Theoretical Computer Science
134, 455–471 (1994)

Stochastic independence, algebraic independence and abstract connectedness

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Abstract. Mutual stochastic independences among σ-algebras and mutual algebraic independences among elements of semimodular lattices are observed to have a very similar behaviour. We suggest abstract independence structures called I-relations describing it. Presented examination of I-relations resembles a theory of abstract connectedness: a dual characterization of I-relations by families of connected sets is found by means of a special Galois connection. Representations of I-relations in the matroid theory sense by σ-algebras and by elements of lattices are discussed.

Introduction

Independence structures can be found in profusion throughout many mathematical disciplines. The concept of matroid which appeared in the works of Whitney [27] and Van der Waerden [24] is a classical model of abstract independence approaches. It describes, for example, linear independence, affine independence and algebraic independences in fields and lattices (cf. Birkhoff [3], Grätzer [8]). Successes of matroid theory (see Welsh [25], White [26]) in diverse areas of geometry, algebra and combinatorics cause that matroids are only in a smaller part of this theory viewed as independence structures. Also various generalizations of the concept (for an account see Kung [11]) bear this feature. Other independence structures were motivated by Marczewski [14] and his notion of independence in abstract algebras. Various kinds of independences in groups are modelled by the dependence structures of Dlab [5] (a review of the fields is in Grätzer [9]).

The aim of this paper is to present a study of the notion of stochastic independence and mainly of its abstract independence structures in spirit of the mentioned conceptions.

Conditional stochastic independence being one of the building stones of probability theory and statistics was thoroughly examined by Dawid [4], Mouchart and Rolin [17], Studený [21] and others from this point of view. Corresponding independence structures developed during last decade by Pearl, Geiger, Malvestuto, Paz, Verma and others (cf. [7], [13], [6], we mention only recent works containing complete lists of relevant references) are strongly influenced by database constraints theory (see Armstrong [1], Armstrong and Delobel [2]). The motivation comes especially from the functional dependence structures, which are also generalizations of matroids and belong not solely to the domain of databases (cf. Malvestuto [12], Matúš [16]). Another approach is brought in by Studený [22].

More precisely speaking, the paper is based on an analysis of mutual stochastic independences among σ-algebras according to Situation 1 on parallel lines with algebraic independences among elements of semimodular lattices according to Situation 2 (see section 1). Apart from these situations we define in section 2 the notion of I-relation in which the essence of both ones is extracted. Situation 2 serves as proper motivation to our examination of the I-relations. The crucial point is the introduction of connected sets: the standard property of connected sets (in topological spaces, matroids or graphs) expressed by the phrase "the union of two connected sets with nonempty intersection is connected" becomes the principal axiom of our concept of C-families (section 3).

A nontrivial bijection between the classes of I-relations and C-families is revealed and elaborated using the methodology of Galois connections from lattice theory (section 4). These results open a study of connectedness on abstract level and enable to recognize the way in which I-relations generalize matroids. Finally, they are applied to representation problems in the matroid theory sense, i.e. to a prescribed I-relation there are constructed σ-algebras which have their mutual stochastic independences described just by the I-relation (Theorem 2, section 5) and similarly I-relations are proved to be an adequate tool for dealing with algebraic independence in Situation 2 (Theorem 3, section 6).

1. Stochastic and algebraic independences

Let N be any finite or infinite set over which we shall consider our independence relations and by the elements i and j of which the objects in question will be indexed. By $\mathcal{F}(N)$ we denote the family of all finite subsets of N, by $\mathcal{P}(N)$ the family of all subsets of N, by $\mathcal{G}(N) = \{(I,J);\ I,J \in \mathcal{F}(N)\}$ the set of all ordered pairs of finite subsets of N and by $\mathcal{E}(N)$ the lattice of all equivalences on N (see Grätzer [8]). To the cardinality of sets we reserve the symbol $|\ |$.

The investigation leading to this paper was started considering the following general situation in the very fundamentals of probability theory.

SITUATION 1. *Let $(\Omega, \mathbf{S}, P)$ be a probability space and $\mathcal{S} = (\mathcal{S}_i)_{i\in N}$ a system of sub-σ-algebras of the σ-algebra $\mathbf{S}$ indexed by N. We denote by $\mathcal{S}_I$, $I \in \mathcal{F}(N)$, the smallest sub-σ-algebra of $\mathbf{S}$ containing the union of $\mathcal{S}_i$ for $i \in I$; in symbols $\mathcal{S}_I = \sigma\left(\bigcup_{i\in I} \mathcal{S}_i\right)$ (note that $\mathcal{S}_\emptyset = \{\emptyset, \Omega\}$). One suggests to analyse mutual stochastic independences among the σ-algebras of the system $\mathcal{S}$ by means of the binary relation $\mathcal{K}_\mathcal{S}^{(1)}$ on $\mathcal{F}(N)$ given by*

$$(I,J) \in \mathcal{K}_\mathcal{S}^{(1)} \iff \forall A \in \mathcal{S}_I \quad \forall B \in \mathcal{S}_J \quad P(A\cap B) = P(A)\ P(B),$$

i.e. a pair (I,J) belongs to $\mathcal{K}_\mathcal{S}^{(1)}$ if and only if the σ-algebras $\mathcal{S}_I$ and $\mathcal{S}_J$ are stochastically independent (see [18]*).*

Our main contribution to the examination of Situation 1 presented below consists likely in the fact that it was related to Situation 2 which will give rise to the same kind of independence structures and will serve as an inspiration for dealing them.

SITUATION 2. *Let Z be a semimodular lattice in which the descending chain condition is valid and let $z = (z_i)_{i \in N}$ be a system of its elements indexed by N. We denote by z_I, $I \in \mathcal{F}(N)$, the join of all elements z_i for $i \in I$, i.e. $z_I = \vee_{i \in I} z_i$ ($z_\emptyset = 0$). The relation $\mathcal{K}_z^{(2)} \subset \mathcal{G}(N)$ determined by*

$$(I, J) \in \mathcal{K}_z^{(2)} \iff z_I \quad \textit{and} \quad z_J \quad \textit{are algebraically independent}$$

is proposed to distinguish mutual algebraic independences among the elements of the system z.

We recall that all maximal chains connecting 0 and $x \in Z$ have the same finite length, called the height $h(x)$ of x, and that the inequality

$$h(x \vee y) + h(x \wedge y) \leq h(x) + h(y), \qquad x, y \in Z\,,$$

is valid due to the semimodularity. Two elements x, y of Z are said to be algebraically independent if and only if $h(x \vee y) = h(x) + h(y)$ (see [3], [8] where equivalent characterizations are discussed).

An important special case of Situation 2, when N is finite and every z_i is an atom of Z, corresponds to the notion of loopless matroid (see [25], the function r on $\mathcal{F}(N)$ given by $r(I) = h(z_I)$ is a rank function). We see now that (I, J) belongs to $\mathcal{K}_z^{(2)}$ if and only if (I, J) is a pair of separated sets of the respective matroid.

2. I-relations

As a roof covering both previous situations we shall give here the definition of I-relations as well as some simple related technicalities.

DEFINITION 1. A nonempty subset $\mathcal{K}$ of $\mathcal{G}(N)$ is called *I-relation on N* if it fulfills these three conditions

1. $(I, I) \in \mathcal{K},\ L \in \mathcal{F}(N) \Longrightarrow (I, L) \in \mathcal{K}$
2. $(I, J) \in \mathcal{K},\ L \subset J \Longrightarrow (I, L) \in \mathcal{K}$
3. $(I \cup J, K) \in \mathcal{K},\ (I, J) \in \mathcal{K} \Longrightarrow (J \cup K, I) \in \mathcal{K}.$

The following two assertions are expected to be, of course, valid.

LEMMA 1. *$\mathcal{K}_{\mathcal{S}}^{(1)}$ is an I-relation on N (cf. Situation 1).*

PROOF. $\mathcal{K}_{\mathcal{S}}^{(1)}$ meets evidently 2. It follows from $(I, I) \in \mathcal{K}_{\mathcal{S}}^{(1)}$ that all events from $\mathcal{S}_I$ have the probabilities 0 or 1 (in this case one says that $\mathcal{S}_I$ is a trivial σ-algebra). Then $\mathcal{S}_I$ is stochastically independent of $\mathbf{S}$ and consequently of any $\mathcal{S}_L$, $L \in \mathcal{F}(N)$.

The assumptions $(I \cup J, K) \in \mathcal{K}_{\mathcal{S}}^{(1)}$ and $(I, J) \in \mathcal{K}_{\mathcal{S}}^{(1)}$ imply the equality

$$P(A \cap B \cap C) = P(A \cap B)\, P(C) = P(A)\, P(B)\, P(C) = P(B \cap C)\, P(A)$$

for events A, B, C belonging to $\mathcal{S}_I$, $\mathcal{S}_J$, $\mathcal{S}_K$, respectively (remark that it expresses the mutual stochastic independence of $\mathcal{S}_I$, $\mathcal{S}_J$, $\mathcal{S}_K$). Hence, the family $\mathcal{S}_{J,K} = \{B \cap C;\ B \in \mathcal{S}_J,\ C \in \mathcal{S}_K\}$

is stochastically independent of $\mathcal{S}_I$ and by [18], IV-4 also $\sigma(\mathcal{S}_{J,K}) = \mathcal{S}_{J\cup K}$ independent of $\mathcal{S}_I$, i.e. $(J \cup K, I) \in \in \mathcal{K}_{\mathcal{S}}^{(1)}$. $\square$

LEMMA 2. $\mathcal{K}_z^{(2)}$ *is an I-relation on N (cf. Situation 2).*

PROOF. If (I, I) belongs to $\mathcal{K}_z^{(2)}$ then $h(z_I) = 0$, whence $z_I = 0$ and

$$h(z_I) + h(z_L) = h(z_L) = h(z_L \vee 0) = h(z_{I\cup L}), \qquad L \in \mathcal{F}(N).$$

To verify the second condition we begin from $h(z_{I\cup J}) = h(z_I) + h(z_J)$ and from the fact $z_{I\cup L} \wedge z_J \geq z_L,\ L \subset J$. Due to the properties of h we get

$$h(z_I) + h(z_L) \geq h(z_{I\cup L}) \geq h(z_{I\cup L} \wedge z_J) + h(z_{I\cup J}) - h(z_J) \geq h(z_L) + h(z_I).$$

The third condition can be easily obtained by adding the equalities

$$\begin{aligned} h(z_{I\cup J}) + h(z_K) &= h(z_{I\cup J\cup K}) \\ h(z_I) + h(z_J) &= h(z_{I\cup J}) \end{aligned}$$

and by the observation

$$h(z_{J\cup K}) + h(z_I) \geq h(z_{I\cup J\cup K}) = h(z_I) + h(z_J) + h(z_K) \geq h(z_{J\cup K}) + h(z_I).$$

$\square$

Simple properties of I-relations are collected below.

LEMMA 3. *Let (I, J) stands here (and in the following proof) for an abbreviation of the statement $(I, J) \in \mathcal{K}$, where $\mathcal{K}$ is any fixed I-relation on N. Then the following implications are valid.*

4. $(I, J) \Longrightarrow (J, I)$
5. $(I, J),\ K \subset I,\ L \subset J \Longrightarrow (K, L)$
6. $I \in \mathcal{F}(N) \Longrightarrow (\emptyset, I)$
7. $(I \cup J, K \cup L),\ (I, J),\ (K, L) \Longrightarrow (I \cup K, J \cup L)$
8. $(I, I),\ (J, J) \Longrightarrow (I \cup J, I \cup J)$
9. $(I, J),\ (K, K),\ (L, L) \Longrightarrow (I \cup K, J \cup L)$

PROOF. The symmetry of I-relations is a consequence of 2. and 3.

$$\begin{aligned} &(I, J) \Longrightarrow (I, \emptyset) \\ &(I \cup \emptyset, J),\ (I, \emptyset) \Longrightarrow (J, I). \end{aligned}$$

From it and from heredity in the second coordinate (2.) the heredity in both coordinates (5.) can be obtained. Applying 5. we get immediately $(\emptyset, \emptyset)$ (since $\mathcal{K} \neq \emptyset$) and by 1. also 6.

The seventh implication can be verified using properly 3. three times

$$\begin{aligned} &(I \cup J, K \cup L),\ (I, J) \Longrightarrow (J \cup K \cup L, I) \\ &(K \cup L, I \cup J),\ (K, L) \Longrightarrow (I \cup J \cup L, K) \\ &(J \cup K \cup L, I),\ (J \cup L, K) \Longrightarrow (I \cup K, J \cup L); \end{aligned}$$

we have not been commenting employements of the symmetry and heredity.

Further, 8. follows from 1. and 3.

$$(I,I),\ (J,J) \Longrightarrow (I\cup J,I)\ (I\cup J,J)$$
$$(I\cup J\cup J,I),\ (I\cup J,J) \Longrightarrow (I\cup J,I\cup J)$$

and 9. does similarly (3. used two times)

$$\begin{aligned}(K,K),\ (L,L) &\Longrightarrow (I\cup J\cup L,K),\ (I\cup J,L)\\ (I\cup J,L),\ (I,J) &\Longrightarrow (J\cup L,I)\\ (J\cup L\cup I,K),\ (J\cup L,I) &\Longrightarrow (I\cup K,J\cup L).\end{aligned}$$

□

We shall denote by $\mathbf{K}(N)$ the class of all I-relations on N. After the observation that intersection of any class of I-relations on N is an I-relation on N, i.e. that $\mathbf{K}(N)$ is a closure system on $\mathcal{G}(N)$ (see [3], [8]), we can find to any $\mathcal{L}\subset\mathcal{G}(N)$ the smallest I-relation on N containing $\mathcal{L}$ as usually

$$\kappa(\mathcal{L})=\bigcap\{\mathcal{K}\in\mathbf{K}(N);\ \mathcal{L}\subset\mathcal{K}\}.$$

The Greek letter κ denotes here the corresponding closure operator on $\mathcal{G}(N)$ (i.e. a mapping of $\mathcal{P}(\mathcal{G}(N))$ onto $\mathbf{K}(N)$ with the property

$$\mathcal{K}\subset\mathcal{L}\subset\mathcal{G}(N)\Longrightarrow\mathcal{K}\cup\kappa(\mathcal{K})\cup\kappa\left(\kappa(\mathcal{K})\right)\subset\kappa(\mathcal{L}),$$

cf. [20]). Another "iterative" form of this closure operator can be written similarly like in abstract algebras (see [9]); for $\mathcal{L}\subset\mathcal{G}(N)$ we set

$$\begin{aligned}\mathcal{L}_0 &= \mathcal{L}\\ \mathcal{L}_{n+1} &= \mathcal{L}_n \cup \{(I,L);\ (I,I)\in\mathcal{L}_n,\ L\in\mathcal{F}(N)\}\cup\\ &\qquad\cup \{(I,L);\ \exists\ J\subset N:\ (I,J)\in\mathcal{L}_n,\ J\supset L\}\cup\\ &\qquad\cup \{(J\cup K,I);\ (I\cup J,K)\in\mathcal{L}_n,\ (I,J)\in\mathcal{L}_n\},\quad n=0,1,\ldots,\end{aligned}$$

and then

$$\kappa(\mathcal{L})=\bigcup_{n\geq 0}\mathcal{L}_n.$$

As we shall see later the following notions are introduced in accordance with the matroid theory terminology. Where $\mathcal{L}$ is any subset of $\mathcal{G}(N)$, a point $i\in N$ belonging to $I\cap J$ for a pair $(I,J)\in\mathcal{L}$ will be called *loop* of $\mathcal{L}$. The set of all loops of $\mathcal{L}$ we denote by $\ell[\mathcal{L}]$. If $\mathcal{L}$ has no loops we say also that $\mathcal{L}$ is *loopless*. We shall use the notation $\mathbf{K}_M(N)$, $M\subset N$, for the class of all I-relations on N having M for the set of their loops.

A few remarks are in order. From the iterative form of κ one can obtain

$$\ell[\mathcal{L}]=\ell[\kappa(\mathcal{L})],\qquad \mathcal{L}\subset\mathcal{G}(N),$$

so that the closure operator κ "does not add loops". Since for any $\mathcal{K}\in\mathbf{K}(N)$

$$(I,J)\in\mathcal{K}\iff (I-\ell[\mathcal{K}],\ J-\ell[\mathcal{K}])\in\mathcal{K}$$

(cf. Lemma 3) we see that $\mathcal{K}$ is uniquely given by its set of loops $\ell[\mathcal{K}]$ and by this loopless I-relation on $N-\ell[\mathcal{K}]$

$$\{I-\ell[\mathcal{K}],\ J-\ell[\mathcal{K}];\ (I,J)\in\mathcal{K}\}.$$

We say that two I-relations $\mathcal{K}, \mathcal{L}$ on N are of the same type if there exists a permutation π on N such that

$$(I, J) \in \mathcal{K} \Longleftrightarrow (\pi(I),\ \pi(J)) \in \mathcal{L}.$$

In Table the numbers of I-relations, loopless I-relations, types of I-relations and types of loopless I-relations (in the four columns respectively) are presented. The rows correspond to the cardinality of N ($|N| = 0, 1, 2, 3, 4$). For example, there are 47 types of loopless I-relations on $N = \{1, 2, 3, 4\}$ (see also [15], where they are visualised).

3. Connected sets

We indicated, after the presentation of Situation 2, that we intended to view pairs (I, J) of an I-relation like "pairs of separated sets", or "separated pairs" shortly. This notion occurs frequently, beside matroid theory, in graph theory (e. g. two disjoint sets of vertices of a graph are separated if there is no edge between them) and in topology (the most general situation seems to be that of Fréchet V-spaces: two disjoint subsets of that space are separated if neither of them has a common point with the derivation of the other, see Sierpinski [20]). Let us remark that the separated pairs can be employed as the primitive term of a topological space definition (see Szymanski [23]). In all mentioned cases connected sets can be defined by use of separated pairs; loosely speaking, a set L is said to be connected if it cannot be partitioned $L = I + J$ into two nonempty subsests I and J such that (I, J) is a separated pair.

To follow the general ideas, we shall introduce with many advantages the binary relation $\circ \subset \mathcal{G}(N) \times \mathcal{F}(N)$ given by: $(I, J) \circ L$ if and only if

$$I \cap J \cap L = \emptyset \quad \text{and} \quad (L - (I \cup J) \neq \emptyset \quad \text{or} \quad I \cap L = \emptyset \quad \text{or} \quad J \cap L = \emptyset)$$

which expresses that "L cannot be partitioned by (I, J)". Now, having declared a set $\mathcal{K} \subset \mathcal{G}(N)$ to be the set of separated pairs, the family of *connected sets with respect to* $\mathcal{K}$ can be defined as

$$\mathcal{K}^* = \{L \in \mathcal{F}(N);\ \forall\ (I, J) \in \mathcal{K} :\ (I, J) \circ L\}\,.$$

Keeping symmetry we denote also for $\mathcal{V} \subset \mathcal{F}(N)$

$$\mathcal{V}^* = \{(I, J) \in \mathcal{G}(N);\ \forall\ L \in \mathcal{V} :\ (I, J) \circ L\}\,.$$

The pair of mappings $\mathcal{K} \rightarrow \mathcal{K}^*$, $\mathcal{V} \rightarrow \mathcal{V}^*$ between $\mathcal{P}(\mathcal{G}(N))$ and $\mathcal{P}(\mathcal{F}(N))$ is nothing but the Galois connection arising from the relation $\circ$, see [3] (confusions from incorrect using of the star in the position of superscripts will not occur if we shall stick strictly, and we will do so, on the context $\mathcal{K},\ \mathcal{L} \subset \mathcal{G}(N)$ and $\mathcal{V},\ \mathcal{W} \subset \mathcal{F}(N)$).

DEFINITION 2. A family $\mathcal{V} \subset \mathcal{F}(N)$ of finite subsets of N containing the empty set is called *C-family on N* if

1. $i \in L,\ L \in \mathcal{V} \Longrightarrow \{i\} \in \mathcal{V}$
2. $L_1, L_2 \in \mathcal{V},\ L_1 \cap L_2 \neq \emptyset \Longrightarrow L_1 \cup L_2 \in \mathcal{V}$

Further notations and remarks are analogical to those concerning I-relations: $\mathbf{V}(N)$ denotes the class of all C-families on N, ν is its closure operator

$$\nu(\mathcal{W}) = \bigcap \{\mathcal{V} \in \mathbf{V}(N);\ \mathcal{W} \subset \mathcal{V}\}\,, \qquad \mathcal{W} \subset \mathcal{F}(N);$$

and also

$$\nu(\mathcal{W}) = \bigcup_{n\geq 0} \mathcal{W}_n$$

where

$$\begin{aligned}
\mathcal{W}_0 &= \mathcal{W} \cup \{\emptyset\} \cup \left\{ \{i\};\ i \in \bigcup \mathcal{W} \right\} \\
\mathcal{W}_{n+1} &= \mathcal{W}_n \cup \{L_1 \cup L_2;\ L_1, L_2 \in \mathcal{W}_n,\ L_1 \cap L_2 \neq \emptyset\}, \qquad n = 0, 1, 2, \ldots .
\end{aligned}$$

Similarly, for $\mathcal{W} \subset \mathcal{F}(N)$ an element $i \in N$ not belonging to $\bigcup \mathcal{W}$ is called loop of $\mathcal{W}$, $\ell[\mathcal{W}]$ denotes the set of all loops of $\mathcal{W}$, etc. Let us remark that

$$\ell[\mathcal{W}] = \ell[\nu(\mathcal{W})], \qquad \mathcal{W} \subset \mathcal{F}(N),$$

and, in addition, the trivial equalities

$$\begin{aligned}
\ell[\mathcal{K}^*] &= \ell[\mathcal{K}], \qquad \mathcal{K} \subset \mathcal{G}(N), \\
\ell[\mathcal{V}^*] &= \ell[\mathcal{V}], \qquad \mathcal{V} \subset \mathcal{F}(N),
\end{aligned}$$

take place.

Some elementary properties of the relation $\circ$ are formulated below without proofs.

LEMMA 4.

$$\begin{gathered}
(I, J) \circ L,\ i \in L \Longrightarrow (I, J) \circ \{i\} \\
(I, J) \circ L_1,\ (I, J) \circ L_2,\ L_1 \cap L_2 \neq \emptyset \Longrightarrow (I, J) \circ (L_1 \cup L_2)
\end{gathered}$$

LEMMA 5.

$$\begin{gathered}
(I, I) \circ M,\ L \in \mathcal{F}(N) \Longrightarrow (I, L) \circ M \\
(I, J) \circ M,\ L \subset J \Longrightarrow (I, L) \circ M \\
(I \cup J, K) \circ M,\ (I, J) \circ M \Longrightarrow (J \cup K, I) \circ M
\end{gathered}$$

From Lemma 4 we can conclude that all connected sets with respect to a set $\mathcal{K} \subset \mathcal{G}(N)$ form a C-family on N, i.e. $\mathcal{K}^* \in \mathbf{V}(N)$. This gives a reason to interpret the elements of a C-family as abstract connected sets and to read the second axiom of C-families in the way mentioned in Introduction: "the union of two connected sets with nonempty intersection is connected". This phrase have already appeared from topological point of view in famous Hausdorff book [10] and the matroid theoretical formulation of it belongs to Whitney [27]. Symmetrical observation implied by Lemma 5 leads to $\mathcal{V}^* \in \mathbf{K}(N)$ for $\mathcal{V} \subset \mathcal{F}(N)$.

Some other classical assertions about connected sets may be brought over C-families too.

DEFINITION 3. Let $\mathcal{V}$ be a loopless C-family on N. For $L \in \mathcal{F}(N)$ we say that $L = \sum_{s=1}^{t} M_s$ is a *$\mathcal{V}$-partition of* L if all sets (blocks) M_s belong to $\mathcal{V}$. The smallest number of blocks of a $\mathcal{V}$-partition of L is called *the degree of decomposability of* L and is denoted by $\rho(L)$. A $\mathcal{V}$-partition of L with $\rho(L)$ blocks is said to be *canonical.*

Canonical $\mathcal{V}$-partitions are defined unambiguously; the topological counterpart sounds "topological space can be uniquely decomposed into the maximal connected components" (see [20]). In fact, if $L = \sum M_s = \sum L_t$ are two canonical $\mathcal{V}$-partitions of a finite set $L \subset N$ then evidently every L_t must be a subset of some M_s and vice versa. More generally, all $\mathcal{V}$-partitions of L viewed as equivalencies on L form an upper sub-semilattice of the lattice $\mathcal{E}(L)$ of all equivalences on L. The canonical $\mathcal{V}$-partition is its greatest element.

We shall finish this section with this auxiliary proposition.

LEMMA 6. *Let $\mathcal{V}$ be a looples C-family on N and ρ the corresponding degree of decomposability. Then*

$$(I, J) \in \mathcal{V}^* \Longleftrightarrow \rho(I \cup J) = \rho(I) + \rho(J).$$

PROOF. Let us assume that $(I, J) \in \mathcal{V}^*$. Then $I \cap J = \emptyset$ since $\mathcal{V}^*$ is a looples I-relation on N. To avoid trivial cases suppose in addition $I \neq \emptyset$, $J \neq \emptyset$. A block L of the canonical $\mathcal{V}$-partition of $I \cup J$ must be a subset of I or J (otherwise $(I, J) \notin \mathcal{V}^*$). Hence, all blocks of the partition may be divided into two groups creating $\mathcal{V}$-partitions of I and J, respectively, what leads to

$$\rho(I \cup J) \geq \rho(I) + \rho(J).$$

However, the reversed inequality is trivial.

On the other hand, let $\rho(I \cup J) = \rho(I) + \rho(J)$, $I, J \in \mathcal{F}(N)$. Then $I \cap J = \emptyset$ otherwise the blocks of the canonical $\mathcal{V}$-partitions of I and J would not form a disjoint cover of $I \cup J$ providing $\rho(I \cup J) < \rho(I) + \rho(J)$. As the sets I, J are disjoint, the union of the canonical $\mathcal{V}$-partitions of I and J equals the canonical $\mathcal{V}$-partition of $I \cup J$and then any subset L of $I \cup J$ belonging to $\mathcal{V}$ must be contained in a block of this partition. Hence $I \cap L = \emptyset$ or $J \cap L = \emptyset$, i.e. $(I, J) \in \mathcal{V}^*$.□

4. Duality of I-relations and C-families

The class $\mathbf{K}(N)$ of I-relations on N viewed as a closure system can be considered to be a complete lattice (the meet of a class of I-relations is equal to its intersection and the join to the (κ) closure of its union, see [3]). Similarly for the class $\mathbf{V}(N)$ of C-families on N.

THEOREM 1. *The closure operator κ is identical with the mapping $\mathcal{K} \to \mathcal{K}^{**}$ defined on $\mathcal{G}(N)$ and, symmetrically, $\nu(\mathcal{V}) = \mathcal{V}^{**}$ for $\mathcal{V} \subset \mathcal{F}(N)$. The complete lattices $\mathbf{K}(N)$ and $\mathbf{V}(N)$ are anti-isomorphical. The Galois connection $\mathcal{K} \to \mathcal{K}^*$, $\mathcal{V} \to \mathcal{V}^*$ restricted on $\mathbf{K}(N)$ and $\mathbf{V}(N)$, respectively, realizes a pair of their mutually inverse anti-isomorphisms.*

PROOF. Plainly $\mathcal{K} \subset \mathcal{K}^{**}$ for any $\mathcal{K} \subset \mathcal{G}(N)$. We shall prove by contradiction that if $\mathcal{K} \in \mathbf{K}(N)$ then the equality takes place here. To this end let us suppose $(I, J) \in \mathcal{K}^{**} - \mathcal{K}$ for an I-relation $\mathcal{K}$ on N and, moreover, let all $(K, L) \in \mathcal{K}^{**}$ satisfying $|K \cup L| < |I \cup J|$ belong to $\mathcal{K}$. If $I \cap J = M \neq \emptyset$ then $(M, M) \in \mathcal{K}$ (remind that $\mathcal{K}$ and $\mathcal{K}^{**}$ have the same sets of loops). Next, $(I - M, J - M)$ belongs to $\mathcal{K}^{**}$, and owing to the assumption also to $\mathcal{K}$. Lemma 3, the implication 9. yields $(I, J) \in \mathcal{K}$. Thus, $I \cap J = \emptyset$, $I \neq \emptyset$, $J \neq \emptyset$ takes place and the same reasoning yields that $I + J$ does not contain loops. Since $(I, J) \o (I + J)$ then one gets $(I + J) \notin \mathcal{K}^*$, i.e. there exists a pair $(K, L) \in \mathcal{K}$ which is not in the relation $\circ$ with $I + J$. There is no loss of generality in assuming $K + L = I + J$, $K \neq \emptyset$, $L \neq \emptyset$. Hence, $|(K \cap I) + (K \cap J)| < |I + J|$ implies $(K \cap I, K \cap J) \in \mathcal{K}$ and in the same way $(L \cap I, L \cap J) \in \mathcal{K}$. Lemma 3, the implication 7. gives $(I, J) \in \mathcal{K}$ what contradicts the assumption.

The equality $\mathcal{V} = \mathcal{V}^{**}$ for $\mathcal{V} \in \mathbf{V}(N)$ will be proved along the same lines. It is plain that $\mathcal{V} \subset \mathcal{V}^{**}$, $\mathcal{V} \in \mathcal{F}(N)$. On the other hand, let $L \in \mathcal{V}^{**} - \mathcal{V}$ for a C-family $\mathcal{V}$ on N. From $\ell[\mathcal{V}] = \ell[\mathcal{V}^{**}]$ we see that L has at least two elements. Denote by I a subset of L belonging to $\mathcal{V} \subset \mathcal{V}^{**}$ with the greatest cardinality among all subsets of L from $\mathcal{V}$; then $I \neq \emptyset$ and $J = L - I \neq \emptyset$. From $(I, J) \o L$ one gets $(I, J) \notin \mathcal{V}^*$ and there exists $M \in \mathcal{V}$ satisfying

$I \cap M \neq \emptyset$, $J \cap M \neq \emptyset$ and $M \subset I + J = L$. But owing to the second property of C-families $I \cup M \in \mathcal{V}$ contradicting the choice of I.

From the iterative form of the closure operator κ and from Lemma 5 one can obtain $\mathcal{K}^* = (\kappa(\mathcal{K}))^*$ for any $\mathcal{K} \subset \mathcal{G}(N)$ and applying the star once again $\mathcal{K}^{**} = (\kappa(\mathcal{K}))^{**} = \kappa(\mathcal{K})$. Analogically by Lemma 4 $\mathcal{V}^{**} = \nu(\mathcal{V})$, $\mathcal{V} \subset \mathcal{F}(N)$. The proof of the first part of Theorem 1 is over. The remaining facts follows from the elementary properties of Galois connections (see [3], Ch. 5). We outline them for readers convenience.

The mappings $\mathcal{K} \to \mathcal{K}^*$, $\mathcal{V} \to \mathcal{V}^*$ restricted on $\mathbf{K}(N)$ and $\mathbf{V}(N)$ (also on $\mathbf{K}_M(N)$ and $\mathbf{V}_M(N)$, $M \subset N$) respectively, are bijections each inverse to the other. Clearly

$$\left(\bigvee \mathcal{K}_\alpha\right)^* = \left(\kappa\left(\bigcup \mathcal{K}_\alpha\right)\right)^* = \left(\bigcup \mathcal{K}_\alpha\right)^* = \bigcap \mathcal{K}_\alpha^* = \bigwedge \mathcal{K}_\alpha^*$$

is valid for any class of subsets of $\mathcal{G}(N)$. Accordingly the equalities

$$\left(\bigwedge \mathcal{K}_\alpha\right)^* = \left(\bigcap \mathcal{K}_\alpha^{**}\right)^* = \left(\bigcup \mathcal{K}_\alpha^*\right)^{**} = \nu\left(\bigcup \mathcal{K}_\alpha^*\right) = \bigvee \mathcal{K}_\alpha^*$$

hold for any class of I-relations on N. The dual case can be treated in the same manner. □

We are now going to discuss the relation between matroids and I-relations. Let r be the rank function of a loopless matroid with (finite) supporting set N. As we mentioned in section 2 disjoint subsets I, J of N are separated if and only if $r(I \cup J) = r(I) + r(J)$. It is easy to recognize directly that the family $\mathcal{K}_r^{(3)} = \mathcal{K}$ of all separated pairs of the matroid (N, r) is a loopless I-relation on N (it follows from Lemma 2 and [25], Ch. 3, too). A subset $L \subset N$ is non-separable or connected due to Whitney [27] if and only if for every nontrivial partition $I + J = L$, $I \neq \emptyset$, $J \neq \emptyset$ of L $r(I) + r(J)$ is greater than $r(L)$; hence $\mathcal{V} = \mathcal{K}^*$ is the very family of all connected sets of this matroid.

Leaving aside the empty sets and all singletons from $\mathcal{V}$ the minimal elements of $\mathcal{V}$ with respect to the inclusion are just the circuits of the matroid (we denote by $\mathcal{V}_{\min}$ the minimal elements in this sense for any loopless $\mathcal{V} \in \mathbf{V}_\emptyset(N)$). Thus

$$I, J \in \mathcal{V}_{\min},\ i \in I \cap J \Longrightarrow \exists\ K \in \mathcal{V}_{\min}\ :\ K \subset (I \cup J) - \{i\}$$

is a reformulation of the circuit axiom (see [25], 1.2.) and

$$I \in \mathcal{V},\ i, j \in I,\ i \neq j \Longrightarrow \exists\ J \in \mathcal{V}_{\min}\ :\ \{i, j\} \subset J \subset I$$

is a standard property of connected sets in matroids ([25], 6.2). A look at the iterative form of the closure operator ν yields that the last implication is equivalent to $\mathcal{V} = \nu(\mathcal{V}_{\min})$ (see [25], Ch. 5, Th. 4).

On the contrary, let $\mathcal{V}$ be a loopless C-family on N such that for $\mathcal{V}_{\min}$ the circuit axiom is valid and $\mathcal{V} = \nu(\mathcal{V}_{\min})$. Then there is exactly one matroid (N, r) having $\mathcal{V}_{\min}$ for its family of circuits and thus $\mathcal{V}$ for its family of connected sets.

From this nonstandard cryptomorphic description of matroids by the families of their connected sets one can conclude that C-families and also I-relations may be regarded as direct generalizations of matroids.

At the end of this section we mention briefly the membership problem for loopless I-relations discussed on the level of Situation 1 in [6] and on the level of databases (for the integrity constraints called crosses) in [19].

Let $\mathcal{K}$ be any loopless subset of $\mathcal{G}(N)$ and I, J finite disjoint subsets of N. Theorem 1 and Lemma 6 imply that (I, J) belongs to $\kappa(\mathcal{K}) = \mathcal{K}^{**}$ if and only if I and J are unions of blocks from the canonical $\mathcal{K}^*$-partition of $I + J$. The decision whether (I, J) is an element of $\kappa(\mathcal{K})$ reduces thus to looking for the canonical $\mathcal{K}^*$-partition of $M = I + J$. Observe that searching it, we can replace $\mathcal{K}$ by $\mathcal{L} = \{(K \cap M, L \cap M);\ (K, L) \in \mathcal{K},\ K \cap M \neq \emptyset,\ L \cap M \neq \emptyset\}$ since the connected subsets of M with respect to $\mathcal{K}$ and $\mathcal{L}$ coincide. But if $(K, L) \in \mathcal{K}$ and $K + L = M$ then the canonical $\mathcal{K}^*$-partition of M is the union of the canonical $\mathcal{K}^*$-partitions of K and L. These two steps (restriction and deletion) used iteratively enable to construct an effective algorithm performing the canonical $\mathcal{K}^*$-partitions (cf. [6]). Remark that having the canonical $\mathcal{K}^*$-partition of M we can decide all membership problems $(K, L) \in \kappa(\mathcal{K})$ for $K + L = M$ at once.

Another membership (accepting) algorithm presented in [19] is based on this construction

$$(I, J)^{\diamondsuit} = \{K \in \mathcal{F}(N); (I, J) \emptyset\, K\} = \mathcal{F}(N) - (I, J)^*$$

(we omitt braces) and on this fact

$$(I, J) \in \kappa(\mathcal{K}) \iff (I, J)^{\diamondsuit} \subset \bigcup \left\{(K, L)^{\diamondsuit};\ (K, L) \in \mathcal{K}\right\}.$$

This is a simple consequence of Theorem 1; observe that

$$(I, J)^{**} \subset \mathcal{K}^{**} \iff (I, J)^* \supset \mathcal{K}^*$$

what means

$$(I, J) \in \kappa(\mathcal{K}) \iff (I, J)^* \supset \bigcap \{(K, L)^*;\ (K, L) \in \mathcal{K}\}.$$

5. Probabilistic representations

We shall return to Situation 1 and settle here the question whether to an arbitrary I-relation $\mathcal{K}$ on N there exist a probability space $(\Omega, \mathbf{S}, P)$ and a system $\mathcal{S} = (\mathcal{S}_i)_{i \in N}$ of sub-σ-algebras of $\mathbf{S}$ such that the relation $\mathcal{K}_{\mathcal{S}}^{(1)}$ equals $\mathcal{K}$. Problems of this kind have been extensively studied not only in matroid theory (see [25]) and in abstract algebra (see [9]) but also in database theories by use of a logical language (cf. [1], [2], [6], [7], [13]). We shall retain the matroid theory terminology and call the I-relations expressed in the form $\mathcal{K} = \mathcal{K}_{\mathcal{S}}^{(1)}$ to be probabilistically representable.

THEOREM 2. *Every I-relation on finite or infinite set is probabilistically representable.*

We supply two proofs. First one operating only for finite N gives nice insight into the problem and brings forward a representation by two-valued random variables (in fact we can generalize it for countable N, see [15]). The second proof (cf. [6]) is based on a standard construction used in the proof of Lemma 7 preceding it. In both cases Theorem 1 plays the key role.

PROOF 1. Let $\mathcal{K}$ be a loopless I-relation on a finite set N (the loops can be omitted because they are representable by trivial σ-algebras). Put $\Omega = \mathcal{P}(N)$ and $\mathbf{S} = \mathcal{P}(\Omega)$; the probability P is given on the elementary event $I \subset N$ of Ω by

$$P(I) = 2^{-|N|} \sum_{J \subset N} (-1)^{|I \cap J|} t^{\rho(J)},$$

where ρ is the degree of decomposability with respect to $\mathcal{K}^*$ and t is a small positive number (e.g. $0 < t \leq 2^{-|N|}$). The σ-algebra $\mathcal{S}_i$ of the system $\mathcal{S}$ (the representation of $\mathcal{K}$) consists of four events: $\{I \subset N;\ i \in I\}$, its complement, $\emptyset$ and Ω. It remains to verify $\mathcal{K} = \mathcal{K}_{\mathcal{S}}^{(1)}$.

First observe that $\mathcal{S}_I$, $I \subset N$, has $2^{|I|}$ atoms (we index them by subsets K of I) of the form $\{J \subset N;\ J \cap I = K\}$ and with the probabilities

$$\begin{aligned} P^I(K) &= \sum_{L \subset N-I} P(K+L) = 2^{-|N|} \sum_{L \subset N-I} \sum_{J \subset N} (-1)^{|L\cap J|+|K\cap J|} t^{\rho(J)} = \\ &= 2^{-|I|} \sum_{J \subset I} (-1)^{|K\cap J|} t^{\rho(J)}. \end{aligned}$$

A pair (I, J) of disjoint subsets of N belongs to $\mathcal{K}_{\mathcal{S}}^{(1)}$ if and only if

$$Q_{I,J}(K,L) = 2^{|I+J|} \left[P^{I+J}(K+L) - P^I(K) P^J(L) \right] = 0, \qquad K \subset I,\ J \subset L.$$

If $(I, J) \notin \mathcal{K}$ then due to Lemma 6 and Theorem 1 $\rho(I+J) < \rho(I) + \rho(J)$ (and evidently $\rho(K+L) \leq \rho(K) + \rho(L)$, $K, L \in \mathcal{F}(N)$) and then

$$Q_{I,J}(\emptyset, \emptyset) = \sum_{K \subset I} \sum_{L \subset J} \left[t^{\rho(K+L)} - t^{\rho(K)+\rho(L)} \right] > 0,$$

i.e. $(I, J) \notin \mathcal{K}_{\mathcal{S}}^{(1)}$. Similarly, if $(I, J) \in \mathcal{K}$ then $(K, L) \in \mathcal{K}$ for $K \subset I$, $L \subset J$ and thus, by Lemma 6 and Theorem 1, $\rho(K + L) = \rho(K) + \rho(L)$. Hence, $Q_{I,J}(K, L) = 0$ is valid for any $K \subset I$ and $L \subset J$, i.e. $(I, J) \in \mathcal{K}_{\mathcal{S}}^{(1)}$. □

We shall try to illustrate the idea of the foregoing proof by the example of a system $\xi = (\xi_i)_{i \in N}$ of real-valued nonconstant random variables which is more familiar to statisticians (here the σ-algebra $\mathcal{S}_i$ is the inverse image of Borel sets in line R according to ξ_i). Let us denote by f_ξ the characteristic function of the vector ξ, i.e. the Fourier-Stieltjes transform of its distribution; f_ξ is a complex-valued function on R^N. It is a known fact from probability theory that $\mathcal{S}_I$ and $\mathcal{S}_J$ (I, J disjoint) are stochastically independent if and only if the function f_ξ admits the decomposition $f_\xi\left(x^{I+J}\right) = f_\xi\left(x^I\right) f_\xi\left(x^J\right)$, $x \in R^N$, where $x^I \in R^N$, $I \subset N$, denotes the system of real numbers obtained from the system $x = (x_i)_{i \in N}$ substituting 0 for x_i as soon as $i \in N - I$.

Now, let $\mathcal{K} = \mathcal{K}_{\mathcal{S}}^{(1)}$ and a nonempty subset $L \subset N$ has the canonical $\mathcal{K}^*$-partition $L = \sum_{s=1}^t L_s$. Then the decomposition

$$f_\xi\left(x^L\right) = \prod_{s=1}^t f_\xi\left(x^{L_s}\right), \qquad x \in R^N,$$

is possible and, moreover, no term of the decomposition can be further decomposed in this way (since L_s are connected with respect to $\mathcal{K}$). That means that the function f_ξ is completely determined by its values on the set

$$\bigcup_{L \in \mathcal{K}^*} \left\{ x \in R^N;\ x_i \neq 0 \ \text{ for } \ i \in L \ \text{ and } \ x_i = 0 \ \text{ for } \ i \in N - L \right\}.$$

A variation on these ideas was used in Proof 1, namely, the probability distribution of the corresponding random variables with values in a two-element group was chosen to be the inverse Fourier-Stieltjes transform of a function with the above decomposition structure (the function was even constantly equal to t on the determination set).

LEMMA 7. *The intersection of any class of probabilistically representable I-relations on a (possibly infinite) set N is probabilistically representable.*

PROOF. Consider the probability spaces $(\Omega_\alpha, \mathbf{S}_\alpha, P_\alpha)$ and the systems $\mathcal{S}^\alpha = (\mathcal{S}_i^\alpha)_{i\in N}$ of their sub-σ-algebras giving rise to the I-relations $\mathcal{K}_\alpha$, $\alpha \in A$ (the case $A = \emptyset$ is trivial and we exclude it). We take $(\Omega, \mathbf{S}, P)$ to be equal to the Cartesian product of these spaces (see [18]) and let $\mathcal{S} = (\mathcal{S}_i)_{i\in N}$, where $\mathcal{S}_i$ is the smallest sub-σ-algebra of $\mathbf{S}$ containing all σ-algebras $\pi_\alpha^{-1}(\mathcal{S}_i^\alpha)$, $\alpha \in A$ (π_α denotes the coordinate projection of $(\Omega, \mathbf{S}, P)$ on $(\Omega_\alpha, \mathbf{S}_\alpha, P_\alpha)$). Note that

$$\mathcal{S}_I = \sigma\left(\bigcup_{i\in I}\mathcal{S}_i\right) = \sigma\left(\bigcup_{i\in I}\bigcup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{S}_i^\alpha)\right) = \sigma\left(\bigcup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{S}_I^\alpha)\right) = \sigma(\mathcal{Q}_I), \quad I \subset N,$$

where $\mathcal{Q}_I$ denotes the class of all finite intersections of sets from $\bigcup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{S}_I^\alpha)$. Hence, a pair (I, J) belongs to $\mathcal{K}_\mathcal{S}^{(1)}$ if and only if the classes $\mathcal{Q}_I$ and $\mathcal{Q}_J$ are stochastically independent (see [18]).

If $(I, J) \in \bigcap_{\alpha\in A}\mathcal{K}_\alpha = \mathcal{K}$ then $P_\alpha(E_\alpha \cap F_\alpha) = P_\alpha(E_\alpha)\, P_\alpha(F_\alpha)$ as soon as $E_\alpha \in \mathcal{S}_I^\alpha$ and $F_\alpha \in \mathcal{S}_J^\alpha$ for all $\alpha \in A$. But then for $B \in \mathcal{F}(A)$

$$P\left(\bigcap_{\alpha\in B}\pi_\alpha^{-1}(E_\alpha) \cap \bigcap_{\beta\in B}\pi_\beta^{-1}(F_\beta)\right) = P\left(\bigcap_{\alpha\in B}\pi_\alpha^{-1}(E_\alpha)\right)\; P\left(\bigcap_{\beta\in B}\pi_\beta^{-1}(F_\beta)\right)$$

and hence $\mathcal{Q}_I$ and $\mathcal{Q}_J$ are stochastically independent, i.e. $(I, J) \in \mathcal{K}_\mathcal{S}^{(1)}$.

On the other hand, if $(I, J) \in \mathcal{K}_\mathcal{S}^{(1)}$ then for all $\alpha \in A$, $E_\alpha \in \mathcal{S}_I^\alpha$, $F_\alpha \in \mathcal{S}_J^\alpha$ the sets $\pi_\alpha^{-1}(E_\alpha)$ and $\pi_\alpha^{-1}(F_\alpha)$ belong to $\mathcal{S}_I$ and $\mathcal{S}_J$, respectively. From

$$P\left(\pi_\alpha^{-1}(E_\alpha) \cap \pi_\alpha^{-1}(F_\alpha)\right) = P\left(\pi_\alpha^{-1}(E_\alpha)\right)\; P\left(\pi_\alpha^{-1}(F_\alpha)\right)$$

one gets $P_\alpha(E_\alpha \cap F_\alpha) = P_\alpha(E_\alpha)\; P_\alpha(F_\alpha)$, i.e. $(I, J) \in \mathcal{K}$. □

PROOF 2. Due to Theorem 1 for any I-relation $\mathcal{K}$

$$\mathcal{K} = \mathcal{K}^{**} = \left(\bigcup\{\{L\};\ L \in \mathcal{K}^*\}\right)^* = \bigcap\{\{L\}^*;\ L \in \mathcal{K}^*\}$$

takes place and the validity of Theorem 2 follows from Lemma 7 supposing every I-relation $\{L\}^*$ on N, $L \in \mathcal{F}(N)$, is probabilistically representable. Hence, throwing away loops it suffices only to find representations of the I-relations $\{N\}^* = \{(I, J);\ I \cup J = \emptyset,\ I \cap J \neq N\}$ on N for finite N and this is trivial (cf. Proof 1 or [6]). □

6. Algebraic representations

An I-relation $\mathcal{K}$ on N will be called algebraically representable if and only if there exist a semimodular lattice Z satisfying the descending chain condition and a system $z = (z_i)_{i\in N}$ of its elements such that $\mathcal{K} = \mathcal{K}_z^{(2)}$.

THEOREM 3. *Every I-relation on a finite set is algebraically representable.*

PROOF. We follow the pattern of Proof 2. We claim that intersection of any (finite) class of algebraically representable I-relations on N is algebraically representable. In fact, it suffices only to consider two I-relations represented by systems $z^{(s)} = (z_i^{(s)})_{i\in N}$ of elements of lattices Z_s, $s = 1, 2$, respectively, and to verify that the system $(z_i^{(1)}, z_i^{(2)})_{i\in N}$ of elements of the Cartesian product $Z = Z_1 \times Z_2$ represents the intersection (note that the height function h on Z has the form

$$h(x, y) = h_1(x) + h_2(y), \qquad x \in Z_1,\ y \in Z_2,$$

where h_s is the height on Z_s, $s = 1,2$).

Omitting loops representable by zeros it rests to find algebraic representations of the I-relations $\{N\}^*$ on N and this is simple. Namely, the atoms of the lattice $\{I \subset N;\ |I| \neq |N| - 1\}$ with the usual inclusion ordering represent $\{N\}^*$, if $|N| \geq 3$. Other cases are trivial. □

As the example below indicates the I-relations on infinite sets need not be algebraically representable. The question which ones do is left open.

EXAMPLE. Let N be the set of positive integers and $[n] = \{1, 2, \ldots, n\}$, $n \geq 1$. Suppose the I-relation $\{[n];\ n \geq 1\}^* = \mathcal{K}$ is representable by a system z of elements of a lattice Z like in Situation 2. Then $([n], \{n+1\}) \notin \mathcal{K}$ what means that

$$h\left(z_{[n]}\right) + h\left(z_{n+1}\right) \geq 1 + h\left(z_{[n+1]}\right), \qquad n \geq 1,$$

and hence

$$\sum_{i=1}^{n+1} h\left(z_i\right) \geq n + h\left(z_{[n+1]}\right) \geq n + h\left(z_{\{2,3,\ldots,n+1\}}\right).$$

On the other hand, since $\mathcal{K} \supset \{(I, J) \in \mathcal{G}(N);\ 1 \notin I \cup J,\ I \cap J = \emptyset\}$ the canonical $\mathcal{K}^*$-partition of the set $\{2, 3, \ldots, n\}$ consists from singletons and then

$$h\left(z_{\{2,3,\ldots,n\}}\right) = \sum_{i=2}^{n} h\left(z_i\right), \qquad n \geq 2.$$

The obtained contradiction $h(z_1) \geq n$, $n \geq 1$, demonstrates that $\mathcal{K}$ has no algebraic representation. □

One can think of an analogy of Situation 2 for modular lattices where the algebraic independence of two elements x, y is defined simply by $x \wedge y = 0$. In this situation also only I-relations appear. A slight modification of reasoning of this section brings that every I-relation on a possibly infinite set has an algebraic representation in this sense (for details see author's thesis [15]).

Acknowledgement

I owe a great debt of gratitude to M. Studený and R. Jiroušek who contributed many suggestions and comments concerning the final form of the paper.

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1	1	1	1
2	1	2	1
5	2	4	2
22	12	10	6
485	420	57	47

TABLE. Numbers of I-relations and their types.

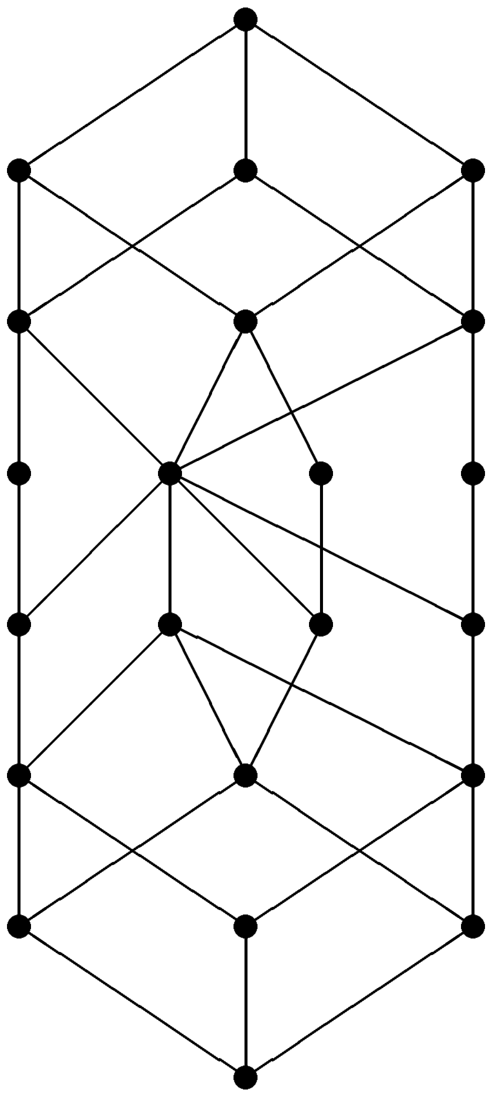

FIGURE. The lattice $\mathbf{K}(N)$ for $N = \{1, 2, 3\}$.