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CONDITIONAL INDEPENDENCES AMONG FOUR RANDOM VARIABLES I.[1]

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The conditional independences within a system of four discrete random variables are studied simultaneously. The problem where independences can occur at the same time, called the problem of probabilistic representability, is attacked by an analysis of cones of polymatroids. New results on the cone of all polymatroids satisfying Ingleton inequalities imply a substantial reduction of the problem and an explicit description of the remaining open cases.[3]

1. Introduction

Let N be a finite set and $\mathcal{S}(N)$ the family of all couples $(ij|K)$ where $K \subset N$ and ij is the union of two, not necessarily different, singletons i and j of $N-K$. Elements and singletons of N are not distinguished and the unions of subsets of N are written simply as juxtapositions. Having a system of random variables $\xi = (\xi_i)_{i \in N}$ with subsystems $\xi_K = (\xi_k)_{k \in K}$, $K \subset N$, we introduce the notation

$$[\![\xi]\!] = \{(ij|K) \in \mathcal{S}(N);\ \xi : i \perp j|K\},$$

where $\xi : i \perp j|K$ is the abbreviation of the statement "ξ_i is conditionally independent of ξ_j given ξ_K". The subsystem $\xi_\emptyset$ is presumed to be constant.

A subfamily $\mathcal{L} \subset \mathcal{S}(N)$ is called *probabilistically (p–) representable* if there exists a system ξ, called its *p–representation*, such that $\mathcal{L} = [\![\xi]\!]$. We prefer to speak about a relation $\mathcal{L}$ for it is in fact a binary relation on the power set of N. The paper is intended as the first installment of a series of papers whose final aim is to characterize the class $\mathbf{P}(N)$ of all p–representable relations for a four-element set N.

First problems of this kind emerged in works of J. Pearl and his collaborators preceding his book [10], where the conditional independences among subsystems ξ_I, ξ_J and ξ_K, $I, J, K \subset N$ disjoint, were studied *en block*. The corresponding families of triplets

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(I, J, K) have been called conditional independence relations or also (in)dependence models. Let us remark that there is no loss of generality when the sets I and J are admitted to be singletons only (cf. [6]) and that the original Pearl's problem to characterize all conditional independence relations is equivalent to the question which relation $\mathcal{L}$ contained in $\mathcal{R}(N) = \{(ij|K) \in \mathcal{S}(N);\ i \neq j\}$ is p–representable. Pearl's conjecture about a simple finite axiomatic framework appeared untrue [12].

The first steps toward the nondisjoint case and the above formulation of the problem of p–representability were done in [7]. Note that the statement $\xi : i \perp i|K$ means that the variable ξ_i is functionally dependent on the subsystem ξ_K and thus our setting includes implicitly also this kind of dependence. In the very formulation the p–representable relations comprise linear matroids (see [5]) and all p–representations of matroids are of a highly symmetric form (see [7]). For other results and references see [5]–[8] and [11]–[14].

The original practical motivation stems from the observation that the knowledge of $\mathbf{P}(N)$, at least for small sets N, might contribute to understanding of conditional inference in expert systems with uncertainty. The theoretical motivation is that of intriguing "probabilistic combinatorial configurations". In this respect our task resembles to constructions of the catalogues of combinatorial or algebraical structures and is analogically intended to serve as a motivating source of examples.

The paper is organized as follows. In order to establish necessary notations we review in Section 2 the methodology of [7] based on the use of Shannon entropy and related cones of polymatroids. Accordingly, all relations from $\mathbf{P}(N)$ are viewed as p–representable semimatroids; these are discussed in Section 3, where also the solution of the problem of p–representability in the cases $|N| \leq 3$ is outlined. After recognizing the role of Ingleton inequalities in the four-element case (Section 4) we introduce the notion of Ingleton semimatroid and prove that these semimatroids are p–representable. All canonical examples are listed consequently. In the last two sections we characterize explicitly all semimatroids that are not Ingleton. This will be the starting point of the next paper with same title.

2. Preliminaries

We shall work with real functions defined on the power set $\mathcal{P}(N)$ of N, with their differences

$$\triangle h(ij|K) = h(iK) + h(jK) - h(ijK) - h(K), \quad h \in \mathbf{R}^{\mathcal{P}(N)}, \ \ (ij|K) \in \mathcal{S}(N),$$

and with the relations

$$[\![h]\!] = \{(ij|K) \in \mathcal{S}(N);\ \triangle h(ij|K) = 0\}\,.$$

The central place of our considerations will be occupied by the convex cone

$$\mathbf{H}(N) = \{h \in \mathbf{R}^{\mathcal{P}(N)};\ h(\emptyset) = 0 \ \text{ and } \ \triangle h(ij|K) \geq 0,\ (ij|K) \in \mathcal{S}(N)\};$$

this means that the set functions from $\mathbf{H}(N)$ are normalized, nondecreasing and semimodular. Note that the requirements $\triangle h(i|K) \geq 0$ for $K \neq N - i$ are superfluous. Every pair (N, h) corresponds to a *polymatroid*, see [15].

A relation $\mathcal{L} \subset \mathcal{S}(N)$ is called *semimatroid* on N if and only if $\mathcal{L} = [\![h]\!]$ for some $h \in \mathbf{H}(N)$; we comment this situation also by saying that $\mathcal{L}$ arises from h. For the class of all semimatroids on N we reserve the symbol $\mathbf{S}(N)$. *Matroids* are the semimatroids arising from the rank functions of matroids, see [7], [15].

If ξ is a system of random variables then we denote the Shannon entropy of the subsystem ξ_I by $h_\xi(I)$, $I \subset N$. It is a well-known fact that the set function h_ξ (called in [2] *entropy function*) belongs to $\mathbf{H}(N)$ and, moreover,

$$\triangle h_\xi(ij|K) = 0 \quad \Leftrightarrow \quad \xi : i \perp j|K, \qquad (ij|K) \in \mathcal{S}(N),$$

which implies $[\![\xi]\!] = [\![h_\xi]\!]$. Hence, every p–representable relation is a semimatroid; in symbols $\mathbf{P}(N) \subset \mathbf{S}(N)$.

The intersection of two semimatroids is a semimatroid ($[\![h_1]\!] \cap [\![h_2]\!] = [\![h_1 + h_2]\!]$ for any $h_1, h_2 \in \mathbf{H}(N)$) and thus $(\mathbf{S}(N), \cap)$ is a semilattice. The intersection of two p–representable semimatroids is p–representable. In fact, if the independent systems ξ^1 and ξ^2 are combined coordinatewise $\xi = ((\xi^1_i, \xi^2_i))_{i\in N}$ then the equality $[\![\xi^1]\!] \cap [\![\xi^2]\!] = [\![\xi]\!]$ follows from $h_\xi = h_{\xi^1} + h_{\xi^2}$. Hence $(\mathbf{P}(N), \cap)$ is a subsemilattice of $(\mathbf{S}(N), \cap)$.

A semimatroid $\mathcal{L}$ is said to be *irreducible in* $\mathbf{S}(N)$ (or $\mathbf{S}$*–irreducible*) if it cannot be written as a nontrivial intersection of two semimatroids from $\mathbf{S}(N)$, thus if $\mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}$ and $\mathcal{L}_1, \mathcal{L}_2 \in \mathbf{S}(N)$ imply $\mathcal{L}_1 = \mathcal{L}$ or $\mathcal{L}_2 = \mathcal{L}$; see [1]. The same notion applies to $(\mathbf{P}(N), \cap)$. Consequently, every $\mathbf{S}$–irreducible semimatroid $\mathcal{L} \in \mathbf{P}(N)$ is also $\mathbf{P}$–irreducible. All semilattices are considered for lattices in the usual sense.

Two relations $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{S}(N)$ will be isomorphic if there exists a permutation π on N such that $(ij|K) \in \mathcal{L}_1$ if and only if $(\pi(i)\pi(j)|\pi(K)) \in \mathcal{L}_2$, where $\pi(K) = \{\pi(k); k \in K\}$. A *type* will be a class of all isomorphic relations. If a relation is semimatroid which is p–representable and irreducible in either of the above two senses then all isomorphic relations are semimatroids with the corresponding properties, respectively. This makes possible to use the introduced notions directly for types.

3. Semimatroids and polymatroids

In this section we shall describe the structure of the lattice of semimatroids and, as an illustration, we solve the p–representability problem for at most three-element sets N.

The cone $\mathbf{H}(N)$ is pointed and has a finite number of extreme rays. By its *face* we understand a nonempty intersection of $\mathbf{H}(N)$ with one of its supporting hyperplanes. Intersection of two faces is a face and the lattice of all faces of $\mathbf{H}(N)$, denoted by $\mathbf{F}(N)$, is known to be finite and atomic. Its atoms are the extreme rays of $\mathbf{H}(N)$. (All these observations are almost trivial as they follow from general properties of cones in Euclidean spaces, see e. g. [4].)

Lemma 1. *The lattices* $\mathbf{S}(N)$ *and* $\mathbf{F}(N)$ *are antiisomorphic.*

Proof. The binary relation $(ij|K) \circ h \Leftrightarrow \triangle h(ij|K) = 0$ between $\mathcal{S}(N)$ and $\mathbf{H}(N)$ gives rise to Galois connection $\mathcal{L} \to \mathcal{L}^*$ and $\mathcal{F} \to \mathcal{F}^*$. Here the closed convex cone

$$\mathcal{L}^* = \{h \in \mathbf{H}(N);\ \forall (ij|K) \in \mathcal{L}\ \ (ij|K) \circ h\} = \{h \in \mathbf{H}(N); [\![h]\!] \supset \mathcal{L}\}, \quad \mathcal{L} \subset \mathcal{S}(N),$$

is a face. To verify this fact the equivalent definition of faces requiring $h_1, h_2 \in \mathcal{F}$ as soon as $ah_1 + h_2 \in \mathcal{F}$ for some $h_1, h_2 \in \mathbf{H}(N)$ and $a > 0$ is to be used. The relation

$$\mathcal{F}^* = \{(ij|K) \in \mathcal{S}(N); \forall h \in \mathcal{F}\ \ (ij|K) \circ h\} = \bigcap \{[\![h]\!]; h \in \mathcal{F}\}, \quad \mathcal{F} \subset \mathbf{H}(N),$$

is clearly a semimatroid.

Next, plainly $\mathcal{L}^{**} = \bigcap\{[\![h]\!];\, [\![h]\!] \supset \mathcal{L}\} = \mathcal{L}$ if $\mathcal{L}$ is a semimatroid. It rests to demonstrate $\mathcal{F} = \mathcal{F}^{**}$ for faces. An appeal to the general properties of Galois connections will then close the proof, cf. [1].

If $\mathcal{F} = \mathbf{H}(N)$ then evidently $\mathcal{F} = \mathcal{F}^{**}$. Let $\mathcal{F} = \{h \in \mathbf{H}(N); (g,h) = 0\}$ be a face different from $\mathbf{H}(N)$, where $g \in \mathbf{R}^{\mathcal{P}(N)}$ is a normal vector of a supporting hyperplane. We can suppose g to satisfy $g(\emptyset) = 0$, $g \neq 0$ and $(g,h) \geq 0$ for any $h \in \mathbf{H}(N)$. Then g belongs to the polar cone of $\mathbf{H}(N)$ and there exist nonnegative numbers $a_{(ij|K)}$ such that

$$(g,h) \;=\; \sum_{(ij|K)\in\mathcal{S}(N)} a_{(ij|K)}\triangle h(ij|K)\ , \quad h \in \mathbf{R}^{\mathcal{P}(N)},\ h(\emptyset) = 0.$$

Denoting by $\mathcal{L} = \{(ij|K) \in \mathcal{S}(N);\, a_{(ij|K)} > 0\} \neq \emptyset$ we can write $\mathcal{F} = \mathcal{L}^*$ and as always $\mathcal{L}^* = \mathcal{L}^{***}$ we have the desired $\mathcal{F} = \mathcal{F}^{**}$. ∎

Consequence 1. *The lattice* $\mathbf{S}(N)$ *is coatomic. Its coatoms arise from the functions generating extreme rays of* $\mathbf{H}(N)$. ∎

A characterization of the extreme rays of the cone $\mathbf{H}(N)$ was found in [9]. It implies that the rank function of a matroid generates an extreme ray of $\mathbf{H}(N)$ if and only if the matroid is connected after deleting all its loops.

If $|N| \leq 3$ then there are no other extreme rays and if $|N| = 4$ then the cone $\mathbf{H}(N)$ has in addition to the 27 matroid extreme rays further 14 extreme rays (see e.g. [8]). They are generated by the functions ($i \in N$)

$$\begin{aligned} g_i^{(2)}(J) &= 2, & J &= i, \qquad & g_i^{(3)}(J) &= |J|, & i &\notin J,\\ &= \min\{2,|J|\}, & J &\neq i, & &= \min\{3,|J|+1\}, & i &\in J, \end{aligned}$$

and ($i,j \in N, i \neq j$)

$$\begin{aligned} f_{ij}(K) &= 3, & & K \in \{ik, jk, il, jl, kl\},\\ &= \min\{4, 2|K|\}, & & \text{otherwise}, \end{aligned}$$

where $kl = N - ij$. The number of semimatroids irreducible in $\mathbf{S}(N)$ is thus 1, 2, 4, 9, 42 and the number of corresponding types is 1, 2, 3, 5, 12 for $|N| = 0, 1, 2, 3, 4$, respectively.

In [5] we proved that every matroid which is linearly representable over a finite field is also p–representable. Hence for $|N| \leq 4$ all matroids are p–representable. Also the irreducible semimatroids $[\![g_i^{(2)}]\!]$ and $[\![g_i^{(3)}]\!]$, $i \in N$, are p–representable, as their free extensions are the uniform matroids of rank 2 and 3 on a five-element set, respectively (for details on extensions see [9], [7]). Their p–representations will be included in the Example below. The semimatroids $[\![f_{ij}]\!]$ were found not to be p–representable in [7].

The above mentioned facts together with Consequence 1 allow us to state the equality $\mathbf{P}(N) = \mathbf{S}(N)$, $|N| \leq 3$. The lattice isomorphisms below may be trivially verified.

$$\begin{aligned} \mathbf{P}(N) &\sim \mathcal{P}(N), & |N| &\leq 1,\\ &\sim \mathcal{U} \times \mathcal{P}(N), & |N| &= 2,\\ &\sim \mathcal{V} \times \mathcal{P}(N), & |N| &= 3, \end{aligned}$$

where $\mathcal{U}$ is a two-element lattice and $\mathcal{V}$ is the 22-element lattice of all semimatroids which are contained in $\mathcal{R}(N)$, see Figure. The decompositions of $\mathbf{S}(N)$ into the Cartesian products are not accidental, cf. [7].

4. Ingleton inequality

From now on we shall assume that the set N has four elements; $|N| = 4$. For brevity we omit N in expressions like $\mathbf{P}(N)$, $\mathbf{H}(N)$, etc. The symbols i, j, k and l will always denote distinct elements of N.

The nonnegativity of the expression

$$\Box h(ij) = h(ik) + h(jk) + h(il) + h(jl) + h(kl) - h(ij) - h(k) - h(l) - h(ikl) - h(jkl)$$

is a necessary condition for a matroid with the rank function h to be linear, see [3], [15]. By $\mathbf{H}^{\Box}$ we denote the subcone of $\mathbf{H}$ defined by means of the six Ingleton inequalities $\Box h(ij) \geq 0$.

One of the main arts in our proofs below is the following chain of equalities exhibiting *five masks* of $\Box h(ij)$. The function h is dropped out so that we work with functionals.

$$\begin{aligned} \Box(ij) &= \triangle(kl|i) + \triangle(kl|j) + \triangle(ij|\emptyset) - \triangle(kl|\emptyset) \\ &= \triangle(kl|i) + \triangle(jl|k) + \triangle(ij|\emptyset) - \triangle(jl|\emptyset) \\ &= \triangle(ij|k) + \triangle(ik|l) + \triangle(kl|j) - \triangle(ik|j) \\ &= \triangle(ij|k) + \triangle(ij|l) + \triangle(kl|ij) - \triangle(ij|kl) \\ &= \triangle(ij|k) + \triangle(ik|l) + \triangle(kl|ij) - \triangle(ik|jl)\,. \end{aligned}$$

Lemma 2. *The cone $\mathbf{H}^{\Box}$ has 35 extreme rays; they coincide with the extreme rays of $\mathbf{H}$ different from those generated by the functions f_{ij}, $i, j \in N$ distinct.*

Proof. We divide the cone $\mathbf{H}$ into two subcones $\mathbf{H}^- = \{h \in \mathbf{H}; \nabla h \leq 0\}$ and $\mathbf{H}^+ = \{h \in \mathbf{H}; \nabla h \geq 0\}$, where

$$\nabla h = \sum_{I \subset N} (-1)^{|I|-1} h(I) = \triangle h(ij|kl) + \triangle h(ij|\emptyset) - \triangle h(ij|k) - \triangle h(ij|l), \quad h \in \mathbf{R}^{\mathcal{P}(N)}$$

(i, j, k, l can be arbitrary distinct elements of N). By examining the fourth mask we may realize the identity

$$\triangle(ij|\emptyset) + \triangle(kl|ij) = \Box(ij) + \nabla,$$

and thus $\mathbf{H}^- \subset \mathbf{H}^{\Box}$. All matroid extreme rays of $\mathbf{H}$ belong clearly to $\mathbf{H}^{\Box}$ as the corresponding matroids are linear and thus satisfy all Ingleton inequalities. From $\nabla g_i^{(2)} = \nabla g_i^{(3)} = -1$ we deduce that $\mathbf{H}^{\Box}$ has at least 35 extreme rays.

The second part of the proof is based on the fact that the cone $\mathbf{H}^-$ has exactly 33 extreme rays (see Example in [8]). These may be obtained by removing from the previously mentioned 35 rays the two rays generated by the rank functions r_1 and r_3 of the uniform matroids of the ranks 1 and 3, respectively. It remains to verify that every function $h \in \mathbf{H}^{\Box}$ is a conical combination of r_1, r_3 and a function $g \in \mathbf{H}^-$. To this end let us set

$$g = h - \min\{\triangle h(ij|\emptyset)\}\, r_1 - \min\{\triangle h(ij|kl)\}\, r_3 \quad ,$$

where the minima range over six-element sets of differences. This function is plainly an element of $\mathbf{H}^{\Box}$ (note that $\Box g = \Box h$ as $\Box r_1 = \Box r_3 = 0$). It has one difference with $\emptyset$ and one difference with a two-element set on the second place of the indexing couple equal to zero. We claim that $g \in \mathbf{H}^-$. Indeed, if $\triangle g(ij|\emptyset) = 0$ and $\triangle g(kl|ij) = 0$ then, due to the above identity, $\nabla g = -\square g(ij) \le 0$. In addition, the equalities $\triangle g(ij|\emptyset) = 0$ and $\triangle g(ij|kl) \cdot \triangle g(ik|jl) = 0$ yield $\nabla g = \triangle g(kl|ij) - \square g(ij) \le 0$, as a consequence of the fourth and fifth masks. ∎

5. Ingleton semimatroids

We say that $\mathcal{L} \in \mathbf{S}$ is an *Ingleton semimatroid* if and only if $\mathcal{L} = [\![h]\!]$ for some $h \in \mathbf{H}^\square$. This notion applies immediately to the types. The lattice of Ingleton semimatroids will be denoted by $\mathbf{S}^\square$.

Theorem 1. *Every Ingleton semimatroid is p–representable; formally* $\mathbf{S}^\square \subset \mathbf{P}$. *There are eleven* $\mathbf{P}$*–irreducible Ingleton types.*

Proof. First assertion follows from Lemma 2 and from the discussion about the p–representability of the coatoms of $\mathbf{S}$ in Section 3. Moreover, the lattice $\mathbf{S}^\square$ is coatomic and its coatoms are also coatoms of $\mathbf{S}$ by Consequence 1. Thus for Ingleton semimatroids the $\mathbf{S}$–irreducibility coincides with the $\mathbf{P}$– and $\mathbf{S}^\square$–irreducibility. ∎

In the following example we list representatives of the $\mathbf{P}$–irreducible Ingleton types and give their p–representations. The employed notation for rank functions will be used later.

Example. Let $N = \{1, 2, 3, 4\}$ and $\Omega = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \ldots\}$ be a finite probability space with the uniform probability distribution; the number of elementary events will be clear from the context. Random variables on Ω are given as partitions corresponding to inverse images.

1. $\mathcal{L} = \mathcal{S} = [\![0]\!]$ is p–representable by constants $\xi_1 = \xi_2 = \xi_3 = \xi_4 = (\mathsf{a})$.
2. $\mathcal{L} = [\![r_1^{\{2,3,4\}}]\!]$ (the matroid of rank 1 with the loops 2, 3 and 4) is p–representable by $\xi_1 = (\mathsf{a})(\mathsf{b}), \xi_2 = \xi_3 = \xi_4 = (\mathsf{ab})$.
3. $\mathcal{L} = [\![r_1^{\{3,4\}}]\!]$ (the matroid of rank 1 with the loops 3 and 4) has the p–representation ξ consisting of $\xi_1 = \xi_2 = (\mathsf{a})(\mathsf{b}), \xi_3 = \xi_4 = (\mathsf{ab})$.
4. $\mathcal{L} = [\![r_1^{\{4\}}]\!]$ (the matroid of rank 1 with the loop 4) has the p–representation $\xi_1 = \xi_2 = \xi_3 = (\mathsf{a})(\mathsf{b}),\ \xi_4 = (\mathsf{ab})$.
5. $\mathcal{L} = [\![r_1]\!]$ (the uniform matroid of rank 1) is represented by four identical nonconstant random variables $\xi_1 = \xi_2 = \xi_3 = \xi_4 = (\mathsf{a})(\mathsf{b})$.
6. $\mathcal{L} = [\![r_2^{\{4\}}]\!]$ (the matroid with the loop 4 and with the uniform submatroid of rank 2 on $\{1, 2, 3\}$) has the p–representation given by $\xi_1 = (\mathsf{ab})(\mathsf{cd})$, $\xi_2 = (\mathsf{ac})(\mathsf{bd})$, $\xi_3 = (\mathsf{ad})(\mathsf{bc})$ and $\xi_4 = (\mathsf{abcd})$.
7. $\mathcal{L} = [\![r_2^{1\|4}]\!]$ (the matroid with two parallel elements 1 and 4 and with the uniform submatroid of rank 2 on $\{1, 2, 3\}$) has the p–representation as in the previous case except from $\xi_4 = \xi_1$.
8. $\mathcal{L} = [\![r_2]\!]$ (the uniform matroid of rank 2) can be represented by $\xi_1 = (\mathsf{abc})(\mathsf{def})(\mathsf{ghi})$, $\xi_2 = (\mathsf{adg})(\mathsf{beh})(\mathsf{cfi})$, $\xi_3 = (\mathsf{aei})(\mathsf{bfg})(\mathsf{cdh})$ and $\xi_4 = (\mathsf{afh})(\mathsf{bdi})(\mathsf{ceg})$.

9. $\mathcal{L} = [\![r_3]\!]$ (the uniform matroid of rank 3) is representable by $\xi_1 = (\mathsf{abcd})(\mathsf{efgh})$, $\xi_2 = (\mathsf{aceg})(\mathsf{bdfh})$, $\xi_3 = (\mathsf{abef})(\mathsf{cdgh})$ and $\xi_4 = (\mathsf{adfg})(\mathsf{bceh})$.

10. $\mathcal{L} = [\![g_4^{(2)}]\!]$ has the p–representation as in 6. but $\xi_4 = (\mathsf{a})(\mathsf{b})(\mathsf{c})(\mathsf{d})$.

11. $\mathcal{L} = [\![g_4^{(3)}]\!]$ is representable as in 9. but $\xi_4 = (\mathsf{ah})(\mathsf{bg})(\mathsf{cf})(\mathsf{de})$.

Every semimatroid above is accompanied by a "standard" p–representation ξ. This p–representation has the property that the cardinality of its probability space is as small as possible. Moreover, any p–representation of a semimatroid from 6.–11. that is defined on a probability space of the same cardinality as the given Ω does not practically differ from the standard ξ. This means that its probability space must be equipped with the uniform distribution and the p–representations (partitions) coincide up to a bijection (cf. also Theorem of [7]). Note that every subsystem ξ_K of every standard ξ, taken as a partition of Ω, consists from blocks of the same cardinality. The corresponding entropy functions are proportional to the rank functions, respectively. ∎

6. More about the cone H

The last two sections of the paper are devoted to an explicit description of the class $\mathbf{S} - \mathbf{S}^{\Box}$ of semimatroids which are not Ingleton. This task calls for further insight into the structure of the cone $\mathbf{H}$.

Lemma 3. *The cone $\mathbf{H}_{ij}^{\Box} = \{h \in \mathbf{H};\ \Box h(ij) \le 0\}$, $i, j \in N$ distinct, is the convex hull of 15 extreme rays. They are generated by the 15 linearly independent functions f_{ij}, r_1^{ijk}, r_1^{ijl}, r_1^{ikl}, r_1^{jkl}, r_1, r_3, r_1^i, r_1^j, r_1^{ik}, r_1^{jk}, r_1^{il}, r_1^{jl}, r_2^k and r_2^l, where $kl = N - ij$.*

Proof. Looking at the five masks of $\Box(ij)$ we observe that the cone $\mathbf{H}_{ij}^{\Box}$ is defined equivalently by the nonnegativity of $-\Box(ij)$, of the four differences $\triangle(m|N-m), m \in N$, and of the ten differences corresponding to the elements of the set

$$\mathcal{M}_{ij} = \{(kl|ij), (ij|\emptyset), (ij|k), (ij|l), (kl|i), (kl|j), (ik|l), (jk|l), (il|k), (jl|k)\}$$

and, of course, by the normalizing equality $h(\emptyset) = 0$ (altogether by 15 inequalities). We recall that the differences $\triangle(m|K)$, $K \ne N - m$, are irrelevant. Next,

$$[\![f_{ij}]\!] = \mathcal{M}_{ij} \cup \{(k|ij), (l|ij), (i|jkl), (j|ikl), (k|ijl), (l|ijk)\}$$

and $-\Box f_{ij}(ij) = 1$.

For the rank function $r_1^{N-m}, m \in N$, only one of the 15 inequalities is strict, namely $\triangle r_1^{N-m}(m|N-m) = 1$. The similar claim is trivial also for r_1 with $\triangle r_1(ij|\emptyset) = 1$ and for r_3 with $\triangle r_3(kl|ij) = 1$. A bit more computation is needed for the rank functions r_1^i with $\triangle r_1^i(kl|i) = 1$ and r_1^{ik} with $\triangle r_1^{ik}(jl|k) = 1$. The most tedious is the case of r_2^k with $\triangle r_2^k(ij|l) = 1$. The easiest way how to imagine these computations is, at least for us, to think about the conditional independences among the p–representations listed in the Example.

We conclude that $-\Box(ij)$, $\triangle(l|ijk)$, $\triangle(k|ijl)$, $\triangle(j|ikl)$, $\triangle(i|jkl)$, $\triangle(ij|\emptyset)$, $\triangle(kl|ij)$, $\triangle(kl|i)$, $\triangle(kl|j)$, $\triangle(jl|k)$, $\triangle(il|k)$, $\triangle(jk|l)$, $\triangle(ik|l)$, $\triangle(ij|l)$ and $\triangle(ij|k)$ are the coordinate functionals of the declared functions, respectively. ∎

Consequence 2. *A relation $\mathcal{L} \subset [\![f_{ij}]\!], i, j \in N$ distinct, is a semimatroid if and only if it satisfies*

$$(k|ij) \in \mathcal{L} \Leftrightarrow \{(kl|ij), (k|ijl)\} \subset \mathcal{L}, \quad k \in N - ij, l = N - ijk.$$

Proof. As $\triangle(k|ij) = \triangle(kl|ij) + \triangle(k|ijl)$, every semimatroid clearly satisfies the above two conditions. But, any semimatroid $\mathcal{L}$ is uniquely determined by its intersection with the class $\mathcal{R}'(N) = \mathcal{R}(N) \cup \{(m|N-m);\ m \in N\}$, cf. [7]. If $\mathcal{L} \subset [\![f_{ij}]\!]$ then $\mathcal{L} \cap \mathcal{R}'(N) = [\![h]\!] \cap \mathcal{R}'(N)$ for the sum h of some of the functions from Lemma 3 and if $\mathcal{L}$ also satisfies the conditions then it is a semimatroid ($\mathcal{L} = [\![h]\!]$). ∎

Consequence 3. *Let $\mathcal{L}_1 \subset \mathcal{L}_2 \subset [\![f_{ij}]\!]$, $i, j \in N$, $i \neq j$, be two semimatroids. If $\mathcal{L}_2$ is p–representable then $\mathcal{L}_1$ is p–representable, too.* ∎

The next assertion is interesting on its own right.

Lemma 4. *The intersection of two different cones $\mathbf{H}^{\square}_{ij}$ is contained in the cone $\mathbf{H}^{\square}$. The cone $\mathbf{H}$ is the disjoint union of $\mathbf{H}^{\square}$ and the six cones $\{h \in \mathbf{H};\ \square h(ij) < 0\}$.*

Proof. The first assertion follows from the identities (cf. masks 3 and 4)

$$\begin{aligned}
\square(ij) + \square(ik) &= \triangle(ik|l) + \triangle(kl|j) + \triangle(ij|l) + \triangle(jl|k), \\
\square(ij) + \square(kl) &= \triangle(ij|k) + \triangle(ij|l) + \triangle(kl|i) + \triangle(kl|j).
\end{aligned}$$

Hence, any $h \in \mathbf{H} - \mathbf{H}^{\square}$ violates exactly one Ingleton inequality. ∎

7. The remainder $\mathbf{S} - \mathbf{S}^{\square}$

Let us denote by

$$\mathcal{M}^{(kl|\emptyset)}_{ij} = \{(kl|i), (kl|j), (ij|\emptyset)\}, \qquad \mathcal{M}^{(jl|\emptyset)}_{ij} = \{(kl|i), (jl|k), (ij|\emptyset)\},$$
$$\mathcal{M}^{(ik|j)}_{ij} = \{(ij|k), (ik|l), (kl|j)\},$$
$$\mathcal{M}^{(ij|kl)}_{ij} = \{(ij|k), (ij|l), (kl|ij)\} \quad \text{and} \quad \mathcal{M}^{(ik|jl)}_{ij} = \{(ij|k), (ik|l), (kl|ij)\}$$

the semimatroids (cf. Consequence 2) corresponding to the differences with the sign + in the five masks of $\square(ij)$, with i, j, k, l distinct. We shall also permute $i \leftrightarrow j$ and $k \leftrightarrow l$ in these notations.

Theorem 2. *A semimatroid $\mathcal{L}$ is not Ingleton if and only if there are $i, j \in N$ distinct such that it is contained in $[\![f_{ij}]\!]$ and contains at least one of the 14 semimatroids $\mathcal{M}^{(\bullet)}_{ij}$, where $(\bullet)$ is an element of $\mathcal{R}(N) - \mathcal{M}_{ij}$ (equivalently, if and only if $\mathcal{M}^{(\bullet)}_{ij} \subset \mathcal{L}$ and $(\bullet) \notin \mathcal{L}$ for at least one $(\bullet)$).*

Proof. If $\mathcal{M}^{(\bullet)}_{ij} \subset \mathcal{L} \subset [\![f_{ij}]\!]$, where $(\bullet)$ is a couple admissible in the upper index position, and $\mathcal{L} = [\![h]\!]$, $h \in \mathbf{H}$, then $\square h(ij) = -\triangle h(\bullet) < 0$ follows from the properly chosen mask. Thus, $\mathcal{L} \notin \mathbf{S}^{\square}$.

Let $\mathcal{L}$ be a semimatroid that is not Ingleton. Using Lemma 3 we derive $\mathcal{L} \subset [\![f_{ij}]\!]$ for some $i, j \in N$ distinct. Let us suppose that $\mathcal{L}$ contains none of the relations $\mathcal{M}^{(\bullet)}_{ij}$. If $\mathcal{L} = [\![h]\!]$, $h \in \mathbf{H}$, $\square h(ij) < 0$, then we consider the new function $g = h + \square h(ij) f_{ij}$ which satisfies, on account of Lemma 3, $g \in \mathbf{H}^{\square}$, $\square g(ij) = 0$ and also $[\![g]\!] \cap [\![f_{ij}]\!] = [\![h]\!]$. The incidence $(\bullet) \in [\![g]\!]$ for a couple $(\bullet) \in \mathcal{R}(N) - [\![f_{ij}]\!]$ would imply $\mathcal{M}_{ij}^{(\bullet)} \subset [\![g]\!]$ and hence contradict $\mathcal{M}_{ij}^{(\bullet)} \subset [\![h]\!] = \mathcal{L}$. The conclusion sounds that the semimatroids $[\![g]\!]$ and $[\![h]\!]$ coincide on $\mathcal{R}'(N)$, whence $[\![g]\!] = \mathcal{L}$ is Ingleton, i.e. again a contradiction. ∎

Consequence 4. *A semimatroid $\mathcal{L} \in \mathbf{P} - \mathbf{S}^{\square}$ is irreducible in $\mathbf{P}$ if and only if it is maximal in $\mathbf{P} - \mathbf{S}^{\square}$ with respect to the inclusion.*

Proof. If $\mathcal{L} \in \mathbf{P} - \mathbf{S}^{\square}$ is maximal in $\mathbf{P} - \mathbf{S}^{\square}$ and $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$ for some $\mathcal{L}_1, \mathcal{L}_2 \in \mathbf{P}$ then at least one of the semimatroids $\mathcal{L}_1$ and $\mathcal{L}_2$ is not Ingleton. But then it equals $\mathcal{L}$.

If $\mathcal{L} \in \mathbf{P} - \mathbf{S}^{\square}$ is not maximal, i.e. $\mathcal{L} \subset \mathcal{L}_1 \in \mathbf{P} - \mathbf{S}^{\square}$ for some $\mathcal{L}_1 \neq \mathcal{L}$, then, by Lemma 3, it is the intersection of an Ingleton semimatroid $\mathcal{L}_2 \in \mathbf{P}$ and $\mathcal{L}_1$. Choosing properly i, j and $(\bullet)$ in Theorem 2 we get $\mathcal{L} \subset [\![f_{ij}]\!]$ and $\mathcal{M}_{ij}^{(\bullet)} \subset \mathcal{L} \subset \mathcal{L}_2$. From $(\bullet) \in \mathcal{L}_2 - \mathcal{L}$ we conclude that $\mathcal{L}$ is not irreducible in $\mathbf{P}$. ∎

Concluding, we summarize our results. For a four-element set we have reduced the p–representability problem to the question which semimatroids from $\mathbf{S} - \mathbf{S}^{\square}$ are p–representable. This question concerns semimatroids restricted by $\mathcal{M}_{ij}^{(\bullet)} \subset \mathcal{L} \subset [\![f_{ij}]\!]$, where $i, j \in N$ are fixed and distinct, and where $(\bullet) \in \{(kl|\emptyset), (jl|\emptyset), (ik|j), (ij|kl), (ik|jl)\}$. Modulo permutations this is, roughly speaking, a reduction to less than $5 \cdot 2^{11}$ cases. In the next paper we promise a reduction of the problem by purely probabilistic methods to less than $3 \cdot 2^5$ cases.

The final aim would be to complete the list from the Example by the $\mathbf{P}$–irreducible types which are not Ingleton. Having the extended list, any p–representable relation could be obtained by means of permutations and intersections.

References

[1] BIRKHOFF, G. (1984). *Lattice Theory.* Nauka, Moscow. (Russian translation)

[2] FUJISHIGE, S. (1978). Polymatroidal dependence structure of a set of random variables. *Information and Control* **39** 55–72.

[3] INGLETON, A.W. (1971). Conditions for representability and transversality of matroids. *Proc. Fr. Br. Conf.* (1970), Springer Lecture Notes **211** 62–67.

[4] LEICHTWEISS, K. (1980). *Konvexe Mengen.* VEB Deutscher Verlag der Wissenschaften, Berlin.

[5] MATÚŠ, F. (1992). Ascending and descending conditional independence relations. *Transactions of the Eleventh Prague Conference on Information Theory, Statistical Decision Functions and Random Processes,* Academia, Prague, Vol. B, 181–200.

[6] MATÚŠ, F. (1992). On equivalence of Markov properties over undirected graphs. *J. Appl. Prob.* **29** 745–749.

[7] MATÚŠ, F. (1994). Probabilistic conditional independence structures and matroid theory: background. *International Journal of General Systems* **22** 185–196.

[8] MATÚŠ, F. (1994). Extreme convex set functions with many nonnegative differences. (to appear in *Discrete Mathematics*)

[9] NGUYEN, H.Q. (1986). Semimodular functions. *Theory of Matroids* (N. White, ed.), Cambridge University Press, Cambridge, London, etc.

[10] PEARL, J. (1988). *Probabilistic Reasoning in Intelligent Systems.* Morgan Kaufman, San Meteo, California.

[11] STUDENÝ, M. (1989). Multiinformation and the problem of characterization of conditional independence relations. *Problems of Control and Inf. Th.* **18** 3–16.

[12] STUDENÝ, M. (1992). Conditional independence relations have no finite complete characterization. *Transactions of the Eleventh Prague Conference on Information Theory etc.,* Academia, Prague, Vol. B, 377–396.

[13] STUDENÝ, M. (1994). Structural semigraphoids. *International Journal of General Systems* **22** 207–217.

[14] STUDENÝ, M. (1994). Description of structures of stochastic conditional independence by means of faces and imsets (in three parts). (to apper in *International Journal of General Sustems*)

[15] WELSH, D.J.A. (1976). *Matroid Theory.* Academic Press, London.

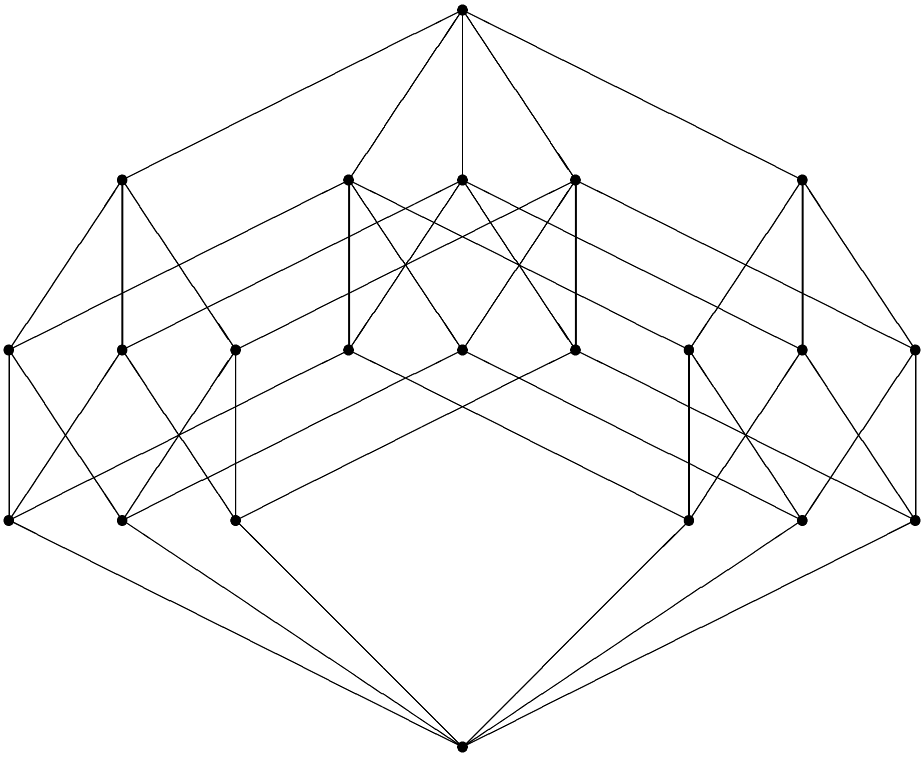

Figure.