Vg * \omega$ is its solution with the smallest norm. Any solution differs from this one by a function f such that $f * \varepsilon = 0$.

Remarks: 1. We point out at the beginning other necessary and sufficient conditions for existence of a solution called also the *projectivity conditions* (mainly in the frame of marginal problems, see [13], [14])

$$V_i g_i * \lambda_j = V_j g_j * \lambda_i \quad i, j \in N.$$

We explain only the necessity here. If $\Lambda f = g$ then

$$V_i g_i * \lambda_j = V_i \Lambda_i f * \lambda_j = f * \lambda_i * \lambda_j = f * \lambda_j * \lambda_i = V_j g_j * \lambda_i \quad i, j \in N.$$

The opposite implication is a nontrivial part of Proof of Theorem 1 in Appendix A.
2. Our solution of Reconstruction problem can be written formally in the same manner as in the classical Radon transform theory. Namely, it can be written as a convolved backprojection $Vg * \omega$, or equivalently (see 5. in Lemma 1), a backprojected convolution $V(g * \Lambda\omega)$. Theorem 1 remains valid, if we consider only real functions on G as ω and ε are real.
3. The function $\lambda_\emptyset$ is nonzero only at the point e and $\lambda_\emptyset(e) = |G|$. It is the identity of the algebra $\mathbf{A}(\mathcal{H})$ similarly like the delta function among distributions in Euclidean plane. If we observe its projections $g = \Lambda\lambda_\emptyset$ then our reconstruction according to the above theorem will be (cf. 9. in Lemma 2)

$$Vg * \omega = V\Lambda\lambda_\emptyset * \omega = \sum_{i \in N} V_i \Lambda_i \lambda_\emptyset * \omega = \sum_{i \in N} (\lambda_\emptyset * \lambda_i) * \omega = \lambda_\emptyset * \sum_{i \in N} \lambda_i * \omega = \lambda_\emptyset * \varepsilon = \varepsilon.$$

More generaly, if we observe the projections Λf of a function f, the reconstruction formula yields, similar to the above, $f * \varepsilon$ (note that the mapping assigning $f * \varepsilon$ to a function f is a projector due to 8. of Lemma 2).
4. If $\lambda_\emptyset \neq \varepsilon$ then the nonzero function $\lambda_\emptyset - \varepsilon$ has zero projections (we remind that V_i is injective)

$$V_i \Lambda_i (\lambda_\emptyset - \varepsilon) = (\lambda_\emptyset - \varepsilon) * \lambda_i = \lambda_i - \lambda_i = 0.$$

On contrary, if all functions of $\mathbf{A}(G)$ are uniquely given by their projections then our reconstruction of $\lambda_\emptyset$ must be equal to $\lambda_\emptyset$, whence $\lambda_\emptyset = \varepsilon$. These considerations show that the equation $\lambda_\emptyset = \varepsilon$ is the necessary and sufficient condition for unique reconstruction in the projection scheme given by the family $\mathcal{H}$.

D. Dual problem

An analysis of the dual operator V may be also of some interest; we shall present here the dual version of Theorem 1.

Theorem 2: The equation $Vg = f$, where $f \in \mathbf{A}(G)$ is given, is solvable if and only if $f * \varepsilon = f$ and in this case $\Lambda(f * \omega) = \Lambda f * \Lambda\omega$ is its solution with the smallest norm. Every solution differs from this one by a $g = (g_i)_{i \in N} \in \mathbf{A}(\mathcal{H})$ satisfying $\Lambda(Vg * \omega) = 0$.

Remarks: 1. Similar to our observation after Theorem 1, we can see that Theorem 2 can be easily reformulated for the algebra of real functions on G.
2. The backprojection V is one-to-one only if $|N| = 1$. Otherwise for $i \neq j$, $i, j \in N$, we set $g_i = 1, g_j = -1$ and $g_k = 0$, $k \in N - \{i, j\}$ having $Vg = 0$ for $g = (g_i)_{i \in N} \neq 0$.

E. Iterative reconstruction

The solution of Reconstruction problem from Theorem 1 can be rewritten in a simple iterative form which is computationally very tractable.

Consider $N = \{1, 2, ..., n\}$, choose arbitrary $f_{(0)} \in \mathbf{A}(G)$ and set

$$f_{(i)} = f_{(i-1)} + V_i(g_i - \Lambda_i f_{(i-1)}), \quad i = 1, 2, \ldots, n,$$

where $g_i \in \mathbf{A}(G/H_i)$, $i \in N$, are given projections.

So, this is nothing but a counterpart of one of the standard algebraic iterative reconstruction techniques well known e.g. in computerized tomography (see [8], [17]): the new iteration $f_{(i)}$ is obtained from the old one $f_{(i-1)}$ by adding the correction in the form of backtraced subtraction of the given and current projections. We claim that $f_{(n)}$ is yet a solution of Reconstruction problem; thus, only n iterations are needed which completely removes the main drawback of the reconstruction methods of this type, namely the slow rate of convergence.

Theorem 3: Let $g = (g_i)_{i \in N}$ be a family of projections satisfying the projectivity conditions. Then $\Lambda f_{(n)} = g$ and, moreover, the choice $f_{(0)} = 0$ gives $f_{(n)} = Vg * \omega$.

III. FINITE RADON TRANSFORMS ON Z_p^2

In this section we describe finite Radon transforms on the group Z_p^2 taking arbitrary families of its nontrivial subgroups. Further we make explicit connections with the finite Fourier transform. Finally, reconstruction from partial projections will be examined, i.e. instead of having given Radon projections g_i, $i \in N$, on whole quotient groups we suppose to know only some values of them.

A. Basic scheme

Let us consider the group $G = Z_p^2$ being the Cartesian product $Z_p \times Z_p$ of two exemplares of the cyclic group $Z_p = \{0, 1, \ldots, p-1\}$ with addition modulo p. We shall suppose that p is prime. This group has $p+1$ nontrivial subgroups

$$\begin{aligned} H_i &= \{(k,l) \in G;\ \ li = k \,(\mathrm{mod}\, p)\}, \quad 0 \le i < p, \\ H_p &= \{(k,0) \in G;\ \ k \in Z_p\}; \end{aligned}$$

we put $N = \{0, 1, \ldots, p\}$, $\mathcal{H} = (H_i)_{i \in N}$. Every group H_i and every factorgroup G/H_i is isomorphical to Z_p. The cosets of the factorgroup G/H_i will be indexed by $j \in Z_p$ in this way

$$\begin{aligned} H_i^j &= \{(k,l) \in G;\ \ li + j = k \,(\mathrm{mod}\, p)\}, \quad 0 \le i < p, \\ H_p^j &= \{(k,j) \in G;\ \ k \in Z_p\}, \end{aligned}$$

thus, $H_i^0 = H_i$, $i \in N$. Having an element $x = (k,l)$ of G and a subgroup H_i, $i \in N$, there is just one coset H_i^j of H_i containing x, let us write $\pi_i(k,l) = j$. In other words, $\pi_p(k,l) = l$ and $\pi_i(k,l) = k - li \,(\mathrm{mod}\, p)$, $0 \le i < p$. The function π_i is nothing but a variant of the factormapping of G on G/H_i.

The Radon projection of a function f on G is now given by

$$\Lambda_i f(H_i^j) = \frac{1}{p} \sum_{k=0}^{p-1} \sum_{l=0}^{p-1} f(k,l)\, \delta_j(\pi_i(k,l)), \quad i \in N, j \in Z_p,$$

where $\delta_j(m)$ equals one or zero according to whether m equals j or not. The backprojection of a function g_i on G/H_i has the form

$$V_i g_i(k,l) = g_i(\pi_i(k,l)), \quad i \in N,\ (k,l) \in G,$$

shortening the expression $g_i(H_i^j)$ to $g_i(j)$.

B. Reconstruction formulae

Now, we shall rewrite Theorem 1 for the presented scheme. Let us suppose more generally that N is a nonempty subset of $\{0,1,\ldots,p\}$ with cardinality n. Observing that $H_I = G$ as soon as I has at least two elements we get immediately

$$\varepsilon = 1 - n + \sum_{i \in N} \lambda_i$$

and after a computation (see Appendix B)

$$\omega = \frac{1}{n} - n + \sum_{i \in N} \lambda_i \,.$$

The Radon transform Λ with respect to $\mathcal{H} = (H_i)_{i \in N}$ is injective according to Theorem 1 and Remark 4 if and only if $\varepsilon = \lambda_\emptyset$. This means $\varepsilon(e) = 1 - n + np = \lambda_\emptyset(e) = p^2$, where $e = (0,0)$ is the identity element of G. We arrive at the necessary condition $n = p+1$ and in this case due to $G - e = \sum_{i \in N}(H_i - e)$ we see that $\varepsilon(x) = 1 - (p+1) + p = 0$ for $x \neq e$. The Radon transform Λ is thus injective just if it consists of all $p+1$ Radon projections.

A family $g = (g_i)_{i \in N}$ of functions on the factorgroups G/H_i, $i \in N$, can be viewed as the family of projections of a function f if it fulfils the projectivity conditions of Remark 1. These conditions require that the expression

$$V_i g_i * \lambda_{i'} = V_i g_i * \lambda_i * \lambda_{i'} = V_i g_i * \lambda_G = \frac{1}{p^2} \sum_{(k,l) \in G} V_i g_i(k,l) = \frac{1}{p} \sum_{j=0}^{p-1} g_i(j), \quad i, i' \in N, i \neq i',$$

does not depend on $i \in N$. Naturally, all projections must have the same averages; in this case we denote them by $a(g)$. The smallest norm function having these prescribed projections is given by

$$\begin{aligned} Vg * \omega = & \sum_{i \in N} V_i g_i * \left(\tfrac{1}{n} - n + \sum_{j \in N} \lambda_j\right) = \\ = & \; n\, a(g) \tfrac{1-n^2}{n} + \sum_{i \in N} V_i g_i + a(g)(n^2 - n) = a(g) + \sum_{i \in N} (V_i g_i - a(g)). \end{aligned}$$

A carefull look at Theorem 3 reveals that this formula can be obtained also easily by the iterative procedure.

For the Radon transform composed of all $p+1$ Radon projections we get the identity

$$f = V\Lambda f * \omega = a(f) + \sum_{i=0}^{p} (V_i \Lambda_i f - a(f)), \quad f \in \mathbf{A}(G),$$

where

$$a(f) = \frac{1}{p^2} \sum_{(k,l) \in G} f(k,l)$$

is the average value of f. Note that if the Radon transform Λ is considered only for the functions with zero average then its dual V is also its inverse.

There is another way of viewing the above situation. Namely, the group G together with all cosets H_i^j, $0 \leq i \leq p$, $0 \leq j < p$, form the affine plane $AG(2,p)$. It has p^2 points, p^2+p lines (cosets), every point $x \in G$ lies on $p+1$ lines (in one coset of every factorgroup G/H_i), every line contains p points and, what is the most noteworthy, every two distinct points lies on just one line. Under some conditions, like the very last one, more general identities were proved (in [2] and [18]) for combinatorial designs.

C. Radon and Fourier transforms

The Fourier transform is closely related to the Radon one in the classical Radon transform theory. Roughly speaking, the one-dimensional Fourier transform of the projection $\hat{f}_q$ is equal to the two-dimensional Fourier transform of the original function f restricted on the line $q^\perp$ which is perpendicular to q and contains the origin, see [7], [18]. This assertion is called in applications *projection slice theorem* ([8], [17]). We shall see here that an analogy is valid also for the group Z_p^2.

Let us remind that the finite Fourier transforms can be written as

$$\mathbf{F}_2 f(k,l) = \frac{1}{p^2} \sum_{k'=0}^{p-1} \sum_{l'=0}^{p-1} f(k',l') \exp\left[-\frac{2\pi \mathrm{i}}{p}(kk' + l\,l')\right]$$

for functions on Z_p^2 and as

$$\mathbf{F}_1 h(j) = \frac{1}{p} \sum_{j'=0}^{p-1} h(j') \exp\left[-\frac{2\pi \mathrm{i}}{p} jj'\right]$$

for functions on Z_p.

The Fourier transform of a projection $\Lambda_i f$ is thus

$$\begin{aligned} \mathbf{F}_1 \Lambda_i f(j) &= \frac{1}{p} \sum_{j'=0}^{p-1} \frac{1}{p} \sum_{k=0}^{p-1} \sum_{l=0}^{p-1} f(k,l)\, \delta_{j'}\left(\pi_i(k,l)\right) \exp\left[-\frac{2\pi \mathrm{i}}{p} jj'\right] \\ &= \frac{1}{p^2} \sum_{k=0}^{p-1} \sum_{l=0}^{p-1} f(k,l) \exp\left[-\frac{2\pi \mathrm{i}}{p} j\, \pi_i(k,l)\right], \end{aligned}$$

i.e. we find

$$\begin{aligned} \mathbf{F}_1 \Lambda_p f(j) &= \mathbf{F}_2 f(0,j), && j \in Z_p, \\ \mathbf{F}_1 \Lambda_i f(j) &= \mathbf{F}_2 f(j, -ij (\operatorname{mod} \mathrm{p})), && 0 \leq i < p,\ \ j \in Z_p. \end{aligned}$$

Denoting by $\rho(0) = p$, $\rho(p) = 0$ and by $\rho(i) = j$ the unique solution of the equation $1 + ij = 0 \,(\operatorname{mod} p)$, we can interpret the above equations verbaly in the following form: knowledge of the Fourier transform $\mathbf{F}_1 \Lambda_i f$ of a projection $\Lambda_i f$ is nothing but knowledge of the Fourier transform $\mathbf{F}_2 f$ restricted on the subgroup $H_{\rho(i)}$, $0 \leq i \leq p$. This claim could be named *discrete projection slice theorem.* We remark that similar assertions can be formulated for compact groups by means of harmonic analysis [14].

D. Reconstruction from partial projections

Let us denote by P_i a subset of the cyclic group Z_p, $0 \leq i \leq p$, with cardinality $0 \leq |P_i| = p_i \leq p$ and by I the set of those i from $N = \{0, 1, \ldots, p\}$ for which P_i equals Z_p.

Having a system $h = (h_i)_{i \in N}$ of functions (h_i defined on P_i) we shall employ the extensions $E_t h_i$ of h_i on Z_p induced by a number t

$$E_t h_i(j) = h_i(j), \qquad\qquad j \in P_i,$$
$$E_t h_i(j) = \frac{1}{p-p_i}\left[pt - \sum_{j \in P_i} h_i(j)\right], \quad j \in Z_p - P_i.$$

Since $E_t h_i$ is constant on $Z_p - P_i$ chosen in such a way to ensure the average of $E_t h_i$ equal to t, one can speak about the *uniform extensions* induced by t. We shall write also $E_t h = (E_t h_i)_{i \in N}$.

Theorem 4: The family of equations $\Lambda_i f(H_i^j) = h_i(j)$ indexed by $i \in N$, $j \in P_i$, where $h = (h_i)_{i \in N}$ is a given system of functions, has a solution if and only if the expression $\frac{1}{p} \sum_{j \in P_i} h_i(j)$ is constant on I; for $I \neq \emptyset$ we denote it by $a(h)$ and for $I = \emptyset$ we take

$$a(h) = \frac{\sum_{i=0}^{p} \frac{1}{p-p_i} \sum_{j \in P_i} h_i(j)}{-p + \sum_{i=0}^{p} \frac{p}{p-p_i}}.$$

If this is fulfilled the function $\Lambda^{-1} E_{a(h)} h$ is the smallest norm solution.

For the proof of Theorem 4 see Appendix B.

Remarks: 1. If we do not have more than one complete projection, the family of equations is always solvable.

2. Thus, the distinguished solution has these values

$$a(h) + \sum_{i=0}^{p} [E_{a(h)} h_i(\pi_i(k,l)) - a(h)], \quad (k,l) \in Z_p^2.$$

IV. NUMERICAL EXPERIMENTS

We performed simple computational studies of the finite Radon transforms and their inversions oriented to digital image processing. These transforms were found to be numerically well tractable and the corresponding image representations interesting from the transform coding standpoint. In this section we shall demonstrate favorable behaviour and properties of the related algorithms and discuss the results of our numerical experiments.

A. Computing finite Radon transforms of images

We worked with 256 gray-levels digital images of size 127×127 pixels. Interpreting them as functions $f(k,l)$, $0 \leq k \leq 126$, $0 \leq l \leq 126$, on the group Z_{127}^2 the finite Radon transform according to the scheme from Section III.A. consists of averaging over cosets of the 128 nontrivial subgroups. They are visualized in Fig. 1; Fig. 1a exhibits the function π_{127} which is constant on cosets of H_{127} and Fig. 1b, 1c and 1d were generated to show π_{126}, π_{125} and π_{124}, respectively. These pictures resemble contour images of discrete representations of analog lines (see[17]). Fig. 1a even coincides with them. However, the visual forms of the functions π_i for a considerable part of projections look like gray unstructured images – the points of lines are as a rule “dissipated” all over the images.

In Fig. 2a we can see the digital image of a portrait which we analysed. Its finite Radon transform $g_i(j) = \Lambda_i f(H_i^j)$, $0 \leq i \leq 127$, $0 \leq j \leq 126$ was arranged into an array of 128 rows every one of lenght 127, see Fig. 2b. Computing the i-th Radon projection, i. e. the i-th row of the array, we need to pass once all pixels of the original image and to employ 127 histogrammers – one for every pixel in the row; the gray level $f(k,l)$ of a pixel (k,l) is added to the $\pi_i(k,l)$ histogrammer. At the end all 127 histogrammed values are divided by 127 to get the average values and the results are rounded off to integer numbers. This is the only source of noise in our model .

To illustrate this algorithm we present the following scheme.

```
begin
    for i = 0, p − 1
        g_i = 0
        n = i
        for l = 0, p − 1
            n = n − i
            if n < 0 then n = n + p
            j = n − 1
            for k = 0, p − 1
                j = j + 1
                if j ≥ p then j = j − p
                g_i(j) = g_i(j) + f(k, l)
            endfor
        endfor
        g_i = g_i/p
    endfor
    g_p = 0
    for l = 0, p − 1
        for k = 0, p − 1
            g_p(l) = g_p(l) + f(k, l)
        endfor
    endfor
    g_p = g_p/p
end
```

Having at disposal a pipeline of 128 stages one pass of the image through it gives all projections, every one on the corresponding stage. Hence, the computing of the finite Radon transform can be pipelined and performed by P^3E architecture. Another, more effective, way of computing this transform is to use the results of III.C. Accordingly, one performs the two-dimensional finite Fourier transform followed by the 128 one-dimensional ones. This is, of course, much more noisy.

B. Computing the inverse

Our original image was adjusted to have the average value $a(f) = 128$. Its reconstruction was obtained column by column as follows. In the l-th column, $0 \leq l \leq 126$, we employ 127 histogrammers, the k-th one taking care about the pixel (k,l), $0 \leq k \leq 126$. The initial value of all histogrammers is taken to be $g_{127}(l)$ (this is the l-th record in the last row of our projection array). This originates from $a(f) + (g_{127}(l) - a(f))$ what is the constant plus the last term in the sum of the reconstruction formula. Now, it suffices to pass the projection array from the first row to the 126-th row and to find to every datum $g_i(j) = \Lambda_i f(j)$, $0 \leq i \leq 126$, $0 \leq j \leq 126$, the uniquely given index k for which $\pi_i(k,l) = j$. The correction $g_i(j) - a(f)$ is then added to the k-th histogrammer.

More formally, the algorithm is described as follows (note that in this and foregoing schemes we do not use multiplications even).

```
begin
    a = 0
    for j = 0, p − 1
        a = a + g_0(j)
    endfor
    a = a/p
    for l = 0, p − 1
        for k = 0, p − 1
            f(k, l) = g_p(l)
        endfor
        n = − l
        for i = 0, p − 1
            n = n + l
            if n ≥ p then n = n − p
            k = n − 1
            for j = 0, p − 1
                k = k + 1
                if k ≥ p then k = k − p
                f(k, l) = f(k, l) + g_i(j) − a
            endfor
        endfor
    endfor
end
```

This algorithm can be performed again by a pipeline of 127 stages every one taking care about one column of the reconstructed image. Evidently only one pass of the projection data is needed. Moreover, under the assumption that we know precisely $a(f)$ (of an integer value) this inversion procedure works without numerical errors.

The result of the reconstruction is shown in Fig. 2c. Apparently, there is no visual difference between the original image and the reconstructed one and the difference, caused exlusively by the noise accompanying the finite Radon transform, is in fact not greater than two gray levels at each pixel.

The histograms of the portrait image in Fig. 2a and its Radon transform array are shown in Fig. 3. Note that the original image contents all 256 gray levels but the array only 96 gray levels.

C. Compression by rows

Typical row profiles of the finite Radon transform are shown in Fig. 4. Intuitively, profiles in Fig. 4a, 4b corresponding to the rows 128 and 64, respectively, retain a lot of information about the original image whereas profiles in Fig. 4c, 4d (rows 116 and 52) are much less informative.

To ground this intuition let us look more carefully at the reconstruction formula

$$a(f) + \sum_{i=0}^{127} \left[V_i \Lambda_i f - a(f)\right].$$

It consists of the constant $a(f)$ and 128 functions on $G = Z_{127}^2$ all being mutually orthogonal. In fact, since $\lambda_G = 1$, $\lambda_G * \lambda_i = \lambda_G$ and $\lambda_i * \lambda_{i'} = \lambda_G$ ($i, i' \in N$ distinct) we get

$$\langle V_i \Lambda_i f - a(f), \lambda_G \rangle = \langle f, V_i \Lambda_i \lambda_G \rangle - a(f) = \langle f, \lambda_G * \lambda_i \rangle - a(f) = 0$$

$$\langle V_i\Lambda_i f - a(f), V_{i'}\Lambda_{i'} f - a(f)\rangle = \langle f * \lambda_i * \lambda_{i'}, f\rangle - \langle a(f), f * \lambda_{i'}\rangle - \langle f * \lambda_i, a(f)\rangle + |a(f)|^2 = 0.$$

Observing that, due to

$$||V_i\Lambda_i f - a(f)||^2 = ||V_i(\Lambda_i f - a(f))||^2 = \langle \Lambda_i V_i(\Lambda_i f - a(f)), \Lambda_i f - a(f)\rangle = ||\Lambda_i f - a(f)||^2,$$

the norms of these functions are just the variances of the Radon projections (rows in Fig. 2b), we can conclude that projections such as those in Fig. 4c, 4d do not contribute substantially to the reconstruction formula and, when omitted, the resulting image differs only mildly (in the square norm) from the original one.

The rows of the Radon transform image were sorted according to their variances; the ordered finite Radon transform is shown in Fig. 2d. Let us note that from the 4th row the diapason of gray levels enables to encode them to six bits and starting from the 40th row to four bits only. In this way, a 59 % compression of the original image can be gained without any lost of information.

Further, we have used for the reconstruction only first n (more informative) rows of Fig. 2d. The results are exhibited in Fig. 5 (from left to right and from up to bottom we utilized $4, 7, 10, 20; 30, 40, 50, 60; 70, 80, 90, 100; 110, 120$ and 128 projections; the last area is occupied by the original image). The achieved compression levels are presented in Table I.

Let us remark that this way of compression is in the main equivalent to the Fourier transform coding as the omission of noninformative rows corresponds to a restriction of the Fourier transform F_2 (cf. discrete projection slice theorem).

D. Compression by pixels

Another look at the reconstruction formula reveals that the pixels with the gray level $128 = a(f)$ do not contribute to the reconstructed image at all. The most informative pixels are thus the most different from the mean value. This claim is based on the identity from the beginning of Proof of Theorem 4 (see Appendix B) which can be rewritten as

$$||f||^2 = \frac{1}{p^2}\sum_{k,l}|f(k,l)|^2 = |a(f)|^2 + \sum_{i=0}^{p}\frac{1}{p}\sum_{j=0}^{p-1}|\Lambda_i f(H_i^j) - a(f)|^2.$$

Of course, we have to save the knowledge about the positions of the chosen pixels, e.g. by a binary mask.

For the reconstruction of the analysed image seven intensity bands defining the partial projections were consecutively used. Table II reports on intensity values, numbers of pixels in each band and compression ratios. The corresponding partial reconstructions are shown in Fig. 6. Under comparision with the above described "row compression" method, the results here are of better visual quality at the same compression level.

E. Two-dimensional convolution

The Radon transform can serve as a tool for reducing the dimensionality of certain image processing techniques, too. In our setting the convolution of an image f with a kernel φ of small size coincides with the convolution $f * \varphi$ in Z_p^2 up to border effects and can be computed due to Lemma 1 also by the formula

$$f * \varphi = \Lambda^{-1}\Lambda(f * \varphi) = \Lambda^{-1}(\Lambda f * \Lambda\varphi)$$

(cf. with the convolution theorem in [17]). In other words, two-dimensional convolutions in image space correspond to series of $p+1$ one dimensional row convolutions in projection representation space. The Fourier methods are, of course, more effective from a computational standpoint. Nevertheless, having compressed projection array by rows, the above formula implies that the convolutions are to be processed only for the informative rows, which might bring additional simplification of computations.

APPENDIX A: PROOFS OF ASSERTIONS FROM SECTION II.

Proof of Lemma 2: For any $x \in G$ we have $\lambda_I * \lambda_J(x) = |H_I|^{-1} \sum_{y \in H_I} \lambda_J(y^{-1}x)$. This yields that if $x \notin H_{I \cup J}$ then for $y \in H_I$ the element $y^{-1}x$ does not belong to $y^{-1}H_{I\cup J} = H_{I\cup J} = H_I H_J \supset H_J$, i.e. $\lambda_I * \lambda_J(x) = 0$. In the opposite case, $x \in H_{I\cup J}$, we observe that $y \in H_I$ and $y^{-1}x \in H_J$ if and only if $y \in (xH_J) \cap H_I$. The last set is nonempty for having x in the form $x = z_1 z_2$, $z_1 \in H_I$, $z_2 \in H_J$, the element z_1 belongs to $(xH_J) \cap H_I = (z_1 H_J) \cap H_I = z_1(H_I \cap H_J)$. Now, it is easily seen that

$$\lambda_I * \lambda_J(x) = \frac{|H_I \cap H_J||G/H_J|}{|H_I|} = \frac{|H_I \cap H_J||G|}{|H_I||H_J|} = \frac{|G|}{|H_{I\cup J}|} = |G/H_{I\cup J}|, \quad x \in H_{I\cup J},$$

takes place employing standard identities of group theory.

Having proved 6. we apply it immediately. If $i \in N$ then

$$\varepsilon * \lambda_i = [\lambda_\emptyset + \sum_{i \in I \subset N} (-1)^{|I|-1} (\lambda_I - \lambda_{I-\{i\}})] * \lambda_i = \lambda_i$$

and straighforward 7. follows for nonempty I, namely if one chooses any $i \in I$ then

$$\varepsilon * \lambda_I = \varepsilon * (\lambda_i * \lambda_I) = (\varepsilon * \lambda_i) * \lambda_I = \lambda_i * \lambda_I = \lambda_I.$$

Now, the statement 8. is a trivial consequence of 7.

$$\varepsilon * \varepsilon = \sum_{\emptyset \neq I \subset N} (-1)^{|I|-1} \varepsilon * \lambda_I = \varepsilon \left[1 - \sum_{I \subset N} (-1)^{|I|}\right] = \varepsilon$$

and the validity of 9. shows this computation ($c_t = \sum_{s=1}^{t} \frac{1}{s}$, $t \geq 0$)

$$\begin{aligned}\sum_{i \in N} \lambda_i * \omega &= \sum_{i \in N} \sum_{\emptyset \neq I \subset N} (-1)^{|I|-1} \lambda_{I \cup \{i\}}\, c_{|I|} = \sum_{\emptyset \neq I \subset N} (-1)^{|I|-1} \left[|I|\, \lambda_I\, c_{|I|} + \sum_{i \in N-I} \lambda_{I \cup \{i\}} c_{|I|} \right] = \\ &= \sum_{\emptyset \neq I \subset N} |I|\, \lambda_I \left[(-1)^{|I|-1} c_{|I|} + (-1)^{|I|-2} c_{|I|-1} \right] = \varepsilon \qquad \square\end{aligned}$$

Proof of Theorem 1: The condition $\Lambda(Vg * \omega) = g$ is trivially sufficient for existence of a solution. If $\Lambda f = g$ then the projectivity conditions hold as we have seen in Remark 1. From them we derive (see Lemma 1 and Lemma 2)

$$\begin{aligned}\Lambda_i(Vg * \omega) = {} & \Lambda_i V_i \Lambda_i (Vg * \omega) = \Lambda_i (Vg * \omega * \lambda_i) = \Lambda_i (\sum_{j \in N} V_j g_j * \lambda_i * \omega) = \\ = {} & \Lambda_i (V_i g_i * \sum_{j \in N} \lambda_j * \omega) = \Lambda_i (V_i g_i * \varepsilon) = \Lambda_i (V_i g_i * \varepsilon * \lambda_i) = g_i, \quad i \in N.\end{aligned}$$

This crucial computation closes the first part of the proof.

The difference of two solutions of the equation belongs to the kernel of the operator Λ and this can be characterized also as $\mathcal{N}_\Lambda = \{f; \Lambda f = 0\} = \{f; f * \varepsilon = 0\}$. In fact, if $\Lambda_i f = 0$

then $V_i\Lambda_i f = f * \lambda_i = 0$ and for $\emptyset \neq I \subset N$ and $i \in I$, moreover, $f * \lambda_I = f * \lambda_i * \lambda_I = 0$; finally $f * \varepsilon = 0$. Conversely, $f * \varepsilon = 0$ implies $f * \varepsilon * \lambda_i = f * \lambda_i = V_i\Lambda_i f = 0$ and due to injectivity of V_i also $\Lambda_i f = 0$, $i \in N$.

The algebra $\mathbf{A}(G)$ can be, as a complex Hilbert space, orthogonally decomposed into $\mathcal{N}_\Lambda$ and $\mathcal{R}_V$ where $\mathcal{R}_V$ is the range of the backprojection V. But, the solution $Vg * \omega = V(g * \Lambda\omega)$ belongs to $\mathcal{R}_V$ and, thus, has the smallest norm among all solutions of the equation. □

Proof of Theorem 2: The condition $f * \varepsilon = f$ is sufficient for if it is satisfied then by 9. of Lemma 2

$$V(\Lambda f * \omega) = \sum_{i \in N} V_i\Lambda_i(f * \omega) = f * \omega * \sum_{i \in N} \lambda_i = f * \varepsilon = f,$$

i.e. $g = \Lambda(f * \omega)$ solves the equation $Vg = f$. In addition, this reasoning

$$Vg * \varepsilon = \sum_{i \in N} V_i\Lambda_i V_i g_i * \varepsilon = \sum_{i \in N} V_i g_i * \lambda_i * \varepsilon = \sum_{i \in N} V_i g_i * \lambda_i = Vg, \quad g \in \mathbf{A}(\mathcal{H}),$$

shows that $\mathcal{R}_V = \{f; f * \varepsilon = f\}$. The mapping $f \to f * \varepsilon$ is the orthogonal projector on $\mathcal{R}_V$ (cf. Remark 3).

Note that $Vg = 0$ if and only if $\Lambda(Vg * \omega) = 0$. In fact, to prove the nontrivial implication we apply the operator V

$$0 = V\Lambda(Vg * \omega) = Vg * \omega * \sum_{i \in N} \lambda_i = Vg * \varepsilon = Vg.$$

The algebra $\mathbf{A}(\mathcal{H})$ can be orthogonally decomposed into the kernel $\mathcal{N}_V$ of the back-projection and the range $\mathcal{R}_\Lambda$ of the Radon projection, remark that $g \to \Lambda(Vg * \omega)$ is the orthogonal projector on $\mathcal{R}_\Lambda$. As $\Lambda(f * \omega)$ belongs to $\mathcal{R}_\Lambda$ it has the smallest norm among all solutions of the equation. □

Proof of Theorem 3: Let us prove $f_{(i+j)} * \lambda_i = V_i g_i$ for $i \in N$ and $j = 0, 1, \ldots, n-i$ by induction on $j = 0, 1, \ldots, n-1$. For $j = 0$ we have

$$f_{(i)} * \lambda_i = f_{(i-1)} * \lambda_i + V_i g_i * \lambda_i - V_i\Lambda_i f_{(i-1)} * \lambda_i = V_i g_i, \quad i \in N,$$

and the induction argument has the form ($0 \leq j < n-1$)

$$\begin{aligned} f_{(i+j+1)} * \lambda_i &= f_{(i+j)} * \lambda_i \ + \ V_{i+j+1}\, g_{i+j+1} * \lambda_i \ - \ f_{(i+j)} * \lambda_{i+j+1} * \lambda_i = \\ &= V_i g_i \ + \ V_{i+j+1}\, g_{i+j+1} * \lambda_i \ - \ V_i g_i * \lambda_{i+j+1} = V_i g_i, \quad 1 \leq i < n-j, \end{aligned}$$

employing the projectivity conditions.

Thus, $\Lambda f_{(n)} = g$. The choice $f_{(0)} \in \mathcal{R}_V$ gives

$$f_{(n)} = f_{(0)} + \sum_{i \in N} V_i(g_i - \Lambda_i f_{(i-1)}) \ \in \mathcal{R}_V.$$

But, from Proof of Theorem 1 we know that the equation $\Lambda f = g$ has only one solution in $\mathcal{R}_V$ what yields $f_{(n)} = Vg * \omega$ whatever $f_{(0)} \in \mathcal{R}_V$ was chosen. □

APPENDIX B: PROOFS OF ASSERTIONS FROM SECTION III.

Derivation of the form of ω: We have realized that for I having more than one element $\lambda_I = \lambda_G = 1$ and then

$$\omega = \sum_{\emptyset \neq I \subset N} (-1)^{|I|-1} \lambda_I \sum_{s=1}^{|I|} \frac{1}{s} = \sum_{i \in N} \lambda_i - n + \sum_{I \subset N} (-1)^{|I|-1} \sum_{s=1}^{|I|} \frac{1}{s}.$$

Rewriting the sum of the fractions into

$$\sum_{s=1}^{r} \frac{1}{s} = \int_0^1 \sum_{s=1}^{r} z^{s-1} dz = \int_0^1 \frac{1-z^r}{1-z} dz$$

we obtain

$$\begin{aligned}\sum_{I \subset N} (-1)^{|I|-1} \sum_{s=1}^{|I|} \frac{1}{s} &= \int_0^1 \frac{1}{1-z} \sum_{I \subset N} (-1)^{|I|-1} (1 - z^{|I|}) dz = \int_0^1 \frac{1}{1-z} \sum_{I \subset N} (-1)^{|I|} z^{|I|} dz = \\ &= \int_0^1 (1-z)^{n-1} dz = \int_0^1 z^{n-1} dz = \frac{1}{n}.\end{aligned}$$

The convolving function ω can be found also by using the finite Fourier transform and Lemma 2; we omit details. □

Proof of Theorem 4: The solvability condition is evidently necessary. If it is valid then $E_{a(h)}h$ satisfies the projectivity conditions and hence $\Lambda^{-1}E_{a(h)}h$ is a solution.

To prove the last assertion of the theorem we shall need this identity

$$\|f\|^2 = -p|a(f)|^2 + \|\Lambda f\|^2, \;\; f \in \mathbf{A}(G).$$

Note that the finite Radon transform conserves the norm of functions with zero average.The supporting computations follow

$$\begin{aligned}\|\Lambda f\|^2 &= \sum_{i=0}^{p} \|\Lambda_i f\|^2 = \frac{1}{p} \sum_{i=0}^{p} \sum_{j=0}^{p-1} |\Lambda_i f(H_i^j)|^2 = \frac{1}{p^3} \sum_{i=0}^{p} \sum_{j=0}^{p-1} |\sum_{k,l} f(k,l) \delta_j(\pi_i(k,l))|^2 = \\ &= \frac{1}{p^3} \sum_{k,l} \sum_{k',l'} f(k,l) \overline{f}(k',l') \sum_{i=0}^{p} \sum_{j=0}^{p-1} \delta_j(\pi_i(k,l)) \delta_j(\pi_i(k',l'));\end{aligned}$$

the double inner sum is equal to 1 for $(k,l) \neq (k',l')$ and to $p+1$ otherwise what yields

$$\|\Lambda f\|^2 = \frac{1}{p^3} \sum_{k,l} \sum_{k',l'} f(k,l) \overline{f}(k',l') + \frac{1}{p^2} \sum_{k,l} |f(k,l)|^2 = p|a(f)|^2 + \|f\|^2.$$

This identity follows also from the discrete projection slice theorem.

Let us suppose now that f is a solution of the family of equations. Then

$$\|f\|^2 = -p|a(f)|^2 + \sum_{i \in I} \|h_i\|^2 + \frac{1}{p} \sum_{i \in N-I} \sum_{j \in P_i} |h_i(j)|^2 + \frac{1}{p} \sum_{i \in N-I} \sum_{j \in Z_p - P_i} |\Lambda_i f(H_i^j)|^2.$$

Having a complete projection ($I \neq \emptyset$) we estimate by Cauchy inequality

$$\sum_{j \in Z_p - P_i} |\Lambda_i f(H_i^j)|^2 \geq \frac{1}{p - p_i} \left| \sum_{j \in Z_p - P_i} \Lambda_i f(H_i^j) \right|^2 = \frac{1}{p - p_i} \left| p\, a(f) - \sum_{j \in P_i} h_i(j) \right|^2, \;\; i \in N - I,$$

and from $a(f) = a(h) = a(f^*)$, where $f^* = \Lambda^{-1}E_{a(h)}h$, we conclude

$$\|f\|^2 \geq -p|a(f^*)| + \sum_{i\in I} \|\Lambda_i f^*\|^2 + \frac{1}{p}\sum_{i\in N-I}\sum_{j=0}^{p-1} |E_{a(h)}h_i(j)|^2 = \|f^*\|^2$$

the equality taking place just if $\Lambda f = \Lambda f^*$, i.e. $f = f^*$.

Having no complete projection we estimate similarly

$$\|f\|^2 \geq -p|a(f)|^2 + \frac{1}{p}\sum_{i\in N}\sum_{j\in P_i} |h_i(j)|^2 + p\sum_{i\in N}\frac{1}{p-p_i}|a(f) - \frac{1}{p}\sum_{j\in P_i} h_i(j)|^2.$$

An elementary computation shows that this function

$$-t^2 + \sum_{i\in N}\frac{|t-t_i|^2}{p-p_i}$$

of the complex variable t takes its strong global minimum at the point

$$t^* = \frac{\sum_{i\in N}\frac{t_i}{p-p_i}}{-1+\sum_{i\in N}\frac{1}{p-p_i}}$$

Thus,

$$\|f\|^2 \geq -p|a(h)|^2 + \sum_{i\in N}\|E_{a(h)}h_i\|^2 = \|f^*\|^2$$

with equality just for $f = f^*$. □

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Captions of Figures and Tables

Fig. 1 Cosets H_i^j of nontrivial subgroups H_i (parallel lines of the geometry)

- **a)** $i = 127$
- **b)** $i = 126$
- **c)** $i = 125$
- **d)** $i = 124$

Fig. 2 **a)** original image, 127×127 pixels, 256 gray levels

- **b)** Radon transform of the image a)
- **c)** reconstruction of the image a) via inverse Radon transform
- **d)** Radon transform with ordered rows according to their variances

Fig. 3 **a)** histogram of the original image

- **b)** histogram of its Radon transform

Fig. 4 Typical row profiles of the Radon transform array

- **a)** row 128
- **b)** row 64
- **c)** row 116
- **d)** row 52

Fig. 5 Partial reconstructions from the most informative Radon transform rows (according to Table I.)

Fig. 6 Partial reconstructions from the most informative pixels (according to Table II.)

Tab.I. Partial reconstructions from the most informative rows

nr – the number of rows that were used

c – compression level of the original image (in percentage)

Tab.II. Partial reconstructions from the most informative pixels

band – the intensity band in the Radon transform array

np – the number of pixels in the band

c – compression level of the original image (in percentage)

nr	c
4	3
7	5
10	7
20	13
30	19
40	24
50	28
60	32
70	36
80	40
90	44
100	48
110	52
120	56
128	59

Table I.

image	band	np	c
a	0 - 119, 137 - 255	1 179	7
b	0 - 122, 134 - 255	2 169	11
c	0 - 124, 132 - 255	4 510	22
d	0 - 125, 131 - 255	6 779	29
e	0 - 126, 130 - 255	10 009	40
f	0 - 127, 129 - 255	14 058	52
g	0 - 255	16 256	59

Table II.