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Conditional Independence Structures Examined via Minors

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The notion of minor from matroid theory is adapted to examination of classes of conditional independence structures. For the classes of semigraphoids, pseudographoids and graphoids, finite sets of their forbidden minors are found. The separation graphoids originating from simple undirected graphs and triangulated graphs are characterized in this way neatly as well. Semigraphoids corresponding to the local Markov property of undirected graphs and to the d-separation in directed acyclic graphs are discussed.

A new class of semimatroids, called simple semimatroids, is introduced and an infinite set of its forbidden minors constructed. This class cannot be characterized by a finite number of axioms. As a consequence, the class of all semimatroids and the classes of conditional independence structures of stochastic variables and of linear subspaces have infinite sets of forbidden minors and have no finite axiomatization. The closure operator of semimatroids is examined by linear programming methods.

All possibilities of conditional independences among disjoint groups of four random variables are presented.

1. Introduction

Models for highly structured stochastic systems have required special assumptions on the conditional independences among their components. Especially, decomposition, inference, reasoning and explanation mechanisms of expert systems with uncertainty have been based on conditional independence (CI-) structures expressing a certain kind of formalized qualitative knowledge on the domain of interest. These impacts, together with a striving for meaningful mathematical models of conditional thinking in artificial intelligence, have resulted in a considerable effort devoted to the study of CI-structures, see [8], [10], [12], [13], [18], [21], [22], [23], [27], [25], [28], [32], [37], [38], [39], [41], [42], [43] and [46]. In a very wide sense, CI-relations are for us ternary relations $\mathcal{M}$ consisting of triples (I, J, K) of subsets I, J and K of a finite set N. The prevailing language for study of CI-relations has been overtaken mainly from the mathematical logic influenced by the integrity constraint theory of databases. CI-relations are, however, algebraical structures rich enough to contain in a natural way graphs and matroids, which blends more algebraical and combinatorial flavour into the subject.

In this work we are going to study localizable CI-relations through the notion of *minor* familiar from the graph and matroid theories. This old and fruitful idea, going back at least

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to Kuratowski's theorem on planar graphs or Tutte's characterization of graphic matroids, was adapted to CI-relations in [23] in accordance with the matroid theory as follows. Given two disjoint subsets L and M of N, the corresponding minor of a ternary relation $\mathcal{M}$ on N is the ternary relation on L consisting of all triples (I,J,K) such that $I \cup J \cup K \subset L$ and $(I,J,K \cup M) \in \mathcal{M}$. If $M = \emptyset$ the minor is called a restriction and if $M+L = N$ a contraction of $\mathcal{M}$. For example, having the relation $\mathcal{M} = \{(\{1\},\{2\},\{4\}),(\{2\},\{3\},\{4\}),(\{1\},\{3\},\{\emptyset\})\}$ on $N = \{1,2,3,4\}$, the relation $\{(\{1\},\{3\},\{\emptyset\})\}$ on $L = \{1,2,3\}$ is a minor of $\mathcal{M}$ (even a restriction of $\mathcal{M}$, $M = \emptyset$), the relation $\{(\{1\},\{2\},\{\emptyset\}),(\{2\},\{3\},\{\emptyset\})\}$ on $L = \{1,2,3\}$ is a minor of $\mathcal{M}$ (even a contraction of $\mathcal{M}$, $M = \{4\}$) and the relation $\{(\{1\},\{2\},\{\emptyset\})\}$ on $L = \{1,2\}$ is a minor of $\mathcal{M}$, $M = \{4\}$, that is neither a restriction nor a contraction of $\mathcal{M}$.

Roughly speaking, minors of CI-relations are their natural subconfigurations. An examination of a class of CI-relations via this notion means that one wants to recognize the relations of the class only through their, possibly small, subconfigurations that are characteristic for and inherent to the class, or equivalently, through their small defects or obstructions – forbidden subconfigurations of the class. Minors should henceforth provide a complementary tool to the axiomatic approach to the field prevailing up to now.

The aim of this paper is a study of well-known classes of CI-relations from the viewpoint of minors with a special emphasis on graphical visualization. After some preliminaries in Section 2 we characterize the classes of semigraphoids, pseudographoids and graphoids by means of their forbidden or compulsory 3-minors in Section 3. Then we provide in the next section a succint minor characterization of separation graphoids and an axiomatic characterization of boundary semigraphoids which originate from simple graphs according to the global and local Markov assumptions, respectively. Further we mention d-separation graphoids and their minors.

In the second part of the paper we concentrate on semimatroids and especially on a new class of semimatroids called simple semimatroids. This class is of special interest for us because it is relatively easy to handle and, at the same time, the presented results have strong consequences for the structure of the classes of all semimatroids and probabilistically (p-) representable semimatroids. Let us recall that the p-representable semimatroids originate from systems of random variables $\xi = (\xi_i)_{i \in N}$: a relation $\mathcal{M}$ is p-representable if there exists ξ such that a triple (I,J,K) belongs to $\mathcal{M}$ if and only if the subsystems $\xi_I = (\xi_i)_{i \in I}$ and ξ_J are stochastically conditionally independent given the subsystem ξ_K. A similar notion of linear (l-) representability is relevant for our approach as well and makes a bridge to the rich linear representation theory of matroids. A relation $\mathcal{M}$ is l-representable if there exists a collection of subspaces $D = (D_i)_{i \in N}$ of a linear space such that a triple (I,J,K) belongs to $\mathcal{M}$ if and only if $D_{I \cup K} \cap D_{J \cup K} = D_K$ where D_K is the inner sum of D_k over $k \in K$.

The class of simple semimatroids is shown to have an infinite number of forbidden minors. From our approach to the forbidden minors we get immediately the same result for the classes of semimatroids, p-representable and l-representable semimatroids and, at the same time, the nonexistence of their finite axiomatizations in the sense of [34] and [41]. At the end of Section 5 we compute the closure operator of semimatroids using linear programming methods what is more effective than the treatment of structural semigraphoids in [42] and [43]. We announce in Section 6 a theorem describing all p-representable CI-relations consisting of triples of disjoint sets living in a four-element set N. This is the only result of this paper requiring, as we believe, a long and complicated treatment and will be proved elsewhere. All other proofs here are short, owing to the carefully-chosen notation and terminology of the paper, and simplify many other proofs known before.

2. Definitions and Notations

Let N be a finite set with elements $i, j, k, l \in N$ and subsets $I, J, K, L \subset N$. Elements of N are not distinguished from singletons and the sign for union of subsets of N is omitted for simplicity. We will work with triples $(I, J|K)$ consisting of disjoint subsets of N, the order of I and J being irrelevant. The set of all these triples is to be denoted by $\mathcal{T}(N)$ and its subset containing all triples $(i, j|K)$ (i, j different, $i \notin K$ and $j \notin K$) by $\mathcal{R}(N)$. The symbol $(i, j|*)$ abbreviates $(i, j|N - ij)$. Ternary relations $\mathcal{M} \subset \mathcal{T}(N)$ are called global and $\mathcal{L} \subset \mathcal{R}(N)$ local, respectively.

Among the global relations $\mathcal{M} \subset \mathcal{T}(N)$ we restrict ourselves to relations having the property

$$(I, J|K) \in \mathcal{M} \iff \Big[(\forall i \in I)\,(\forall j \in J)\,(\forall L \supset K)\ \ L \subset IJK - ij \Rightarrow (i, j|L) \in \mathcal{M}\Big] \ .$$

We call them *localizable*. The restriction means that, without any loss of generality, we can equivalently work with the local relations $\mathcal{L} \subset \mathcal{R}(N)$ and we will do it starting in Section 4.[†] In fact, any localizable global relation $\mathcal{M}$ is fully reconstructible from $\mathrm{loc}(\mathcal{M}) = \mathcal{M} \cap \mathcal{R}(N) = \mathcal{L}$ by the formula

$$\mathcal{M} = \mathrm{gl}(\mathcal{L}) = \{(I, J|K) \in \mathcal{T}(N); (I, J|K)_\star \subset \mathcal{L}\}$$

where

$$(I, J|K)_\star = \{(i, j|L) \in \mathcal{R}(N)\ ;\ i \in I\,,\ j \in J \text{ and } K \subset L \subset IJK - ij\} \ .$$

Note that a global relation $\mathcal{M}$ is localizable if and only if $\mathrm{gl}(\mathrm{loc}(\mathcal{M})) = \mathcal{M}$.

Every localizable ternary relation $\mathcal{M} \subset \mathcal{T}(N)$ contains the triples $(I, J|K)$ with empty I or J and satisfies obviously the implication

$$(I, JK|L) \in \mathcal{M} \Rightarrow \Big[(I, J|KL) \in \mathcal{M} \text{ and } (I, K|L) \in \mathcal{M}\Big] \ .$$

Global relations with these two properties will be called *hereditary* here. *Semigraphoids* are hereditary relations that obey the opposite implication (to be labeled as the semigraphoid axiom) and *pseudographoids* are hereditary relations obeying the pseudographoid axiom (see [32])

$$\Big[(I, J|KL) \in \mathcal{M} \text{ and } (I, K|JL) \in \mathcal{M}\Big] \Rightarrow (I, JK|L) \in \mathcal{M} \ .$$

Graphoids are semigraphoids that are at the same time pseudographoids.

The following lemma is presented without proof since we believe that it is easy and straightforward. Its semigraphoid part follows from Lemma 3 in [22], cf. also [2].

Lemma 1. *Semigraphoids, pseudographoids and graphoids are localizable.*

For the purposes of this paper, and we believe that also in a wider context, it is advantageous to introduce visual diagrams depicting relations $\mathcal{L} \subset \mathcal{R}(N)$.

Example 1. Let $N = \{1, 2, 3, 4\}$ and assume that a local relation $\mathcal{L}$ consists of the triples

$$(1,2|\emptyset), (1,2|3), (1,2|4), (1,3|\emptyset), (1,3|2), (1,3|4),$$
$$(1,4|2), (1,4|3), (1,4|23), (2,3|\emptyset), (2,3|1), (2,3|4), (2,3|14), (2,4|\emptyset) \ .$$

The reader will have noticed that e.g. instead of $\{2, 3\}$ we write in our triples and diagrams simply 23. This relation (and the localizable relation $\mathrm{gl}(\mathcal{L})$ as well) will be represented by the left diagram of Fig. 1 that contains four circles corresponding to the elements of N connected by dotted or full lines with labels. □

In general, the diagrams of this kind are governed by the following rules. Every two circles are connected by a line of one kind and the lines may be labeled by lists of subsets. If for some

[†] Ignorance of this fact may cause substantial notational and technical problems. If the topic is worked out first for the local relations then, in our opinion, it is a trivial matter to translate it to the global relations afterwards.

i and j the triples $(i,j|K)$ are in $\mathcal{L}$ whenever $K \subset N - ij$ then the circles i and j are connected exclusively by a dotted line (in Fig. 1 between 2 and 3). If for all $K \subset N - ij$ the triple $(i,j|K)$ is not in $\mathcal{L}$ then i and j are connected by a full line (above between 3 and 4). If no one of the preceding two cases takes place we can decide freely whether the dotted or full line is to be used. Using the dotted line between i and j the nonempty list of all K such that $(i,j|K) \in \mathcal{L}$ must be attached to it. Using the full line, the complement of this list must be provided. The symbol $*$ occurs in lists attached to lines to abbreviate the set $N - ij$ on the line between i and j. Dotted and full lines match naturally with 'conditional independences and dependences' prescribed in the relation $\mathcal{L}$.

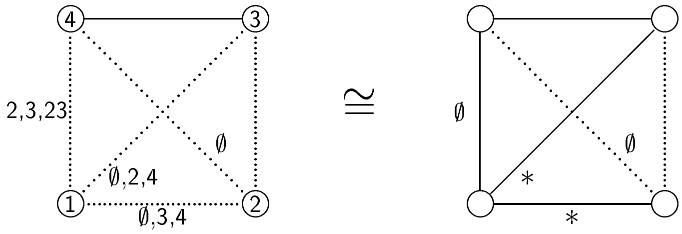


Figure 1. An example of visualization of a local(izable) relation.

Thinking about isomorphism classes of our ternary relations, we believe that the notion of isomorphism is intuitively clear enough not to waste time with a formal definition, we can sometimes omit the notation for elements of N in the circles of our diagrams. We will do this if no ambiguity is possible. The trivial relation $\mathcal{L} = \emptyset$ is thus visualized as a set of empty circles every two connected by a full line and the trivial relation $\mathcal{L} = \mathcal{R}(N)$ ($\mathcal{M} = \mathcal{T}(N)$) similarly replacing all full lines by the dotted ones. The right diagram of Fig. 1 exhibits the isomorphism class of the relation $\mathcal{L}$ from Example 1 or, if suitable, one of its representatives.

3. Minors of localizable relations

In this section we define minors through restrictions and contractions and discuss the classes of semigraphoids, pseudographoids and graphoids from this point of view.

Definition 1. *Let $\mathcal{M} \subset \mathcal{T}(N)$ be a global ternary relation and $L \subset N$. The* restriction *of $\mathcal{M}$ to L is given by*

$$\mathrm{re}_L\, \mathcal{M} = \mathcal{M} \cap \mathcal{T}(L) = \{(I,J|K) \in \mathcal{M};\ I \cup J \cup K \subset L\}$$

and its contraction *to L is*

$$\mathrm{co}_L\, \mathcal{M} = \{(I,J|K) \in \mathcal{T}(L);\ (I,J|K(N-L)) \in \mathcal{M}\}\ .$$

Where L and M are disjoint subsets of N, the relation $\mathrm{co}_L\, \mathrm{re}_{LM}\, \mathcal{M}$ on L is called a minor *of $\mathcal{M}$.*

One observes immediately that the above minor can be equivalently given by

$$\mathrm{re}_L\, \mathrm{co}_{N-M}\, \mathcal{M} = \{(I,J|K) \in \mathcal{T}(L);\ (I,J|KM) \in \mathcal{M}\}\ ,$$

whence minors of a ternary relation $\mathcal{M}$ on N are ternary relations on subsets of N constructed from $\mathcal{M}$ by any sequence of restrictions and contractions. It can be frequently convenient to say 'contraction (restriction) by $N - L$' instead of 'contraction (restriction) to L'. All minors can be constructed by performing restrictions and contractions by one-element sets. We speak of *proper* minors if $L \neq N$ and of n-minors, n-graphoids, etc. if they are relations on a set of n elements.

Example 2. Let $\mathcal{L}$ be the local 4-relation from Example 1. Diagrams of two minors of $\mathcal{L}$ are visualized on Fig. 2. □

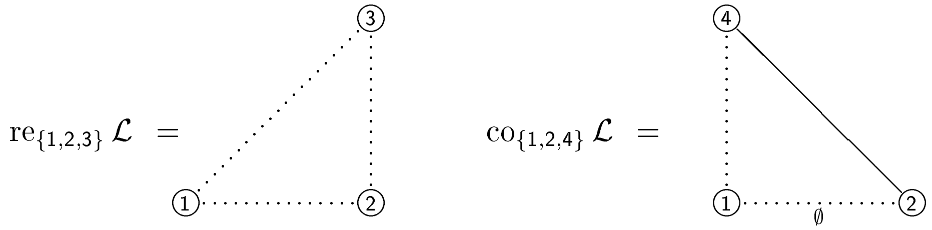


Figure 2. An example of restriction and contraction.

Generally, the restriction by an element i of N in a diagram of $\mathcal{L}$ can be found swifty. It suffices to erase the circle i with all its adjacent edges (and their corresponding lists) and omit all sets that contain i in the remaining lists. An additional correction can be necessary. It may happen namely that a line between some j and k gets the empty or complete (all subsets of $N - ijk$) list by this omitting procedure. In the first case one has to switch between the dotted and the full line and in the second one the complete list must be deleted; this was done in Fig. 2 between 1 and 2 and between 1 and 3. Contraction by i is easy to perform as well. After erasing the circle i and its adjacent lines with their labels we omit all sets in the remaining lists that do not contain i and, finally, delete all i in the diagram (i.e. replace every remaining K by $K - i$). If necessary the same correction is to be done as above; e.g. in Fig. 2 we got the empty list between 2 and 4 and switched therefore to the full line.

Lemma 2. *If a relation $\mathcal{M}$ is hereditary (localizable) then all its minors are hereditary (localizable). Minors of semigraphoids, pseudographoids and graphoids are semigraphoids, pseudographoids and graphoids, respectively.*

The proof of Lemma 2 is a matter of easy technicalities and we omit it.

All minors of relations from the five mentioned classes of CI-relations remain in the same class. Such classes are called *minor-closed.* If a CI-relation does not belong to a minor-closed class of relations but all its proper minors are in the class then the CI-relation is called a *forbidden minor* of the class. Any minor-closed class can be uniquely described by (the isomorphism classes of all) its forbidden minors. A relation belongs to such a class if and only if no its minor is isomorphic to a forbidden minor of the class.

In our cases the number of forbidden minors can be pretty great and it is sometimes more convenient to speak about *compulsory minors.* This notion is defined as follows. If a minor-closed class of CI-relations has a forbidden n-minor for a fixed number n then the n-relations of this class are called the compulsory n-minors. For different values of n compulsory or forbidden minors will be examined according to what is more economical.

Our first result in this direction says that to specify semigraphoids, pseudographoids or graphoids among the localizable relations it is sufficient to specify their 3-minors.

Proposition 1. *A ternary localizable relation $\mathcal{M} \subset \mathcal{T}(N)$ is a graphoid (semigraphoid, pseudographoid) if and only if all its 3-minors are graphoids (semigraphoids, pseudographoids, respectively).*

Proof. The case of graphoids follows from the remaining two cases. The trivial implications are consequences of Lemma 2.

Let $\mathcal{M}$ be a localizable relation such that all its 3-minors are semigraphoids. If $(i,j|kL)$ and $(i,k|L)$ are in $\mathcal{M}$ then $(i,j|k)$ and $(i,k|\emptyset)$ are in the 3-minor $\mathrm{co}_{ijk}\mathrm{re}_{ijkL}\,\mathcal{M}$. Since the 3-minor is a semigraphoid it contains $(i,k|j)$ and $(i,j|\emptyset)$ and then $(i,j|kL)$ and $(i,k|L)$ belong to $\mathcal{M}$ and, by localizability, $(i,jk|L) \in \mathcal{M}$. Thus $\mathcal{M}$ satisfies the semigraphoid axiom for $I = i$, $J = j$ and $K = k$. But owing to Proposition 1 from [21] $\mathcal{M}$ is then a semigraphoid.

The reasoning for pseudographoids is completely analogous. The only fact we have to prove is (we do not have a proper reference for it as above) that if a localizable $\mathcal{M}$ satisfies the pseudographoid axiom for any $I = i$, $J = j$ and $K = k$ then it satisfies the axiom in its full strength for arbitrary I, J and K.

If I has more than one element and $(I, J|KL)$ and $(I, K|JL)$ are both in $\mathcal{M}$ we can deduce that $\mathcal{M}$ contains $(I - i, J|KL)$, $(I - i, K|JL)$, $(I - i, J|KiL)$ and $(I - i, K|JiL)$ for any $i \in I$. By induction on the cardinality of I, $(I - i, JK|L) \in \mathcal{M}$ and $(I - i, JK|iL) \in \mathcal{M}$ for all $i \in I$. Since $\mathcal{M}$ is localizable the formula

$$(I, J|K)_\star = \bigcup_{i \in I} \; (I - i, J|K)_\star \cup (I - i, J|iK)_\star \,, \qquad |I| \geq 2 \,,$$

yields $(I, JK|L) \in \mathcal{M}$.

We can now suppose that $I = i$ and proceed by induction on the sum of cardinalities $|J| + |K|$. Omitting trivial cases, let $J = J_1 + J_2$ and $K = K_1 + K_2$ where J_1, J_2 and K_1 are nonempty subsets of N. If $(i, J|KL)$ and $(i, K|JL)$ belong to $\mathcal{M}$ then also $(i, J_1|KJ_2L) \in \mathcal{M}$ due to heredity of $\mathcal{M}$, and by induction $(i, J_1K|J_2L) \in \mathcal{M}$. Symmetrically, $(i, J_2K|J_1L) \in \mathcal{M}$ and then induction gives $(i, JK_2|L) \in \mathcal{M}$ and $(i, JK_2|K_1L) \in \mathcal{M}$. We can derive also $(i, J_1|KL) \in \mathcal{M}$ and $(i, K|J_1L) \in \mathcal{M}$, implying again by induction $(i, J_1K|L) \in \mathcal{M}$. To summarize, we obtain $(i, (J - j)K|jL) \in \mathcal{M}$, $(i, (J - j)K|L) \in \mathcal{M}$, $(i, J(K - k)|kL) \in \mathcal{M}$ and $(i, J(K - k)|L) \in \mathcal{M}$ for any $j \in J$ and $k \in K$. The above formula yields $(i, JK|L) \in \mathcal{M}$, closing the induction step. ∎

Example 3. The hereditary relation $\mathrm{gl}((12, 34|\emptyset)_\star) - \{(12, 34|\emptyset)\}$ on $N = \{1, 2, 3, 4\}$ is not localizable and all its 3-minors are graphoids. The assumption of localizability in Proposition 1 is therefore substantial. □

An examination of all local(izable) 3-relations provides the characterization of the previous classes of CI-relations via minors in Consequences 1., 2. and 3. Let us remark that the number of isomorphism classes of local 3-relations is 20.

Consequence 1. A localizable relation $\mathcal{M} \subset \mathcal{T}(N)$ is a graphoid if and only if each nontrivial 3-minor of $\mathcal{M}$ is in one of the six isomorphism classes of Fig. 3.

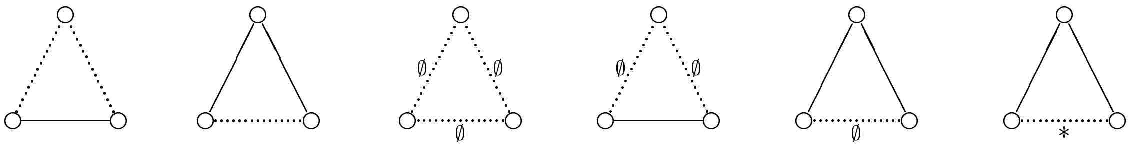


Figure 3. Nontrivial 3-graphoids: nontrivial compulsory 3-minors of graphoids.

Consequence 2. A localizable relation $\mathcal{M} \subset \mathcal{T}(N)$ is a semigraphoid if and only if each its nontrivial 3-minor is isomorphic to a 3-semigraphoid from Fig. 3 or from Fig. 4.

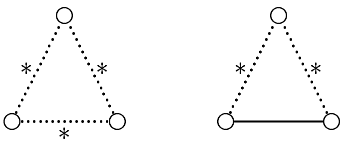


Figure 4. 3-semigraphoids that are not graphoids.

Consequence 3. A localizable relation $\mathcal{M} \subset \mathcal{T}(N)$ is a pseudographoid if and only if it has no 3-minor isomorphic to the relations from Fig. 4 and Fig. 5.

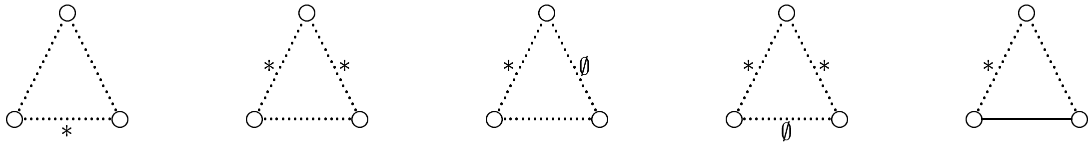


Figure 5. Localizable forbidden 3-minors of pseudographoids.

We can conclude this section by stating that, within localizable relations, the class of graphoids has a minor characterization by eight compulsory 3-minors, the class of semigraphoids by ten compulsory 3-minors, and the class of pseudographoids by seven forbidden (13 compulsory) 3-minors.

4. CI-relations over graphs

In this section we concentrate on special semigraphoids related to the global and local Markov properties of random fields over undirected graphs, see [7], [13], [18], [22], [32], [36] and [46]. The material will be presented directly by graph-theoretical notions. At the end we deal with the conditional independence relations of acyclic directed graphs, see [10], [12], [20] and [30].

From now on we are going to work with local relations. With a little abuse of language we say that $\mathcal{L} \subset \mathcal{R}(N)$ is a semigraphoid if it satisfies

$$\Big[(i,j|kL) \in \mathcal{L} \text{ and } (i,k|L) \in \mathcal{L}\Big] \Rightarrow \Big[(i,k|jL) \in \mathcal{L} \text{ and } (i,j|L) \in \mathcal{L}\Big]$$

and a pseudographoid if it satisfies

$$\Big[(i,j|kL) \in \mathcal{L} \text{ and } (i,k|jL) \in \mathcal{L}\Big] \Rightarrow \Big[(i,j|L) \in \mathcal{L} \text{ and } (i,k|L) \in \mathcal{L}\Big] \ .$$

Graphoids obey both implications as before.

4.1. Let G be an undirected simple (no loops and multiple edges) graph with the vertex set N and let

$$\langle G \rangle = \{(i,j|K) \in \mathcal{R}(N) \, ; \, K \text{ is a cutset of } i \text{ and } j \text{ in } G\} \ .$$

We recall that K is a cutset of i and j if all paths connecting i and j intersect K. One uses equivalently also the phrase 'K separates i and j'. It is easy to see that $\langle G \rangle$ is a graphoid. A local relation $\mathcal{L} \subset \mathcal{R}(N)$ is called a *separation graphoid* if $\mathcal{L} = \langle G \rangle$ for some graph G; note that the graph G is then unique. If this graph is chordal or is a forest we speak about *chordal-separation graphoids* or *forest-separation graphoids*, respectively. Separation graphoids can be characterized by five axioms in their global form [32] or by three axioms in their local form [21]. Forest-separation graphoids were studied in [4] and chordal-separation graphoids in [5].

Lemma 3. *Minors of (chordal-) separation graphoids are (chordal-) separation graphoids. The class of forest-separation graphoids is only contraction-closed.*

Proof. Let $\mathcal{L} = \langle G \rangle$, $k \in N$ and let G_{N-k} be the vertex-induced subgraph of G (constructed by deleting the vertex k and its adjacent edges). It is easy to see that $\mathrm{co}_{N-k}\, \mathcal{L} = \langle G_{N-k} \rangle$. Similarly, $\mathrm{re}_{N-k}\, \mathcal{L} = \left\langle G^{N-k} \right\rangle$ where G^{N-k} is obtained from G_{N-k} by adding all edges of the complete graph on the boundary $\partial_G k$ of the vertex k. Note that if G is chordal then G_{N-k} and G^{N-k} are both chordal and that if G is a forest then its subgraphs are forests. ∎

The restriction of a forest-separation graphoid $\langle G \rangle$ by $k \in N$ is forest-separation if and only if there are at most two edges in G adjacent with k.

Proposition 2. *A ternary relation $\mathcal{L}$ is a separation graphoid if and only if each of its nontrivial 3-minors is isomorphic to one of the two graphoids of Fig. 6.*

Figure 6. Nontrivial compulsory 3-minors of separation graphoids.

Proof. One implication follows from Lemma 3. Let every 3-minor of a relation $\mathcal{L} \subset \mathcal{R}(N)$ be a separation graphoid and let G be a graph on N having an edge between i and j if and only if $(i,j|*)$ is not in $\mathcal{L}$. We are going to show that $\mathcal{L} = \langle G \rangle$. By contradiction, if $\mathcal{L} - \langle G \rangle$ is nonempty we can choose $(i,j|K)$ in this difference such that the cardinality of K is maximal. From the construction of G we know that $K \neq N - ij$. Taking arbitrary $k \in N - ijK$ we look at the 3-minor $\mathrm{re}_{ijk}\mathrm{co}_{N-K}\,\mathcal{L}$ and see that it is isomorphic to $\langle\ \rangle$ or to $\langle\ \rangle$. Hence, $(i,j|kK) \in \mathcal{L}$ and then $(i,j|kK) \in \langle G \rangle$. The minor $\mathrm{re}_{ijk}\mathrm{co}_{N-K}\ \langle G \rangle$ must be now isomorphic to $\langle\ \rangle$. If not, it would contain $(i,j|\emptyset)$, leading to the contradiction $(i,j|K) \in \langle G \rangle$. A comparision of the three mentioned diagrams yields that for a permutation $\tilde{i}$, $\tilde{j}$ and $\tilde{k}$ of i, j and k the triple $(\tilde{i},\tilde{j}|\tilde{k})$ belongs to the minor of $\mathcal{L}$ and not to the minor of $\langle G \rangle$. Thus, $(\tilde{i},\tilde{j}|\tilde{k}K)$ is in $\mathcal{L}$ and not in $\langle G \rangle$, contradicting to the choice of $(i,j|K)$. If $\langle G \rangle - \mathcal{L} \neq \emptyset$ the argumentation is symmetric because we have employed only the properties of minors of $\mathcal{L}$ and $\langle G \rangle$. ∎

Separation graphoids are obviously pseudographoids that are *ascending*

$$\Big[(i,j|K) \in \mathcal{L} \text{ and } K \subset L \subset N - ij\Big] \ \Rightarrow\ (i,j|L) \in \mathcal{L}$$

and *complementary transitive*

$$\Big[(i,j|L) \notin \mathcal{L} \text{ and } (j,k|L) \notin \mathcal{L}\Big] \ \Rightarrow\ (i,k|L) \notin \mathcal{L}\ .$$

(Note that ascending pseudographoids are graphoids.) Since the class of complementary transitive and ascending pseudographoids is obviously minor-closed and the only nontrivial 3-pseudographoids in this class are those of Fig. 6, we can conclude by Proposition 2 that it coincides with the class of separation graphoids. The class of separation graphoids has thus a characterization by means of three axioms. A translation of this result to the global language is straightforward (see [32], p. 94).

From the viewpoint of decompositions in databases and expert systems, the class of chordal-separation graphoids seems to be of considerable importance. Its description via minors is as follows.

Consequence 4. The class of chordal-separation graphoids is specified by four compulsory 3-minors $\langle\ \rangle$, $\langle\ \rangle$, $\langle\ \rangle$ and $\langle\ \rangle$ and one forbidden 4-minor $\langle\ \rangle$. A separation graphoid $\mathcal{L} \subset \mathcal{R}(N)$ is a chordal-separation graphoid if and only if it obeys

$$\{(i,j|klL),(k,l|ijL)\} \subset \mathcal{L} \quad \Rightarrow \quad \{(i,k|jlL),(i,l|jkL),(j,k|ilL),(j,l|ikL)\} \cap \mathcal{L} \neq \emptyset\ .$$

Proof. If a graph G is not chordal and $L \subset N$ is its chordless cycle containing i, j, k and l as consequent nodes then $\mathrm{re}_{ijkl}\mathrm{co}_{N-L}\ \langle G \rangle \cong \langle\ \rangle$. If G is chordal then by Lemma 3 the graphoid $\langle G \rangle$ has no such 4-minor. This yields the first desired assertion.

Let $\mathcal{L} = \langle G \rangle$ for a chordal graph G and assume that the intersection above is empty. Then there exist paths in G connecting i with k, i with l, j with k, and j with l, each one passing not through L and not through the complementary pair of vertices. The paths are taken as short as possible. If two of the four paths have a common interior node then immediately $(i,j|klL) \notin \mathcal{L}$ or $(k,l|ijL) \notin \mathcal{L}$. If not then their union is a cycle containing $ikjl$ in this order. As the cycle must have a chord and the chord is inside of none of the four paths, we arrive at $(i,j|klL) \notin \mathcal{L}$ or $(k,l|ijL) \notin \mathcal{L}$. Hence, the chordal-separation graphoids obey the implication.

If a separation graphoid $\mathcal{L}$ satisfies the implication then all its minors satisfy it as well. It is then easy to see that no 4-minor of $\mathcal{L}$ is isomorph to $\langle\ \rangle$. Hence, the class of these separation graphoids coincides with the class of chordal-separation graphoids. ∎

In [5] the class of chordal-separation graphoids was axiomatized, in the global language, by six inference rules that are equivalent to the following statement. A relation $\mathcal{L} \subset \mathcal{R}(N)$ is a chordal-separation graphoid if and only if it is complementary transitive and ascending pseudographoid that satisfies

$$\{(i,j|klL),(k,l|*)\} \subset \mathcal{L} \quad \Rightarrow \quad \{(i,j|kL),(i,j|lL)\} \cap \mathcal{L} \neq \emptyset \ .$$

Consequence 5. A chordal-separation graphoid is forest-separation if and only if none of its contractions is isomorphic to $\langle\ \rangle$, or axiomatically, if and only if it satisfies

$$\Big[(i,j|*) \notin \mathcal{L} \text{ and } (i,k|*) \notin \mathcal{L}\Big] \ \Rightarrow \ (j,k|*) \in \mathcal{L} \ .$$

Proof. The first assertion follows from Lemma 3 and Proposition 2. If a chordal-separation graphoid $\mathcal{L}$ satisfies the implication then each of its contractions satisfies the implication as well, and then $\mathcal{L}$ has no contraction isomorphic to $\langle\ \rangle$. This entails that $\mathcal{L}$ is a forest-separation graphoid and the opposite is trivial. ∎

The class of forest-separation graphoids was described, in the global language, through six axioms in [4]. This result can be reformulated here by saying: $\mathcal{L} \subset \mathcal{R}(N)$ is a forest-separation graphoid if and only if it is a complementary transitive and ascending pseudographoid that satisfies

$$\Big[(i,j|*) \notin \mathcal{L} \text{ and } (i,k|*) \notin \mathcal{L}\Big] \ \Rightarrow \ (j,k|i) \in \mathcal{L} \ .$$

4.2. Having a simple undirected graph, the construction

$$\langle G\rangle_{\mathrm{loc}} = \{(i,j|K) \in \mathcal{R}(N)\ ;\ K \supset \partial_G i \ \text{ or } \ K \supset \partial_G j\}$$

stems from the local Markov property of random fields (see [36] and [7]); for an examination of the cases when $\langle G\rangle = \langle G\rangle_{\mathrm{loc}}$ see [22]. Obviously, every $\langle G\rangle_{\mathrm{loc}}$ is an ascending semigraphoid and not necessarily a graphoid, as visible from Fig. 7. The class of all semigraphoids of this type, to be called here *boundary semigraphoids*, is not closed under restrictions. For example, any restriction by one element of the boundary semigraphoid from Fig. 7 is not a boundary semigraphoid. Nevertheless, the class is closed under contractions ($\mathrm{co}_{N-k}\ \langle G\rangle_{\mathrm{loc}} = \langle G_{N-k}\rangle_{\mathrm{loc}}$, $k \in N$). We present its description by a few simple axioms below.

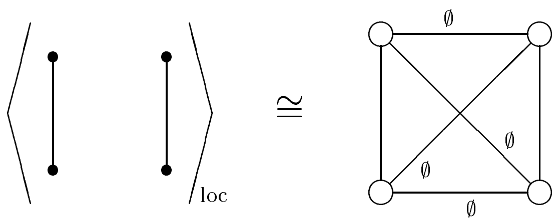


Figure 7. A boundary semigraphoid that is not a graphoid.

Proposition 3. *A ternary relation $\mathcal{L} \subset \mathcal{R}(N)$ is a boundary semigraphoid if and only if $\mathcal{L}$ is ascending, satisfies the pseudographoid axiom for $L = N - ijk$, and the following three implications:*

$$\Big[(i,j|kL) \in \mathcal{L}\,,\ (i,k|*) \in \mathcal{L} \text{ and } (j,k|*) \in \mathcal{L}\Big] \ \Rightarrow \ (i,j|L) \in \mathcal{L} \ ,$$

$$\Big[(i,j|kL) \in \mathcal{L}\,,\ (i,k|*) \in \mathcal{L} \text{ and } (j,l|*) \notin \mathcal{L}\ ,\ l \notin ijkL\Big] \ \Rightarrow \ (i,j|L) \in \mathcal{L} \ ,$$

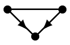
$$\Big[(i,k|*) \notin \mathcal{L} \text{ and } (j,l|*) \notin \mathcal{L}\ ,\ i \neq j\Big] \ \Rightarrow \ (i,j|N - ijkl) \notin \mathcal{L} \ .$$

Proof. Every boundary semigraphoid fulfils evidently the displayed implications. Let an ascending relation $\mathcal{L} \subset \mathcal{R}(N)$ satisfy the four above properties and let G be a graph on N having an edge between i and j if and only if $(i,j|*)$ is not in $\mathcal{L}$. We show that $\mathcal{L} = \langle G \rangle_{\mathrm{loc}}$. The inclusion $\subset$ is derived by means of the last implication. In fact, if $(i,j|K) \notin \langle G \rangle_{\mathrm{loc}}$ then there exist k and l out of K such that ik and jl are edges of G, and then $(i,j|N - ijkl) \notin \mathcal{L}$, whence by ascendingness $(i,j|K) \notin \mathcal{L}$.

To prove the inclusion $\supset$ it suffices to show that if there is no edge in G between some i and j then $(i,j|\partial_G i) \in \mathcal{L}$. We distinguish two cases. If $\partial_G i \supset \partial_G j$ then starting from $(i,j|*) \in \mathcal{L}$ a repeated use of the first implication gives the desired incidence $(i,j|\partial_G i) \in \mathcal{L}$. If not, then for any $l \in \partial_G j - \partial_G i$ we have $(i,l|*) \in \mathcal{L}$ and $(j,l|*) \notin \mathcal{L}$. By the weakened pseudographoid axiom, $(i,j|N - ijl) \in \mathcal{L}$. Since $N - ijl \supset \partial_G i$ we are ready if the inclusion is not strict. In the opposite case we take $k \in N - ijl\partial_G i$ and apply the second implication to conclude $(i,j|N - ijkl) \in \mathcal{L}$. This argumentation can be repeated until we get $(i,j|\partial_G i) \in \mathcal{L}$. ∎

4.3. The last construction we want to mention in this section concernes acyclic directed graphs. In such a graph G with vertex set N, one can consider

$$\langle G \rangle_{\mathrm{d}} = \{(i,j|K) \in \mathcal{R}(N)\ ;\ K \text{ separates } i \text{ and } j \text{ in } \underline{\mathrm{mor}}\, G_{\underline{\mathrm{an}}(ijK)}\}$$

as in [32] and [20]. Here for $L \subset N$ the symbol $\underline{\mathrm{an}}(L)$ denotes the set of all vertices of directed paths in G that end in L, and G_L is the vertex-induced subgraph as before. (A set $L \subset N$ is *ancestral* if $L = \mathrm{an}(L)$.) The symbol $\underline{\mathrm{mor}}$, applied to a directed graph, adds to every its configuration  the horizontal undirected edge  and converts all directed edges into the undirected ones. This procedure is termed *moralization*. Every $\langle G \rangle_{\mathrm{d}}$ is trivially a graphoid and every such graphoid is here called a *d-separation graphoid*.

Example 4. Directed graphs of the shape as in Fig. 8 deserve a special name, *path-with-tails*. It is our strategy to understand first the d-separation graphoids of paths-with-tails and then to describe general d-separation graphoids via these simpler graphoids.

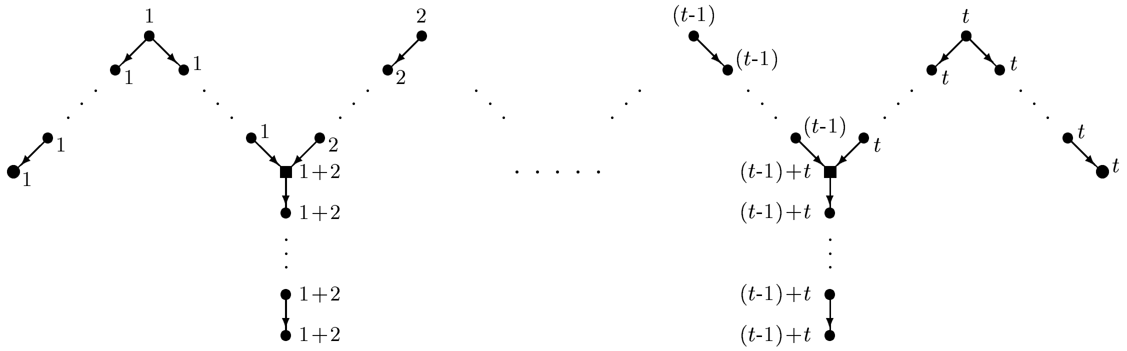


Figure 8. A sketch of a path-with-tails.

First, a more detailed description of paths-with-tails is in order. The orientation of all arrows is downwards. The vertical directed paths are tails and they can have arbitrary lengths, possibly zero. Every tail starts in its root ■-vertex and the number of all tails is possibly zero. The uttermost left and right vertices are two different endpoints. Between the endpoints there are $t \geq 1$ peaks; an endpoint can be the top point of a peak. The vertices of one peak are labeled by the same symbol. Note that a directed path of length at least one is also a path-with-tails having no tail and being one peak itself.

If i and j are arbitrary two different points of a path-with-tails G with the vertex set N, then there is exactly one vertex-induced subgraph of G that is a path-with-tails with the endpoints i and j and which has tails as long as possible. We denote it by G^{ij} and its vertex set by N^{ij}. Note that if i (or j) was chosen in a tail of G the segment of that tail upwards of i (j) should be sloped appropriately and it ceases to be a tail in G^{ij}. Tails of G^{ij} are tails of G otherwise. Now it is easy to see that a triple $(i,j|K) \in \mathcal{R}(N)$ does not belong to $\langle G \rangle_{\mathrm{d}}$ if and only if $K \cap N^{ij}$ is covered by the tails of G^{ij} and contains a vertex of each tail of G^{ij}. $\square$

Lemma 4. *If $\mathcal{L} = \langle G \rangle_d$ is a d-separation graphoid and K is an ancestral set of G then the restriction of $\mathcal{L}$ to K and the contraction of $\mathcal{L}$ by K are d-separation graphoids.*

Proof. The case of the restriction is trivial. Let k be an ancestral singleton. A triple $(i,j|L)$ from $\mathcal{R}(N-k)$ is not in $\mathrm{co}_{N-k}\ \langle G \rangle_{\mathrm{d}}$ if and only if $(i,j|kL) \notin \langle G \rangle_{\mathrm{d}}$, which means that there exists an undirected path in $\underline{\mathrm{mor}}\, G_{\underline{\mathrm{an}}(ijkL)}$ with the endpoints i and j out of kL. This path is however also in $\underline{\mathrm{mor}}\, (G_{N-k})_{\underline{\mathrm{an}}(ijL)}$ out of L, which reads as $(i,j|L) \notin \langle G_{N-k} \rangle_{\mathrm{d}}$. We conclude the proof by stating $\mathrm{co}_{N-k}\ \langle G \rangle_{\mathrm{d}} = \langle G_{N-k} \rangle_{\mathrm{d}}$. $\blacksquare$

The class of d-separation graphoids is neither closed under restrictions nor under contractions, as shown in the diagrams of Fig. 9. No satisfactory description of this class has been found up to now (cf. [12]).

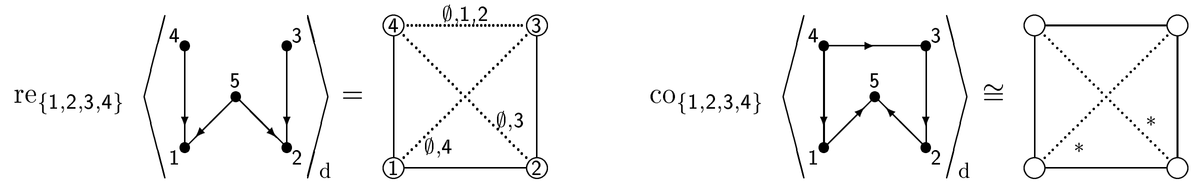


Figure 9. d-separation graphoids are not closed under restrictions and contractions.

If a class of CI-relations is not minor-closed it can be often appealing to complete it by all its minors and look for the forbidden minors of this larger minor-closed class. This would be especially interesting for the d-separation graphoids as their minors can be interpreted as subconfigurations of causal structures, i.e. hidden causal structures.

Let us mention at the end of this section that in multivariate statistical analysis there are a lot of other sophisticated assumptions on the conditional independence structure of random vectors (see e.g. [1],[17] and [19]) which are related to various types of graphs.

5. Simple semimatroids and semimatroids

Minor-closed subclasses of semigraphoids originate in a natural way from polymatroids. We examine their forbidden and compulsory minors in this section and discuss the corresponding closure operators and their properties.

5.1. First, we fix some relevant notions in our notation. A real function h defined on the subsets of N is said to be (the rank function of) a *polymatroid* if $h(\emptyset) = 0$ and the differences

$$\Delta h(i,j|K) = h(iK) + h(jK) - h(ijK) - h(K)$$

are nonnegative for all $(i,j|K) \in \mathcal{S}(N)$, where the symbol $\mathcal{S}(N)$ denotes the union of $\mathcal{R}(N)$ and $\{(i,i|K)\,; K \subset N \text{ and } i \in N-K\}$. If h is in addition integer-valued and $h(i) \leq 1$, $i \in N$, then h is a *matroid*, see [31] or [45]. The functions $r_m^L(K) = \min\{m, |K \cap L|\}$, $K \subset N$, parametrized by $L \subset N$ and an integer $0 \leq m \leq |L|$ are the most trivial examples of matroids. If $L = N$ the superindex is omitted and the matroids r_m are *uniform*.

Semimatroids were introduced in [23] to be relations $\mathcal{L} \subset \mathcal{S}(N)$ of the form $\mathcal{L} = [\![h]\!]$ where h is a polymatroid and the emphasized square bracket

$$[\![h]\!] = \{(i,j|K) \in \mathcal{S}(N);\ \Delta h(i,j|K) = 0\}$$

picks up all equalities in the defining inequalities of polymatroids. For example

$$[\![r_m^L]\!] = \{(i,j|K) \in \mathcal{R}(N);\ ij \not\subset L \text{ or } |K \cap L| \neq m-1\} \cup \{(i,i|K);\ i \not\subset L \text{ or } |K \cap L| \geq m\}\ .$$

In particular, $[\![r_0]\!] = [\![0]\!] = \mathcal{S}(N)$ and $[\![r_{|N|}]\!] = \mathcal{R}(N)$ are semimatroids. (In fact they are 'matroids', because the mapping $h \to [\![h]\!]$ is for matroids injective, see [23]. Thence, matroids can be taken as special semimatroids.) Intersections of semimatroids are semimatroids since the sum of two polymatroids h_1 and h_2 is a polymatroid and $[\![h_1]\!] \cap [\![h_2]\!] = [\![h_1 + h_2]\!]$.

In this paper we are concentrating on ternary relations that are subsets of $\mathcal{R}(N)$ and restrict mainly to the class of all semimatroids contained in $\mathcal{R}(N)$. To every semimatroid $[\![h]\!] \subset \mathcal{S}(N)$ the intersection $[h] = [\![h]\!] \cap \mathcal{R}(N) = [\![h + r_{|N|}]\!]$ is again a semimatroid, which means that we examine mainly traces on $\mathcal{R}(N)$ of semimatroids. Semimatroids contained in $\mathcal{R}(N)$ (we say also under $\mathcal{R}(N)$) are semigraphoids due to the equality

$$\Delta h(i,j|kL) + \Delta h(i,k|L) = \Delta h(i,k|jL) + \Delta h(i,j|L)\ .$$

Let us remark in passing that the class of all semimatroids under $\mathcal{R}(N)$ coincides with the class of structural semigraphoids defined sophistically in [42] or [43].

Definition 2. *A relation $\mathcal{L} \subset \mathcal{R}(N)$ is called a* simple semimatroid *if it can be written as the intersection of semimatroids $[r_m^L]$. If only semimatroids $[r_m]$ occur in the intersection we speak about* homogeneous semimatroids.

The class of homogeneous semimatroids is trivial. A relation $\mathcal{L} \subset \mathcal{R}(N)$ is a homogeneous semimatroid if and only if

$$\Big[(i,j|J) \in \mathcal{L},\ (k,l|L) \in \mathcal{R}(N) \text{ and } |J| = |L|\Big] \ \Rightarrow\ (k,l|L) \in \mathcal{L}\ .$$

Minors of homogeneous semimatroids are evidently homogeneous. A relation $\mathcal{L} \subset \mathcal{R}(N)$ is homogeneous if and only if all its 3-minors are homogeneous. In fact, if all 3-minors of $\mathcal{L}$ are homogeneous and $(i,j|L) \in \mathcal{L}$ then $(i,k|L) \in \mathcal{L}$ for every $k \in N - ijL$ (because the 3-minor $\mathrm{re}_{ijk}\mathrm{co}_{N-L}\,\mathcal{L}$ is homogeneous) and $(i,k|j(L-k)) \in \mathcal{L}$ for every $k \in L$ (look at $\mathrm{re}_{ijk}\mathrm{co}_{N-(L-k)}\,\mathcal{L}$). A repeated use of these two facts proves the above equivalence. Thus, the class of homogeneous semimatroids can be defined alternatively by four compulsory 3-minors.

Lemma 5. *The separation graphoids and boundary semigraphoids are simple semimatroids.*

Proof. The semigraphoids from both these classes are ascending and satisfy

$$\Big[(i,j|kL) \in \mathcal{L},\ (i,j|lL) \in \mathcal{L} \text{ and } (k,l|L) \in \mathcal{L}\Big] \ \Rightarrow\ (i,j|L) \in \mathcal{L}\ .$$

The class of ascending semigraphoids with this property was recognized in [21] to coincide with the class of all intersections of semimatroids $[r_1^L]$, $L \subset N$. Let us remark that intersections of semimatroids $[r_{|L|-1}^L]$, $L \subset N$, form an interesting class as well and are in some sense dual to the previous ones. ∎

The only separation graphoids (boundary semigraphoids) that are homogeneous are $\emptyset$ and $\mathcal{R}(N)$, corresponding to the complete graph and to the graph without edges, respectively.

Another trivial class of simple semimatroids originates from simple undirected graphs if we interpret a missing edge $ij \subset N$ as the set of all possible independences $(i,j|K) \in \mathcal{R}(N)$ for $K \subset N - ij$. The corresponding independence structure of a graph on N is then the intersection of the semimatroids $[r_1^{ij}]$ over the edges ij of the graph. This class of semimatroids is minor-closed and can be characterized by four compulsory 3-minors.

Example 5. Let $N = \{1,2,3,4,5\}$ and let $\mathcal{L}$ be the d-separation graphoid $\langle\ \rangle_{\mathrm{d}}$ where the vertices are labeled from left to right in ascending order. Note that the graph is a path-with-tails having two tails of zero length. We claim that $\mathcal{L}$ is not a simple semigraphoid. To see this, we start with the observation that $(1,5|\emptyset) \in \mathcal{L}$ implies that $\mathcal{L}$ is not contained in $[r_1^L]$ once $L \supset 15$. Further, $(1,5|3) \in \mathcal{L}$ entails that $\mathcal{L} \not\subset [r_2^L]$ if $L \supset 135$. Since $(1,5|23) \in \mathcal{L}$ we have $\mathcal{L} \not\subset [r_2^{125}]$, $\mathcal{L} \not\subset [r_2^{1245}]$ and by symmetry $\mathcal{L} \not\subset [r_2^{145}]$. Again, $(1,5|23)$ is in $\mathcal{L}$ and neither in $[r_3^{1235}]$ nor in $[r_3^N]$ nor, by symmetry, in $[r_3^{1345}]$. And finally, the triple $(1,5|234)$ is in $\mathcal{L}$ but out of $[r_3^{1245}] \cap [r_4^N]$. We can conclude that if a semimatroid $[r_m^L] \neq \mathcal{R}(N)$ contains $\mathcal{L}$ then $L \not\supset 15$. The smallest simple semimatroid containing $\mathcal{L}$ must contain therefore the triple $(1,5|24)$ which is, however, not in $\mathcal{L}$. $\square$

From now on we will use the notions of restriction, contraction and minor also for relations $\mathcal{L} \subset \mathcal{S}(N)$. They are defined verbatim as in Definition 1.

Lemma 6. *The classes of semimatroids and simple semimatroids are minor-closed.*

Proof. If $[\![h]\!]$ is a semimatroid and $L \subset N$ then $\mathrm{re}_L[\![h]\!] = [\![g]\!]$ where g is the restriction of the function h on the power set of L. Similarly, $\mathrm{co}_L[\![h]\!] = [\![g]\!]$ where the function g is given by $g(K) = h(K(N{-}L))$, $K \subset L$. In both cases g is a polymatroid and in this way the restriction and contraction have been defined for polymatroids. The remaining assertions follow from the facts that minors of intersections are intersections of minors and that restrictions and contractions of the matroids r_m^L are matroids of the same type. $\blacksquare$

Proposition 4. *The two graphoids of Fig. 10 (for any $n \geq 3$ and $n \geq 4$, respectively) are not semimatroids and, at the same time, all their proper minors are simple semimatroids.*

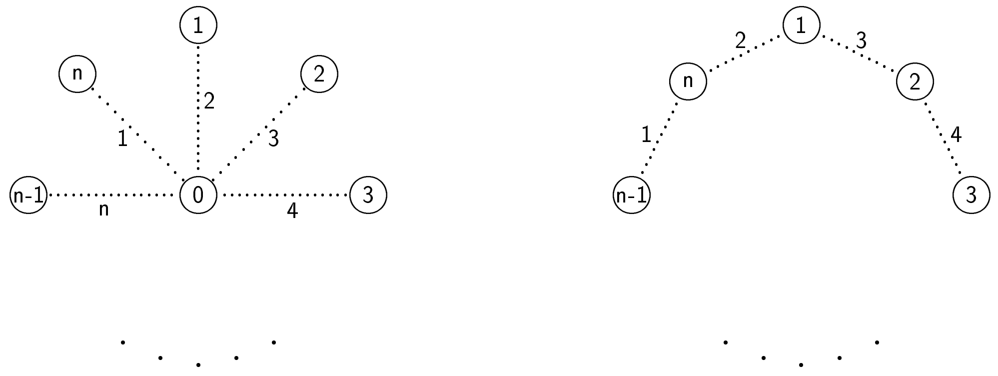


Figure 10. Forbidden minors of simple semimatroids (full lines are suppressed).

The left configuration in Fig. 10 appeared in [34] to prove that embedded multivalued dependencies have no finite axiomatization and later appeared in [41] for an analogous result on stochastic conditional independences. The related situation will be commented later.

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the graphoids represented by the left and right diagrams of Fig. 10, respectively. If $\mathcal{L}_1$ were a semimatroid $\mathcal{L}_1 = [\![h]\!]$, for a polymatroid h, then the equality

$$\Delta h(0,1|2) + \Delta h(0,2|3) + \ldots + \Delta h(0,\mathsf{n}|1) = \Delta h(0,2|1) + \Delta h(0,3|2) + \ldots + \Delta h(0,1|\mathsf{n})$$

would be violated, having the left-hand side equal to zero and the right-hand side positive. This kind of Δ-trick was used in [39] and [41] in a slightly different context and notation. The corresponding equality for the right diagram is

$$\Delta h(1,2|3) + \Delta h(2,3|4) + \ldots + \Delta h(\mathsf{n},1|2) = \Delta h(2,3|1) + \Delta h(3,4|2) + \ldots + \Delta h(1,2|\mathsf{n})\ .$$

We first examine minors of $\mathcal{L}_1$. The restriction and the contraction of $\mathcal{L}_1$ by $\mathsf{0}$ are equal to the empty homogeneous semimatroid $\emptyset$. The restriction of $\mathcal{L}_1$ by n is $\{(\mathsf{0},\mathsf{1}|\mathsf{2}),\dots,(\mathsf{0},\mathsf{n}-\mathsf{2}|\mathsf{n}-\mathsf{1})\}$, being a relation on $N=\{\mathsf{0},\mathsf{1},\dots,\mathsf{n}-\mathsf{1}\}$. It can be written as the intersection of the homogeneous semimatroid $\mathcal{R}_1(N)=\{(i,j|K)\in\mathcal{R}(N);\ |K|=1\}$ with

$$\left[\sum r_2^I+\sum_{\mathsf{k}=2}^{\mathsf{n}-1}\left(r_1^{\{\mathsf{k}-1,\mathsf{k}\}}+r_1^{\{\mathsf{k},\mathsf{k}+1,\dots,\mathsf{n}-1,\mathsf{0}\}}\right)\right].$$

Here the first summation is over $I\subset N$, $|I|=3$ and $|I|\notin\{\{\mathsf{0},\mathsf{1},\mathsf{2}\},\dots,\{\mathsf{0},\mathsf{n}-\mathsf{2},\mathsf{n}-\mathsf{1}\}\}$. The contraction of $\mathcal{L}_1$ by $\mathsf{2}$ is $\{(\mathsf{0},\mathsf{1}|\emptyset)\}$, which is expressible as the intersection of the homogeneous semimatroid $\mathcal{R}_0(N)=\{(i,j|\emptyset)\in\mathcal{R}(N)\}$ with

$$\left[\sum\{r_1^I;\ I\subset N,\ |I|=2\,\text{and}\,I\neq\{\mathsf{0},\mathsf{1}\}\}\right].$$

All other restrictions and contractions by one element are obtained by symmetry.

Contraction of $\mathcal{L}_2$ by $\mathsf{3}$ is $\{(\mathsf{1},\mathsf{2}|\emptyset)\}$, which was just found to be a simple semimatroid. The restriction of $\mathcal{L}_2$ by n is $\{(\mathsf{1},\mathsf{2}|\mathsf{3}),\dots,(\mathsf{n}-\mathsf{3},\mathsf{n}-\mathsf{2}|\mathsf{n}-\mathsf{1})\}$ and this is again intersection of the homogeneous semimatroid $\mathcal{R}_1(N)$ with the simple semimatroid

$$\left[\sum r_2^I+\sum_{k=2}^{\mathsf{n}-2}\left(r_1^{\{\mathsf{k}-1,\mathsf{k}+1\}}+r_1^{\{\mathsf{k},\mathsf{k}+1,\dots,\mathsf{n}-1\}}\right)\right],$$

where the first summation is over all $I\subset N=\{\mathsf{1},\mathsf{2},\dots,\mathsf{n}-\mathsf{1}\}$ of the cardinality 3 with the exception of $\{\mathsf{1},\mathsf{2},\mathsf{3}\},\dots,\{\mathsf{n}-\mathsf{3},\mathsf{n}-\mathsf{2},\mathsf{n}-\mathsf{1}\}$. A symmetry argument finishes the proof. ∎

The attentive reader will have noticed that we have used only the simple semimatroids $[r_1^L]$ and $[r_{|L|-1}^L]$ and the homogeneous semimatroids $\mathcal{R}_0(N)$, $\mathcal{R}_1(N)$ and $\mathcal{R}_0(N)\cap\mathcal{R}_1(N)=\emptyset$ in the intersections of the previous proof. Since $\mathcal{R}_0(N)$ is the intersection of $[r_2^L]$ for all $L\subset N$, $|L|=3$ and $\mathcal{R}_1(N)$ is the intersection of $[r_1]$ with all $[r_3^L]$, $L\subset N$, $|L|=4$, every proper minor of the two graphoids of Fig. 10 can be written as the intersection of the simple semimatroids $[r_1^L]$ and $[r_{|L|-1}^L]$.

Consequence 6. The classes of semimatroids and simple semimatroids cannot be characterized by finite sets of forbidden minors.

5.2. It is easy to verify that every 3-semigraphoid is a simple semimatroid. The forbidden 3-minors of semimatroids are thus the same as the forbidden 3-minors of semigraphoids, see Consequence 2. By Proposition 1 the forbidden 4-minors of the class of semimatroids are exactly thouse 4-semigraphoids that are not semimatroids. Two examples of such 4-(semi)graphoids are in Fig. 10. Another forbidden 4-minors for this class are in Fig. 11. To see that these three graphoids are not semimatroids one has to use Δ-tricks as in the previous proof. The symmetries of the diagrams are sufficient hints for the task, cf. [42] and [44]. It is clear now that there are quite a lot of forbidden 4-minors for the class of semimatroids (even under $\mathcal{R}(N)$) and we turn our attention preferably to the compulsory 4-minors, i.e. to the 4-semimatroids.

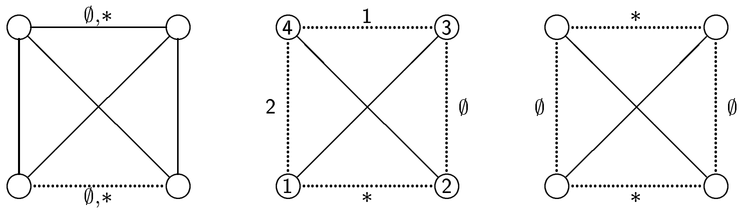


Figure 11. Examples of forbidden 4-minors for semimatroids.

It is an elementary fact that the set of all polymatroids on N is a closed pointed cone $\boldsymbol{H}(N)$ in a Euclidean space and that this cone has a finite number of extreme rays, see e.g. [29]. On each extreme ray a (nonzero and integer-valued) representing polymatroid f can be chosen. Every polymatroid h on N can be written as $\sum\{\alpha_f f;\ f \in \boldsymbol{H}_{\mathrm{ext}}(N)\}$ where f ranges over the set $\boldsymbol{H}_{\mathrm{ext}}(N)$ of the fixed representing polymatroids and α_f are nonnegative numbers. Then the semimatroid $[\![h]\!]$ ($[h]$) is the intersection of the semimatroids $[\![f]\!]$ ($[f]$) over those $f \in \boldsymbol{H}_{\mathrm{ext}}(N)$ for which α_f is positive, for a deeper result see [27], Lemma 3.1. The knowledge of all extreme rays would therefore substantialy increase insight into the class of semimatroids. However, the extreme rays in general are far from being completely understood, see [29], [14] and [24].

The extreme rays of the cone of 4-polymatroids are known (see e.g. [24], in a disguise of the game theory in [35] and [40]). They give rise to the semimatroids $[r_m^L]$ and to the semimatroids of Fig. 12 and Fig. 13 that are not simple semimatroids. Relying on [27] we can formulate without proof the following characterization of compulsory 4-minors of the class of semimatroids contained in $\mathcal{R}(N)$.

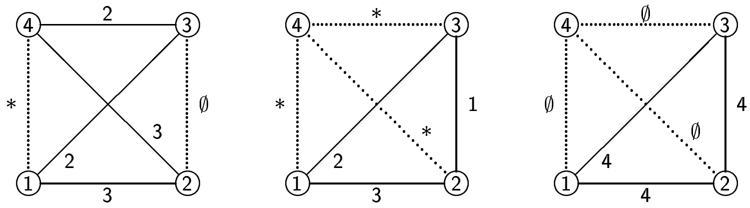


Figure 12. Examples of 4-semimatroids that are not simple.

Proposition 5. *A 4-relation $\mathcal{L} \subset \mathcal{R}(N)$ is a semimatroid if and only if it can be expressed as the intersection of a simple semimatroid with isomorphic copies of the semimatroids from Fig. 12 and Fig. 13.*

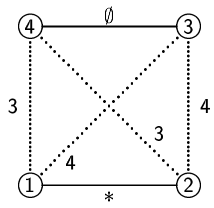


Figure 13. Example of a 4-semimatroid that is not p-representable.

5.3. The classes of semigraphoids, pseudographoids, graphoids, semimatroids and simple semimatroids are examples of classes that are closed under intersections. For such classes one can construct the closure operators in the usual way, see [3]. For example, given a relation $\mathcal{L} \subset \mathcal{S}(N)$ there exists the smallest semimatroid on N containing $\mathcal{L}$. This one, being the intersection of all semimatroids on N containing $\mathcal{L}$, is denoted by $\mathfrak{m}(\mathcal{L})$. Note that if $\mathcal{L} \subset \mathcal{R}(N)$ then $\mathfrak{m}(\mathcal{L}) \subset \mathcal{R}(N)$. The mapping $\mathfrak{m}$ on the power set of $\mathcal{S}(N)$ ($\mathcal{R}(N)$) is the closure operator of semimatroids (under $\mathcal{R}(N)$). A relation $\mathcal{L}$ is a semimatroid if and only if $\mathcal{L} = \mathfrak{m}(\mathcal{L})$.

The closure operator $\mathfrak{g}$ of the class of graphoids has the following plausible property. There exists a positive integer number m such that for every $\mathcal{L} \subset \mathcal{R}(N)$

$$\mathcal{L} = \mathfrak{g}(\mathcal{L}) \iff \mathcal{L} = \bigcup\{\mathfrak{g}(\mathcal{K});\ \mathcal{K} \subset \mathcal{L}\,,\ |\mathcal{K}| \le m\}\ .$$

Expressing the core of this equivalence in words, $\mathcal{L}$ is a graphoid if for all at most m-element $\mathcal{K} \subset \mathcal{L}$ also $\mathfrak{g}(\mathcal{K}) \subset \mathcal{L}$. (More correctly, we should speak about the class of closure operators $\mathfrak{g}_N$ on the power sets of $\mathcal{R}(N)$ where the finite set N varies; the number m cannot depend on the size of N.) If a closure operator has this property we will call it *finitary*. The minimal m with this property is for the classes of semigraphoids, pseudographoids and graphoids obviously $m = 2$. For the class of all intersections of $[r_1^L]$ (or $[r_{|L|-1}^L]$), $L \subset N$, the minimal m is 3, see [21]. However, for the classes of simple semimatroids and semimatroids under $\mathcal{R}(N)$ with the closure operators $\mathfrak{f}$ and $\mathfrak{m}$, respectively, the situation is quite different. The following consequence is the very essence of what would be in a more logical language easily reformulated as the nonexistence of finite axiomatization (i.e. of a finite number of inference rules) for a class of relations; see [34] and [41].

Consequence 7. Let $\mathfrak{c}$ be the closure operator of an intersection-closed class of semimatroids. If $\mathfrak{c}$ is between $\mathfrak{f}$ and $\mathfrak{m}$, i.e. $\mathfrak{m}(\mathcal{L}) \subset \mathfrak{c}(\mathcal{L}) \subset \mathfrak{f}(\mathcal{L})$ for all $\mathcal{L} \subset \mathcal{R}(N)$ (and finite N), then $\mathfrak{c}$ is not finitary.

Proof. For every positive $m \geq 3$ the left diagram of Fig. 10 provides a graphoid $\mathcal{L}_m$ containing m triples. Then $\mathcal{L}_m - \{(0, \mathsf{m}|1)\}$ is the restriction of $\mathcal{L}_{m+1}$ by $\mathsf{m}+1$. By Proposition 4, this is a simple semimatroid. Due to symmetry, every relation $\mathcal{L}_m - \{(i, j|K)\}$, $(i, j|K) \in \mathcal{L}_m$, is a simple semimatroid, and then every proper subset of $\mathcal{L}_m$ is a simple semimatroid. Hence, $\mathcal{L}_m$ is equal to the union of $\mathfrak{c}(\mathcal{K})$ over $\mathcal{K} \subset \mathcal{L}$, $|\mathcal{K}| \leq m-1\}$, because $\mathcal{K} \subset \mathfrak{c}(\mathcal{K}) \subset \mathfrak{f}(\mathcal{K}) = \mathcal{K}$ here. At the same time $\mathcal{L}_m$ is not a semimatroid and therefore must be strictly contained in $\mathfrak{m}(\mathcal{L}_m) \subset \mathfrak{c}(\mathcal{L}_m)$. ∎

At the end of this section we add some practical remarks regarding the closure operator $\mathfrak{m}$ of the class of semimatroids. If one has a relation $\mathcal{L} \subset \mathcal{S}(N)$ and wants to find the smallest semimatroid $\mathfrak{m}(\mathcal{L})$ containing $\mathcal{L}$, or wants at least to say whether a given triple $(i, j|K) \in \mathcal{S}(N)$ belongs to $\mathfrak{m}(\mathcal{L})$, one could theoretically resort to

$$\mathfrak{m}(\mathcal{L}) = \bigcap \{[\![f]\!];\ [\![f]\!] \supset \mathcal{L} \text{ and } f \in \boldsymbol{H}_{\text{ext}}(N)\} \ .$$

This obvious formula is computationally unrealistic if $|N| \geq 6$. Besides, the number of extreme rays of the cone $\boldsymbol{H}(N)$ grows at least superexponentially (see [16]; every up-to-loops-connected matroid generates an extreme ray [29]). The same assertion concerns the size of skeletons from [42] and [43].

An alternative appealing way how to recognize whether $(i, j|K) \in \mathfrak{m}(\mathcal{L})$, motivated by [47], is to consider the linear programming problem max $\Delta h(i, j|K)$ with the variable h subject to the linear constrains $h \in \boldsymbol{H}(N)$ and

$$\sum \{\Delta h(k, l|L);\ (k, l|L) \in \mathcal{L}\} \leq 0 \ .$$

(For a background in linear programming we refer to [6].) The simplex algorithm can say that either the maximum is 0 or the problem is unbounded, and, correspondingly, $(i, j|K)$ belongs to the semimatroid closure of $\mathcal{L}$ or not. The matrix of the problem is very sparse and all computations can be done in integer arithmetic excluding numerical noise. The above sum can be, in the case $(I, J|K)_\star \subset \mathcal{L}$, $(I, J|K) \in \mathcal{T}(N)$, simplified by the identity

$$\Delta h(I, J|K) = h(IK) + h(JK) - h(IJK) - h(K) = |I|^{-1}|J|^{-1} \sum \{\Delta h(i, j|L); (i, j|L) \in (I, J|K)_\star\}$$

(a proof is by induction on $|I| + |J|$). This makes the processing of the last constraint more effective. If one wants to decide whether $(I, J|K)_\star \subset \mathfrak{m}(\mathcal{L})$, the objective function is to be replaced by $\Delta h(I, J|K)$. If $\mathcal{L} \subset \mathcal{R}(N)$ one can diminish the number of variables by supposing $h(N) = h(N - i)$, $i \in N$. There are probably a lot of other simplifications that would deserve special attention. This linear programming approach is more effective than the methods of [43].

6. Linear and probabilistic semimatroids

In this section we start to investigate special semimatroids related to the lattice of linear subspaces of a linear space. They provide classes of CI-relations which have not been investigated up to now but which, in our opinion, might be fundamental for a systematic approach to CI-relations. In addition, they provide a convenient and comfortable tool for a basic study of the notion of p-representability to be done in the second part of this section.

6.1. It is a bit surprising that we have not found a reference to the following natural definition. Some preliminary knowledge of the matroid representation theory could bring the reader more insight.

Definition 3. *A relation $\mathcal{L} \subset \mathcal{S}(N)$ is* linearly (l-) representable *(over a field F) if there exists a collection $D = (D_i)_{i \in N}$ of finite-dimensional linear subspaces of a linear vector space E (over F) such that $\mathcal{L} = [\![D]\!]_{\rm lin}$, where*

$$[\![D]\!]_{\rm lin} = \{(i,j|K) \in \mathcal{S}(N)\ ;\ D_{iK} \cap D_{jK} = D_K\}$$

and where $D_K = \bigoplus_{k \in K} D_k$, $K \subset N$, is the inner sum of subspaces.

If we denote by $h_D(K)$ the dimension of the linear space D_K, $K \subset N$, the set function h_D is a polymatroid because

$$\Delta h_D(i,j|K) = h_D(iK) + h_D(jK) - h_D(ijK) - h_D(K) = \dim(D_{iK} \cap D_{jK}) - \dim(D_K) \geq 0$$

and $[\![h_D]\!] = [\![D]\!]_{\rm lin}$. Every l-representable relation $\mathcal{L} \subset \mathcal{S}(N)$ is thus a semimatroid. If all D_k are trivial (contain only the zero vector) we have $[\![h_D]\!] = \mathcal{S}(N)$. If all D_k are nontrivial and $D_k \cap D_{N-k}$ is trivial, $k \in N$, then $[\![h_D]\!] = \mathcal{R}(N)$. It is easy to see that $h_D = h_D(N)\, r_1^L$, $L \subset N$, if all subspaces D_k, $k \in L$, are equal to a nontrivial subspace $D_0 \subset E$ and all D_k, $k \in N - L$, are trivial. Thus, the semimatroids $[\![r_1^L]\!]$ and $[r_1^L]$ are l-representable.

Let us mention that a matroid is vectorial, or linearly representable in the sense of matroid theory, over a field, see [31] or [45], if it(s rank funtion) is equal to h_D for some collection $D = (D_i)_{i \in N}$; the subspaces must have their dimensions at most 1 and one speaks equivalently about their spanning vectors or simply about the columns of a representing matrix. Our notion of l-representability for matroids, considered for subsets of $\mathcal{S}(N)$, is thus weaker.

Example 6. Let $N = \{1,2,3,4,5\}$ and let $E = \{0,1\}^4$ be the linear space of dimension 4 over the two-element field. The following five groups of columns exhibit nonzero vectors of five two-dimensional subspaces $D_1, \ldots, D_5$ of E.

$$\left|\begin{array}{ccc|ccc|ccc|ccc|ccc}
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1
\end{array}\right|$$

It is easy to see that $h_D = 2\,r_2$ where $D = (D_1, \ldots, D_5)$, whence the (semi)matroid $[\![r_2]\!] = [\![D]\!]_{\rm lin}$ is l-representable over the two-element field. At the same time r_2 is not vectorial over the field (is not binary). □

Lemma 7. *The class of l-representable semimatroids over any fixed field is minor-closed and intersection-closed.*

Proof. Let $\mathcal{L}' = [\![D']\!]_{\rm lin}$ and $\mathcal{L}'' = [\![D'']\!]_{\rm lin}$ be two l-representable semimatroids where D' and D'' are collections of subspaces of two linear spaces E' and E'' over the same field, respectively. If we take E to be the outer sum of E' and E'' and D_i to be the outer sum of D'_i and D''_i, $i \in N$, then $D = (D_i)_{i \in N}$ is a collection of subspaces of E and $\mathcal{L}' \cap \mathcal{L}'' = [\![D]\!]_{\rm lin}$. In fact, $h_D = h_{D'} + h_{D''}$.

If $\mathcal{L} = [\![D]\!]_{\rm lin}$ then the restrictions of $\mathcal{L}$ are trivialy l-representable. The contraction of $\mathcal{L}$ by an element $k \in N$ can be represented by the factorspaces $(D_{ik}/D_k)_{i\in N-k}$ living in E/D_k. ∎

Lemma 8. *Simple semimatroids are l-representable over an arbitrary field.*

Proof. By Lemma 7 it is enough to show that all semimatroids $[\![r_m]\!]$, $1 < m < |N|$, are l-representable over any field. There is an easy argument explaining that r_m is vectorial over all fields F of sufficienly large cardinality, say $|F| \geq s_{m,N}$. Hence, we need to construct l-representations of $[\![r_m]\!]$ only over fields F having the cardinality smaller than $s_{m,N}$. For any such F let F' be an extension of F of the finite cardinality at least $s_{m,N}$. Then we know that $r_m = h_{D'}$ for some $D' = (D'_i)_{i\in N}$ consisting of one-dimensional subspaces of a linear space E' over F'. If we forget the multiplication by elements of $F' - F$ in E' we obtain a new linear space $E = E'$ over F. The sets $D_i = D'_i$, $i \in N$, are also linear subspaces of E with the dimension $|F'||F|^{-1}$. Even, $r_m = |F'||F|^{-1} h_D$ where $D = (D_i)_{i\in N}$. Thence the semimatroid $[\![r_m]\!]$ is l-representable over the field F. ∎

As a consequence, we see that separation graphoids, boundary semigraphoids and homogeneous semimatroids are l-representable over any field, cf. Lemma 5.

Lemma 9. *Every d-separation graphoid is l-representable over arbitrary field.*

Proof. First, we reformulate the definition of d-separation graphoids by means of paths-with-tails from Example 5, cf. [13] and [32]. Let G be an acyclic directed graph with the vertex set N. We claim that a triple $(i,j|K) \in \mathcal{R}(N)$ is not in $\langle G\rangle_{\rm d}$ if and only if there exists a path-with-tails $H \subset G$ with the endpoints i and j such that K intersects the vertex set of H exactly in the set of all its tail endpoints. (The relation $H \subset G$ means that vertices of H are among the vertices of G and arrows of H among the arrows of G.) This claim is slightly stronger than Proposition 3 in [20].

In one direction, if such a path-with-tails exists then it is contained in $G_{\underline{\rm an}(ijK)}$ and using the edges added by the moralization of this graph it is easy to construct an undirected path in $\underline{\rm mor}\, G_{\underline{\rm an}(ijK)}$ with the endpoints i and j avoiding K. On the other hand, if $(i,j|K) \notin \langle G\rangle_{\rm d}$ then there exists a shortest undirected path in $\underline{\rm mor}\, G_{\underline{\rm an}(ijK)}$ with endpoints i and j avoiding K. Then there exists a path-with-tails $H \subset G_{\underline{\rm an}(ijK)}$ with the same endpoints, with tails of zero length and such that K can contain only some roots of H. In fact, if $kl \subset \underline{\rm an}(ijK)$ is an edge of the undirected path obtained by the moralization then no common child of k and l belongs to the undirected path and at least one their common child belongs to $\underline{\rm an}(ijK)$; if $k_1 l_1 \subset \underline{\rm an}(ijK)$ and $k_2 l_2 \subset \underline{\rm an}(ijK)$ are two edges of the undirected path obtained by the moralization then a common child of k_1 and l_1 cannot be a common child of k_2 and l_2. We chose such a path-with-tails H to have the minimal number of peaks. Note that this H need not be related to the original undirected path. Starting at an arbitrary root of H, there is at least one directed path of G ending in K and one fixes it to be as short as possible. Thus only its endpoint is in K. The fixed directed paths can intersect neither the vertex set of H out of its (zero-length) tails, nor each other, otherwise our minimality assumption on the number of peaks would be violated. If we attach them to H as tails we obtain the desired path-with-tails.

In this way we have recognized the following assertion. Every d-separation graphoid $\langle G\rangle_{\rm d}$ is equal to the intersection of d-separation graphoids $\langle H\rangle_{\rm d}$ over all $H \subset G$ which are paths-with-tails after removal of their isolated vertices. We write alternatively over $H \in \mathcal{H}$; the vertex set of every such H is the same as the vertex set of G. Note that the inclusion $\langle G\rangle_{\rm d} \subset \bigcap_{H\in\mathcal{H}} \langle H\rangle_{\rm d}$ is trivial as $H \subset G$ implies, generally, $\langle G\rangle_{\rm d} \subset \langle H\rangle_{\rm d}$.

Let a path-with-tail of some $H \in \mathcal{H}$ has $t \geq 1$ peaks and let $e_1, \ldots, e_t$ be linearly independent vectors in a linear space. We set D_i to be the trivial subspace if $i \in N$ is an isolated vertex of H, D_i to be the one-dimensional linear subspace spanned by the vector e_r if i is in r-th peak, $1 \leq r \leq t$, and D_i to be the one-dimensional linear subspace spanned by the vector $e_r + e_{r+1}$ if i is on the tail between r-th and $(r+1)$-th peak, $1 \leq r < t$, cf. [11]. The assignments of indices of the vectors correspond to the labels in Fig. 8. If $D = (D_i)_{i\in N}$ then it is not difficult to see that $\langle H\rangle_{\mathrm{d}} \subset [D]_{\mathrm{lin}} = \mathcal{R}(N) \cap [\![D]\!]_{\mathrm{lin}}$. Moreover, for the endpoints i and j of the path-with-tails in H a triple $(i,j|K)$ belongs to $[D]_{\mathrm{lin}}$ if and only if $(i,j|K) \in \langle H\rangle_{\mathrm{d}}$.

So far we have constructed for every $H \in \mathcal{H}$ an l-representable relation $\mathcal{L}_H \subset \mathcal{R}(N)$ containing $\langle H\rangle_{\mathrm{d}}$ such that $(i,j|K) \in \mathcal{L}_H$ if and only if $(i,j|K) \in \langle H\rangle_{\mathrm{d}}$ once i and j are the endpoints of the path-with-tails contained in H. But then $\langle G\rangle_{\mathrm{d}}$ is the intersection of $\mathcal{L}_H$ over $H \in \mathcal{H}$ and it is by Lemma 7 l-representable. ∎

Revisiting Lemma 7, Lemma 8, Proposition 4 and Consequence 7 we obtain the following assertions.

Consequence 8. The class of l-representable semimatroids (over any fixed field) has an infinite number of forbidden minors. Its closure operator is not finitary.

By Lemma 8 every 3-semigraphoid is l-representable over any field. If we want to say more about the l-representable 4-semimatroids contained in $\mathcal{R}(N)$ we should have a look again at Fig. 12 and Fig. 13. It is not difficult to recognize that the 4-semimatroids of Fig. 12 are l-representable over any field, too. For example, if (D_1, D_2, D_3) is an l-representation of $[\![r_2]\!]$ on $\{1,2,3\}$ then $D = (D_1, D_2, D_3, D_1)$ gives rise to $[D]_{\mathrm{lin}} = \mathcal{R}(N) \cap [\![D]\!]_{\mathrm{lin}}$, which is the left 4-semimatroid of Fig. 12. If $(D_1, D_2, D_3, D_4, D_5)$ represents $[\![r_2]\!]$, see Example 6, ($[\![r_3]\!]$) on $\{1,2,3,4,5\}$ then $(D_1, D_2, D_3, D_4 \oplus D_5)$ leads to the midle (right) 4-semimatroid of Fig. 12.

The 4-semimatroid of Fig. 13 is, however, l-representable over no field. In fact, if D were its l-representation consisting of four linear subspaces of a linear space then

$$\begin{aligned} 0 &> \Delta h_D(3,4|1) + \Delta h_D(3,4|2) + \Delta h_D(1,2|\emptyset) - \Delta h_D(3,4|\emptyset) = \\ &= h_D(13) + h_D(23) + h_D(14) + h_D(24) + h_D(34) - h_D(12) - h_D(3) - h_D(4) - h_D(134) - h_D(234) \end{aligned}$$

because the first three Δ's are equal to zero. But, due to a result by Ingleton, see [15] and [31], this expression must be nonnegative for any four linear subspaces. Similar arguments prove that the 4-semimatroids of Fig. 14 are not l-representable as well. A complete description of all non-l-representable 4-semimatroids was found in [27] (l-representable 4-semimatroids were called Ingleton semimatroids there). It yields the following characterization of compulsory 4-minors for the class of l-representable semimatroids under $\mathcal{R}(N)$.

Proposition 6. *A 4-semimatroid $\mathcal{L} \subset \mathcal{R}(N)$ is l-representable if and only if it can be written as the intersection of a simple semimatroid with isomorphic copies of the semimatroids from Fig. 12.*

Note that if a 4-semimatroid is l-representable over some field than it is also l-representable over any other field. We have, however, examples of semimatroids that are l-representable only over special fields.

6.2. Our last topic is p-representability. We repeat the definition of this notion and after some examples and the useful Lemma 10 we simply revise our knowledge to arrive at nontrivial consequences.

Definition 4. *A relation $\mathcal{L} \subset \mathcal{S}(N)$ is* probabilistically (p-) representable *if there exists a system of random variables $\xi = (\xi_i)_{i\in N}$ taking only finite number of values such that $\mathcal{L} = [\![\xi]\!]_{\mathrm{pr}}$ where*

$$[\![\xi]\!]_{\mathrm{pr}} = \{(i,j|K) \in \mathcal{S}(N)\ ;\ \xi_i \text{ is conditionally independent of } \xi_j \text{ given } \xi_K = (\xi_k)_{k\in K}\}\ .$$

If we denote by $h_\xi(K)$ the Shannon entropy of $(\xi_k)_{k\in K}$, $K \subset N$, then the set function h_ξ, called the *entropy function* of ξ, is a polymatroid, see [9], and the conditional independence of ξ_i and ξ_j given ξ_K holds if and only if $\Delta h_\xi(i,j|K) = 0$. A straightforward conclusion is that $[\![h_\xi]\!] = [\![\xi]\!]_{\rm pr}$ whence $[\![\xi]\!]_{\rm pr}$ is always a semimatroid. Let us remark that $(i,i|\emptyset)$ belongs to some $[\![\xi]\!]_{\rm pr}$ if and only if ξ_i is a constant random variable in the system ξ. Similarly, $(i,i|K)$ belongs to some $[\![\xi]\!]_{\rm pr}$ if and only if ξ_i is a deterministic function of the subsystem ξ_K. Thus $\mathcal{S}(N) = [\![\xi]\!]_{\rm pr}$ if ξ consists exclusively of constant variables and $\mathcal{R}(N) = [\![\xi]\!]_{\rm pr}$ if all variables of ξ are nonconstant and mutually independent.

Example 7. Let N, E and D be as in the previous example and let P be the uniform probability measure on E, $P(e) = 1/16$ for all $e \in E$. Every subspace D_i induces the partition of E into four cosets of D_i, e.g. for D_3 the space E is partitioned into the four blocks each consisting of four column vectors

$$\left|\begin{array}{cccc|cccc|cccc|cccc} 0&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0\\ 0&0&1&1&0&0&1&1&1&1&0&0&1&1&0&0\\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \end{array}\right| .$$

Every partition can be considered for a random variable mapping $e \in E$ onto the the coset of the partition containing e. In this way we have defined five random variables $\xi = (\xi_1, \ldots, \xi_5)$ on the probability space (E, P), every one taking four values. It is easy to see that $h_\xi = \log 4 \cdot r_2$ whence $[\![r_2]\!] = [\![\xi]\!]_{\rm pr}$ is p-representable. Note that any two of the five random variables are pairwise independent and determine the values of the remaining three variables. □

Intersections of p-representable relations are known to be p-representable, see e.g. [32], and minors of p-representable relations are p-representable as well, see [23] Lemma 2.

Lemma 10. *If a semimatroid is l-representable over a field then it is p-representable.*

Proof. Let a semimatroid $\mathcal{L}$ be l-representable by $D = (D_i)_{i\in N}$ in E and let $e_i^1, \ldots, e_i^{m_i}$, where $m_i = h_D(i)$, be a base of D_i for every $i \in N$. If E is not over a finite field then, by Proposition 6.8.2. in [31], there exist vectors $\underline{e}_i^1, \ldots, \underline{e}_i^{m_i}$, $i \in N$, in a linear space $\underline{E}$ over a finite field that give rise to the same vectorial matroid as the starting vectors did. Then, by [21] Theorem 2, there exist random variables $\xi_i^1, \ldots, \xi_i^{m_i}$, $i \in N$, taking a finite number of values such that their entropy function is proportional to the rank function of the vectorial matroid. Then $\mathcal{L}$ is p-representable by $(\xi_i)_{i\in N}$, where $\xi_i = (\xi_i^1, \ldots, \xi_i^{m_i})$, $i \in N$. ■

We see immediately that owing to Lemma 8 and Lemma 10, simple semimatroids and homogeneous semimatroids are p-representable. Also, separation graphoids and boundary semigraphoids are p-representable by Lemma 5. Lemma 9 and Lemma 10 imply that the d-separation graphoids are p-representable, see [12], [11] and [32]. Now recall Proposition 4 and its Consequence 7 and notice that the closure operator $\mathfrak{p}$ of the class of p-representable semimatroids is between $\mathfrak{f}$ and $\mathfrak{m}$.

Consequence 9. The class of p-representable semimatroids cannot be defined by a finite set of forbidden minors. The closure operator $\mathfrak{p}$ is not finitary.

The nonexistence of a finite axiomatization for p-representable relations was proved in [41] by use of the configuration from [34] where an analogical axiomatization problem for databases was settled in negative. Our proof of this result by use of simple semimatroids removes complicated constructions of probability measures needed before. We stress, we used only $[r_1^L]$ and $[r_{|L|-1}^L]$ in the proof of Proposition 4 and these two semimatroids are obviously p-representable. Hence, for this purpose, one need not plunge into l-representability, and Lemma 8 and Lemma 10 can be avoided here. The same strategy can be used to generate a swifty proof of the original finite axiomatization problem, [34], in databases.

The reader will have noticed that all the results with strong probabilistic flavour have been proved exclusively by algebraic methods and this shows, in fact, that these results on p-representability are not going to the core of this notion. The last result of this paper is a step in this direction. Its proof is long and sophisticated, stretching over [27], [25] and [26].

Theorem. *A semimatroid $\mathcal{L} \subset \mathcal{R}(N)$ on a four-element set N is p-representable if and only if it is equal to the intersection of a simple semimatroid with isomorphic copies of the three semimatroids of Fig. 12 and of the four semimatroids of Fig. 14.*

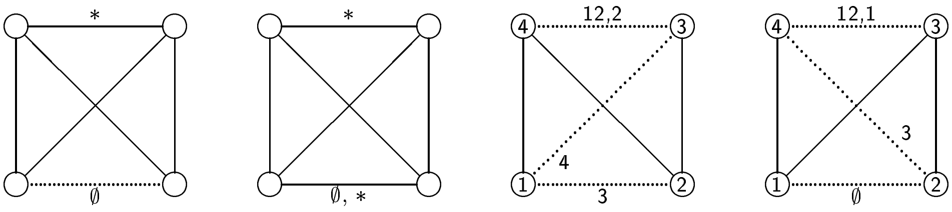


Figure 14. Nonlinear p-representable 4-semimatroids.

We conclude this section by formulating two challenging conjectures on the complexity of the class of p-representable relations. They concern infinite sets of forbidden minors, but analogous conjectures on infinitarity of closure operators are attractive, too. At the moment we have only one result in this direction that will be presented elsewhere, namely, we can prove that there exists an infinite set of semimatroids (they are matroids and not semigraphoids) which are not p-representable but all their proper minors are p-representable.

Conjecture 1. There is an infinite set of semimatroids under $\mathcal{R}(N)$ which are not p-representable and, at the same time, which have all proper minors p-representable.

Conjecture 2. There is an infinite set of p-representable semigraphoids which are not l-representable and, at the same time, which have all proper minors l-representable.

7. Remarks

We believe that human conditional thinking under uncertainty is based on experience with small groups of quantities and qualities and that the main features of such groups have to be recognized and modeled preferentialy. Then going to a larger world configuration, a human expert seems to extrapolate from his experience with the small models. After the extrapolation, all small instances of a large model must be in consistence with the adopted small models. This condition should, in fact, define the possible large models if no further synergy is assumed by the extrapolation.

When CI-relations are used to model the conditional thinking then, first, small CI-relations must be sorted into two classes, say, plausible and nonacceptable. Then a class of CI-relations that models large configurations should not contain a CI-relation having a small nonacceptable CI-relation as a subconfiguration. Moreover, by absence of synergy, every CI-relation without small undesirable obstructions should belong to the class. Then the class is uniquely defined and closed to subconfigurations. We think that the technique of minors, studied rigorously in this paper, is a very natural way of specifying what the subconfigurations are and what the extrapolation is about. Proposition 1 illustrates this approach very clearly: starting from four small 3-semigraphoids the minor-closed class of separation semigraphoids can be obtained.

It is a nice archievement if a set of the features is melted into an easily understandable axiom on CI-relations and, for example, the semigraphoid axiom is undoubtedly a deep axiom. But, we find it not natural if a class of CI-relations pretending to model conditional thinking of a human expert in a restricted domain is defined by a greater number of axioms. A greater number of small configurations seems to be, if unavoidable, more acceptable. This was well demonstrated e.g. on the class of chordal-separation graphoids which can be defined by two nontrivial 3-minors and one forbidden 4-minor, see Consequence 4. We do not find the equivalent definition by four axioms comparably eloquent: a few compulsory and forbidden minors give a clear small-case explanation of the conditional thinking behind the simplicial decompositions of chordal graphs.

One of the most attractive and theoretically interesting classes of CI-relations, the class of p-representable relations in our language, is definitely of a very high complexity. It is worthwhile to state explicitly that a CI-relation is a semigraphoid if and only if each of its 3-minors is p-representable. The class of semigraphoids can be thus claimed to be the 3-minor approximation of the class of p-representable relations and we have seen it is the 3-minor approximation of many other meaningful classes of CI-relations. The 4-minor-approximation of the p-representable relations is achieved by our Theorem, i.e. by simple 4-semimatroids and a few additional diagrams. The currently best test for p-representability of a small CI-relation $\mathcal{L} \subset \mathcal{R}(N)$ is to check whether it is a semimatroid as we proposed in 5.3. and, if it is, then to check whether all its 4-minors are p-representable relying on Theorem. But this is the most probably not enough.

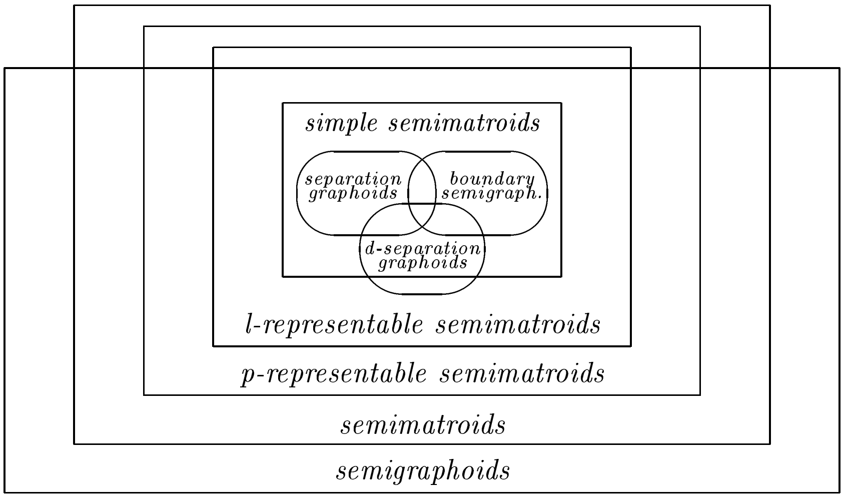


Figure 15. An overview of classes of CI-relations.

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