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Infinitely Many Information Inequalities

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***Abstract*— When finite, Shannon entropies of all subvectors of a random vector are considered for the coordinates of an entropic point in Euclidean space. A linear combination of the coordinates gives rise to an unconstrained *information inequality* if it is nonnegative for all entropic points. With at least four variables no finite set of linear combinations generates all such inequalities. This is proved by constructing explicitly an infinite sequence of new linear information inequalities and a curve in a special geometric position to the halfspaces defined by the inequalities. The inequalities are constructed recurrently by adhesive pasting of restrictions of polymatroids and the curve ranges in the closure of a set of the entropic points.**

I. Introduction

Given a random vector $\xi = (\xi_i)_{i \in N}$ with indices in a finite set N, let $h_\xi(I)$ denote the Shannon entropy of the subvector $\xi_I = (\xi_i)_{i \in I}$, $I \subseteq N$. The collection $h_\xi = (h_\xi(I))_{I \subseteq N}$ of these entropies gives rise to the *entropy function* of ξ, mapping the subsets I of N into $[0, +\infty]$. If finite, the function can be considered for a point of the Euclidean space $\mathbb{R}^{\mathcal{P}(N)}$ where $\mathcal{P}(N)$ denotes the power set of N.

It has been of interest which point of this space corresponds to the entropy function of a vector taking a finite number of values. Such points or functions are called *entropic* and the limits of sequences of such points *asymptotically entropic;* for the sets of these points and limits the notations $\boldsymbol{H}_N^{\mathsf{ent}}$ and $\mathsf{cl}(\boldsymbol{H}_N^{\mathsf{ent}})$ are reserved, respectively. While $\boldsymbol{H}_N^{\mathsf{ent}}$ has a sophisticated shape [9], its closure is a convex cone [15, Theorem 1], not depending on the base of logarithms used in the Shannon entropy. The difference between them is only on a boundary of the latter [11, Theorem 1].

The main result of this contribution is formulated as follows.

Theorem 1. *The cone $\mathsf{cl}(\boldsymbol{H}_N^{\mathsf{ent}})$ is not polyhedral provided N has at least four elements.*

Equivalently rephrased, this cone is not expressible as the intersection of finitely many closed halfspaces, for a background on convexity see [13].

A point $c = (c_I)_{I \subseteq N}$ of $\mathbb{R}^{\mathcal{P}(N)}$ gives rise to a linear (unconstrained) *information inequality* if for any random vector $\xi = (\xi_i)_{i \in N}$ with a finite entropy

$$\sum\nolimits_{I \subseteq N} c_I \, h_\xi(I) \geqslant 0\,.$$

The sum can be interpreted as the standard scalar product of c with the entropy function h_ξ, and thus the inequality expresses that a closed halfspace of $\mathbb{R}^{\mathcal{P}(N)}$ defined by such $c \neq 0$ contains $\boldsymbol{H}_N^{\mathsf{ent}}$. Yet, in another viewing of the situation, $-c$ belongs to the polar cone $(\boldsymbol{H}_N^{\mathsf{ent}})^\circ$ of $\boldsymbol{H}_N^{\mathsf{ent}}$, for this notion see [13, Section 14]. This polar obviously coincides with $(\mathsf{cl}(\boldsymbol{H}_N^{\mathsf{ent}}))^\circ$. By [13, Theorem 14.1], $(\boldsymbol{H}_N^{\mathsf{ent}})^{\circ\circ} = \mathsf{cl}(\boldsymbol{H}_N^{\mathsf{ent}})$. These observations and the fact that the polar of a polyhedral cone is polyhedral [13, Corollary 19.2.2] imply the equivalence of Theorem 1 to the assertion that $-(\boldsymbol{H}_N^{\mathsf{ent}})^\circ$ is not polyhedral when $|N| \geqslant 4$. Informally rephrased, no finite set of linear information inequalities generates all of them.

For the proof of Theorem 1 presented in Section IV, an infinite sequence of new linear information inequalities and a curve in the cone $\mathsf{cl}(\boldsymbol{H}_N^{\mathsf{ent}})$ are constructed such that the halfspaces defined by the inequalities and the curve, defined in Lemma 6, are in a geometric position that excludes the cone to be polyhedral, see Lemma 5 and Figure 1. The information inequalities are found recurrently by adhesive extensions of restrictions of polymatroids; the adhesivity of the polymatroids is introduced and studied in [10]. The essence of this recurrence is captured in a new notion of the *inner adhesivity* of a polymatroid, presented in Section II. In particular, the finite entropy functions have this property by Lemma 2. The polymatroids with inner adhesivity satisfy three sequences of linear inequalities, proved when $|N| = 5$ in Theorem 2 of Section III. A special instance (4) of one of the sequence, that applies in the proof of Theorem 1, rewrites to

$$s\big[I(\xi_3;\xi_4|\xi_1) + I(\xi_3;\xi_4|\xi_2) + I(\xi_1;\xi_2) - I(\xi_3;\xi_4)\big]$$
$$I(\xi_2;\xi_3|\xi_4) + \tfrac{s(s+1)}{2}\big[I(\xi_2;\xi_4|\xi_3) + I(\xi_3;\xi_4|\xi_2)\big] \geqslant 0$$

where s is any positive integer. The case $s = 1$ is the inequality of Zhang-Yeung [17, Theorem 3] while $s = 2$ is the inequality *(i)* of Dougherty-Freiling-Zeger [2, Theorem III.1].

Entropy functions and the information theoretical inequalities emerge in various problems of the information theory and multivariate analysis. An introductory material is available in [14]. Recent works involve [1], [2], [6], [8], [11], [16], pointing to further references. Quantum analogues of the entropy functions are studied in [12].

II. Inner adhesivity of polymatroids

Recall that a *polymatroid* (N, g) consists of a finite ground set N and rank function $g\colon \mathcal{P}(N) \to [0, +\infty)$ that is normalized, $g(\emptyset) = 0$, nondecreasing, $g(I) \leqslant g(J)$ for $I \subseteq J \subseteq N$, and submodular, $g(I) + g(J) \geqslant g(I \cup J) + g(I \cap J)$ for all subsets I,J of N, see [4], [5]. If clear from context, a polymatroid is identified with its rank function. Considered for points of $\mathbb{R}^{\mathcal{P}(N)}$, the polymatroids exhaust bijectively a pointed polyhedral cone $\boldsymbol{H}_N$.

Basic properties of Shannon entropy imply that any finite entropy function h_ξ is a polymatroid [3], called entropic. Therefore $\mathit{cl}(\boldsymbol{H}_N^{\mathsf{ent}}) \subseteq \boldsymbol{H}_N$. By Zhang-Yeung inequality, the inclusion is strict if $|N| \geqslant 4$.

If two polymatroids (N,g) and (M,h) have their ground sets in the inclusion $N \subseteq M$ and satisfy $g(I)=h(I)$ for $I \subseteq N$ then g is the restriction of f, $g=\mathit{re}_N(h)$, and f is an extension of g. Thinking about points, the restriction is a linear map of $\mathbb{R}^{\mathcal{P}(M)}$ onto $\mathbb{R}^{\mathcal{P}(N)}$.

Two polymatroids (N,g) and (M,h) are *adhesive* or *adhere* if a polymatroid $(N\cup M, f)$ extends each of them and

$$\varDelta_{N,M}\, f \triangleq f(N)+f(M)-f(N\cup M)-f(N\cap M)$$

vanishes. Any such polymatroid f is called an adhesive extension of g and h. Note that the submodularity of f implies $\varDelta_{N,M}\, f \geqslant 0$ while the adhesivity demands this inequality to be tight.

The following simple assertion goes back to [8, Lemma 1].

Lemma 1. *Two restrictions of an entropic polymatroid have an adhesive extension that is entropic.*

More technically, for an entropic polymatroid (N,g) and subsets I,J of N there exists an entropic polymatroid $(I\cup J, f)$ such that

$$\mathit{re}_I(g)=\mathit{re}_I(f)\,,\quad \mathit{re}_J(g)=\mathit{re}_J(f) \quad \text{and} \quad \varDelta_{I,J}\, f=0\,.$$

As a consequence, any pair of restrictions of an entropic polymatroid is adhesive, and thus the entropic polymatroids belong to the class od polymatroids with adhesive restrictions, studied in detail in [10].

Definition 1. A polymatroid (N,g) has the *inner adhesivity of order* $s\geqslant 0$ if for any subsets $I_1,J_1,\dots,I_s,J_s$ of N satisfying $I_t\cup J_t=N$ for $1\leqslant t\leqslant s$ there exist polymatroids $f_0,\dots,f_s$ with the common ground set N such that $f_0=g$ and f_t is an adhesive extension of the restrictions of f_{t-1} to I_t and J_t for $1\leqslant t\leqslant s$.

Let $\boldsymbol{H}_N^{\mathsf{ia}}(s)$ denote the set of polymatroids with the ground set N that have the inner adhesivity of order $s\geqslant 0$. Obviously, $\boldsymbol{H}_N^{\mathsf{ia}}(0)=\boldsymbol{H}_N$ and $\boldsymbol{H}_N^{\mathsf{ia}}(1)$ is the set of polymatroids with adhesive restrictions.

Every $\boldsymbol{H}_N^{\mathsf{ia}}(s)$ is a polyhedral cone since it is a projection to first coordinate of the set of $(s+1)$-tuples $(f_0,\dots,f_s)$ of polymatroids that satisfy the linear equalities

$$\begin{array}{ll} \mathit{re}_{I_t}(f_t)=\mathit{re}_{I_t}(f_{t-1}),\ \mathit{re}_{J_t}(f_t)=\mathit{re}_{J_t}(f_{t-1}) & \\ \text{and}\ \ \varDelta_{I_t,J_t}\, f_t=0 & \text{for } 1\leqslant t\leqslant s\,. \end{array}$$

Note that $\boldsymbol{H}_N^{\mathsf{ia}}(s)$ contains $\boldsymbol{H}_N^{\mathsf{ia}}(s+1)$ by definition.

Definition 2. A polymatroid (N,g) has the *inner adhesivity* if it has the inner adhesivity of any order $s\geqslant 0$.

Hence,

$$\boldsymbol{H}_N^{\mathsf{ia}}(\infty)\triangleq \bigcap\nolimits_{s\geqslant 0} \boldsymbol{H}_N^{\mathsf{ia}}(s)$$

introduces a notation for the polymatroids with the inner adhesivity.

Lemma 2. *Any asymptotically entropic polymatroid has the inner adhesivity, $\mathit{cl}(\boldsymbol{H}_N^{\mathsf{ent}}) \subseteq \boldsymbol{H}_N^{\mathsf{ia}}(\infty)$.*

For a proof observe that for an entropic polymatroid the assertion obtains by a repeated use of Lemma 1 and that $\boldsymbol{H}_N^{\mathsf{ia}}(\infty)$ is obviously closed.

III. Inequalities for polymatroids with inner adhesivity

For $I\subseteq N$ let $\delta_I\in\mathbb{R}^{\mathcal{P}(N)}$ have the I-coordinate equal to 1 and the remaining ones equal to 0. In this section mostly $N=\{1,2,3,4,5\}$ with its subsets denoted by concatenations of elements; for example 345 denotes $\{3,4,5\}$. Writing

$$\varDelta_{I,J} \triangleq \delta_I+\delta_J-\delta_{I\cup J}-\delta_{I\cap J}\,,$$

the expression $\varDelta_{I,J}\, f$ encountered previously can be interpreted as the standard scalar product of $\varDelta_{I,J}$ with $f\in\mathbb{R}^{\mathcal{P}(N)}$. The alternative notation $\varDelta_{I\setminus J,J\setminus I|I\cap J}$ is used sometimes for $\varDelta_{I,J}$; for example $\varDelta_{12,5|34}$ abbreviates $\varDelta_{1234,345}$. If $I\setminus J$ and $J\setminus I$ are singletons even the comma is omitted, thus $\varDelta_{34|5}$ abbreviates $\varDelta_{3,4|5}=\varDelta_{35,45}$.

The following expressions

$$\square_{12,34} \triangleq \delta_{13}+\delta_{23}+\delta_{14}+\delta_{24}+\delta_{34}-\delta_{12}-\delta_{3}-\delta_{4}-\delta_{134}-\delta_{234}$$

and

$$\varDelta^{345} \triangleq \varDelta_{34|5}+\varDelta_{35|4}+\varDelta_{45|3}$$

play a crucial role in what follows.

Theorem 2. *For $N=\{1,2,3,4,5\}$, $s\geqslant 0$, $g\in\boldsymbol{H}_N^{\mathsf{ia}}(s)$ and $\boxminus$ equal to any of $\square_{12,34}$, $\square_{13,24}$ and $\square_{14,23}$, the scalar product of*

$$s\big[\,\boxminus+\varDelta_{34|5}+\varDelta_{45|3}\,\big]+\varDelta_{35|4}+\tfrac{s(s-1)}{2}\big[\varDelta_{24|3}+\varDelta_{34|2}\,\big]$$

with g is nonnegative.

Two lemmas and their corollaries are needed for a proof.

Lemma 3.

$$\begin{aligned}\square_{12,34}+\varDelta^{345} &= \varDelta_{15|3}+\varDelta_{15|4}+\varDelta_{25|3}+\varDelta_{25|4}\\ &\quad+\varDelta_{12|5}+\varDelta_{34|15}+\varDelta_{34|25}+\varDelta_{34,5|12}\\ &\quad-\big[\varDelta_{15|34}+\varDelta_{25|34}+\varDelta_{12,5|34}\,\big]\end{aligned}$$

Corollary 1. *If a polymatroid f adhesively extends the restrictions of a polymatroid g to 1234 and 345 then*

$$\square_{12,34}\, g+\varDelta^{345}\, g\geqslant \varDelta_{15|3}\, f+\varDelta_{15|4}\, f+\varDelta_{25|3}\, f+\varDelta_{25|4}\, f\,.$$

Proof. The left-hand side does not change when g is replaced by f since the restrictions of f and g to 1234 and 345 coincide. After the replacement, the inequality follows from Lemma 3. In fact, the scalar product of the bracket in Lemma 3 with f vanishes since $\Delta_{12,5|34}\, f = 0$ by adhesivity, and the scalar products of the Δ 's in the second line of the identity with f are nonnegative by submodularity of f. $\square$

Lemma 4.

$$\begin{aligned} \square_{13,24} + \Delta^{345} &= \Delta_{15|2} + \Delta_{15|4} + \Delta_{25|3} + \Delta_{25|4} \\ &+ \Delta_{13|5} + \Delta_{24|15} + \Delta_{34|25} + \Delta_{35|124} \\ &+ \Delta_{24,5|13} - \big[\Delta_{25|34} + 2\Delta_{12,5|34}\big] \end{aligned}$$

The derivation of Corollary 1 from Lemma 3 has an obvious analogue that provides the following inequality.

Corollary 2. *Under the assumptions of Corollary 1*

$$\square_{13,24}\, g + \Delta^{345}\, g \geqslant \Delta_{15|2}\, f + \Delta_{15|4}\, f + \Delta_{25|3}\, f + \Delta_{25|4}\, f\,.$$

Proof of Theorem 2. Induction on $s \geqslant 0$ is applied, the case $s = 0$ being trivial. If $s \geqslant 1$ assume that the assertion holds for $s-1$. The assumption $g \in \boldsymbol{H}_N^{\mathsf{ia}}(s)$ means that there exists $f \in \boldsymbol{H}_N^{\mathsf{ia}}(s-1)$ such that

$$re_{1234}(f-g) = 0\,,\; re_{345}(f-g) = 0 \;\text{ and }\; \Delta_{12,5|34}\, f = 0\,.$$

The induction assumption applies to f, transposing *2* and *3*,

$$\begin{aligned} (s-1)\big[\, \boxbar\, f + \Delta_{24|5}\, f + \Delta_{45|2}\, f\big] + \Delta_{25|4}\, f \\ + \tfrac{(s-1)(s-2)}{2}\big[\Delta_{24|3}\, f + \Delta_{34|2}\, f\big] \geqslant 0\,. \end{aligned} \tag{1}$$

By Corollary 1 and submodularity of f,

$$\boxbar\, g + \Delta^{345}\, g \geqslant \Delta_{25|4}\, f \tag{2}$$

for $\boxbar$ equal to $\square_{12,34}$. This inequality holds also when $\boxbar$ equals $\square_{13,24}$ by Corollary 2 and when $\boxbar$ equals $\square_{14,23}$ by Corollary 2 with the roles of *3* and *4* interchanged. Then (1) and (2) add to

$$\begin{aligned} s\, \boxbar\, g + \Delta^{345}\, g + (s-1)\big[\Delta_{24|5}\, f + \Delta_{45|2}\, f\big] \\ + \tfrac{(s-1)(s-2)}{2}\big[\Delta_{24|3}\, g + \Delta_{34|2}\, g\big] \geqslant 0 \end{aligned} \tag{3}$$

because the restrictions of f and g to *1234* coincide.

The simple identity

$$\Delta_{34|5} + \Delta_{24|3} + \Delta_{25|34} = \Delta_{24|5} + \Delta_{34|25} + \Delta_{25|3}$$

submodularity of f and $\Delta_{25|34}\, f = 0$, an obvious consequence of $\Delta_{12,5|34}\, f = 0$, imply

$$\Delta_{34|5}\, f + \Delta_{24|3}\, f \geqslant \Delta_{24|5}\, f\,.$$

Therefore, since the restrictions of f and g coincide,

$$\Delta_{34|5}\, g + \Delta_{24|3}\, g \geqslant \Delta_{24|5}\, f\,.$$

This inequality, its instance with *2* interchanged with *5*

$$\Delta_{34|2}\, g + \Delta_{45|3}\, g \geqslant \Delta_{45|2}\, f$$

and (3) combine to

$$\begin{aligned} s\, \boxbar\, g &+ \Delta^{345}\, g \\ &+ (s-1)\big[\Delta_{34|5}\, g + \Delta_{45|3}\, g + \Delta_{24|3}\, g + \Delta_{34|2}\, g\big] \\ &+ \tfrac{(s-1)(s-2)}{2}\big[\Delta_{24|3}\, g + \Delta_{34|2}\, g\big] \geqslant 0 \end{aligned}$$

which completes the induction step. $\square$

Remark 1. If $s = 1$ and $\boxbar$ equals $\square_{12,34}$ then the inequality of Theorem 2 has the form $\square_{12,34}\, g + \Delta^{345}\, g \geqslant 0$ which follows for entropy functions from [6, Theorem 1], equivalent to [16, Theorem 3], with five random variables. If instead $\square_{13,24}$ is on the place of $\boxbar$ then $\square_{13,24}\, g + \Delta^{345}\, g \geqslant 0$ which follows for the polymatroids with adhesive restrictions from [10, Corollary 5]. The last choice $\boxbar = \square_{14,23}$ and a symmetry imply the latter inequality as well.

Corollary 3. *For $s \geqslant 1$ and any asymptotically entropic polymatroid f on $\{1,2,3,4\}$*

$$s\, \square_{12,34}\, f + \Delta_{23|4}\, f + \tfrac{s(s+1)}{2}\big[\Delta_{24|3}\, f + \Delta_{34|2}\, f\big] \geqslant 0 \tag{4}$$

and

$$\begin{aligned} s\, \square_{12,34}\, f + \Delta_{13|4}\, f + s\big[\Delta_{34|1}\, f + \Delta_{14|3}\, f\big] \\ + \tfrac{s(s-1)}{2}\big[\Delta_{24|3}\, f + \Delta_{34|2}\, f\big] \geqslant 0\,. \end{aligned}$$

Proof. If f is the entropy function of $\xi_1, \ldots, \xi_4$ then Theorem 2 with $\boxbar$ equal to $\square_{12,34}$ applies to the entropy function of $\xi = (\xi_1, \ldots, \xi_5)$ with $\xi_5 = \xi_2$ or $\xi_5 = \xi_1$ and implies the above inequalities. For asymptotically entropic polymatroids the inequalities hold by straightforward limiting. $\square$

Remark 2. If $s = 1$ then both inequalities of Corollary 3 are equivalent to Zhang-Yeung inequality $\square_{12,34}\, f + \Delta^{134}\, f \geqslant 0$. If $s = 2$ then they take the forms

$$2\, \square_{12,34}\, f + \Delta_{23|4}\, f + 3\Delta_{24|3}\, f + 3\Delta_{34|2}\, f \geqslant 0$$

$$2\, \square_{12,34} f + \Delta_{13|4} f + 2\Delta_{34|1} f + 2\Delta_{14|3} f + \Delta_{24|3} f + \Delta_{34|2} f \geqslant 0$$

and go back to [2, Theorem III.1], see *(i)* and *(ii)*, respectively. Other non-Shannon-type inequalities of [2] have been challenging for an interpretation.

IV. A CURVE OF ASYMPTOTICALLY ENTROPIC POLYMATROIDS

This section presents a proof of Theorem 1, invoking the following elementary geometrical fact.

Lemma 5. *If a curve $p \mapsto g_p$ with p in a right neighborhood of 0 ranges in a polyhedral set and has the one-sided tangent $\dot{g}_{0+} = \lim_{p \searrow 0+} \frac{1}{p}\,[g_p - g_0]$ at the endpoint g_0 then this set contains the point $g_0 + \varepsilon \dot{g}_{0+}$ for some $\varepsilon > 0$.*

Proof. Assume C is the polyhedral set of those $x \in \mathbb{R}^d$ that satisfy $\nabla_k\, x \leqslant t_k$, $1 \leqslant k \leqslant \ell$, where ℓ is a nonnegative integer, $t_k \in \mathbb{R}$ and $\nabla_k \in \mathbb{R}^d$. The endpoint g_0 of a curve in C satisfies the inequalities $\nabla_k\, g_0 \leqslant t_k$ of which, after a renumbering if necessary, first ℓ' are assumed to be tight and the remaining ones strict. Then, $\nabla_k\, \dot{g}_{0+} \leqslant 0$ for $k \leqslant \ell'$, and if ε is sufficiently small $\nabla_k\,(g_0 + \varepsilon \dot{g}_{0+}) < t_k$ for $k > \ell'$. These observations imply that the inequalities

$$\nabla_k\,(g_0 + \varepsilon \dot{g}_{0+}) \leqslant t_k\,, \qquad 1 \leqslant k \leqslant \ell\,,$$

hold for some $\varepsilon > 0$, which implies the assertion. $\square$

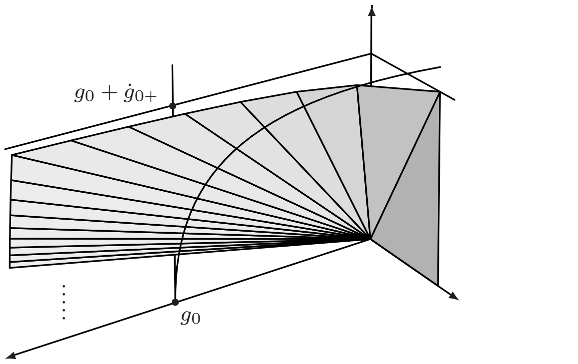

Fig. 1. A projection to $\mathbb{R}^3$ of the hyperplanes given by the equalities in (4) and the curve $p \mapsto g_p$.

From now on let $N = \{\mathit{1},\mathit{2},\mathit{3},\mathit{4}\}$ up to the proof of Theorem 1. For $I \subseteq N$ and $0 \leqslant t \leqslant |N \setminus I|$ let

$$r_t^I(J) = \min\{t, |J \setminus I|\}\,, \quad J \subseteq N\,,$$

which is a polymatroid majorized by $J \mapsto |J|$, thus a matroid. For example, r_2^I with $I = \{\mathit{4}\}$ takes the value 2 for J equal to *123* or *124*, and $r_1^{\mathit{14}}(\mathit{23}) = 1$.

Let ξ_{1} and ξ_{2} be independent $\{0,1\}$-valued random variables such that $\xi_{\mathit{1}} = 0$ with the probability $2p$ and $\xi_{\mathit{2}} = 0$ with the probability $\frac{1}{2}$, and let $\xi_{\mathit{3}} = \xi_{\mathit{1}}\xi_{\mathit{2}}$ and $\xi_{\mathit{4}} = (1-\xi_{\mathit{1}})(1-\xi_{\mathit{2}})$. This four-tuple ξ of random variables, see [7, Example 1] for its conditional independence structure, gives rise to the entropy function denoted below by $h_{\xi,p}$. The entropies are based on the natural logarithms.

The following assertion obtains by simple calculus.

Lemma 6. *Let $p \mapsto g_p$ with $0 \leqslant p \leqslant \frac{1}{2}$ be the curve of polymatroids given by*

$$\ln 2 \cdot g_p = h_{\xi,p} + \beta(p)\, r_1^{\mathit{14}} + [\,\ln 2 + 2p \ln 2 - \tfrac{1}{2}\beta(2p)\,]\,[\,r_1^{\mathit{23}} + r_2^{\mathit{4}}\,]$$

where β denotes the function $p \mapsto -p \ln p - (1-p)\ln(1-p)$.

Then g_0 equals $r_1^{\mathit{14}} + r_1^{\mathit{23}} + r_2^{\mathit{4}}$ and the one-sided tangent $\dot g_{0+}$ at this point equals $f_{\mathit{12}} + r_1^{\mathit{24}} + r_2^{\mathit{3}}$ where f_{12} is the polymatroid given by

$$f_{\mathit{12}}(J) = \begin{cases} 3\,, & J \in \{\mathit{13},\mathit{23},\mathit{14},\mathit{24},\mathit{34}\}, \\ \min\{4, 2|J|\}\,, & \textit{otherwise.} \end{cases}$$

Proof of Theorem 1. As well known $\mathsf{cl}(\boldsymbol{H}_N^{\mathsf{ent}})$ is equal to $\boldsymbol{H}_N$ for $|N| \leqslant 3$. Otherwise, it suffices to consider N of the cardinality four since obviously $\mathsf{re}_N(\mathsf{cl}(\boldsymbol{H}_M^{\mathsf{ent}})) = \mathsf{cl}(\boldsymbol{H}_N^{\mathsf{ent}})$ for $N \subseteq M$ and polyhedral sets are closed to linear mappings. For $N = \{\mathit{1},\mathit{2},\mathit{3},\mathit{4}\}$ the curve $p \mapsto g_p$ of Lemma 6 has the tangent $\dot g_{0+}$ and ranges in the closed convex cone $\mathsf{cl}(\boldsymbol{H}_N^{\mathsf{ent}})$ because $h_{\xi,p}$ and multiples of r_1^{14}, r_1^{23} and r_2^{4} are entropic. For every positive ε, it is easy to verify by Lemma 6 that the left-hand side of (4) with $g_0 + \varepsilon \dot g_{0+}$ in the role of f equals $1 - \varepsilon s$. Hence, for s sufficiently large the polymatroid $g_0 + \varepsilon \dot g_{0+}$ violates (4), and thus does not belong to $\mathsf{cl}(\boldsymbol{H}_N^{\mathsf{ent}})$ by Corollary 3. This observation and Lemma 5 imply that $\mathsf{cl}(\boldsymbol{H}_N^{\mathsf{ent}})$ is not polyhedral. $\square$

Remark 3. Analogously, it is possible to prove that the cone $\boldsymbol{H}_N^{\mathsf{ia}}(\infty)$ is not polyhedral if N has at least five elements; results of [10] imply $\boldsymbol{H}_N^{\mathsf{ia}}(\infty) = \boldsymbol{H}_N$ otherwise.

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