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ON NONNEGATIVITY OF SYMMETRIC POLYNOMIALS

F. MATÚŠ

ABSTRACT. Symmetric polynomials with unit coefficients in the field of complex numbers are shown to be simultaneously nonnegative if and only if all values of the variables are nonnegative. Symmetric polynomials in commuting complex matrices and their positive semidefiniteness are discussed, too.

Introduction. The nonnegativity of elementary symmetric polynomials (cf. [5])

$$x_1 + x_2 + \ldots + x_n, \;\; x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n, \;\; \ldots \;\;, \;\; x_1x_2\cdots x_n$$

implies the nonnegativity of all $x_1, x_2, \ldots, x_n$ in the domain of real numbers, see [3], Problem 10C. For complex numbers this ceases to be valid and thus to get the same conclusion we must assume more. The nonnegativity of the polynomials below in all powers $k \geq 1$ of the arguments

$$x_1^k + x_2^k + \ldots + x_n^k, \;\; x_1^kx_2^k + x_1^kx_3^k + \ldots + x_{n-1}^kx_n^k, \;\; \ldots \;\;, \;\; x_1^kx_2^k\cdots x_n^k$$

is also not yet enough. Namely, the choice of all n-th roots of unity, $n \geq 3$ prime, renders $x_1^k, x_2^k, \ldots, x_n^k$ to be the all roots or ones and having in mind the Viète formulae for roots of unity we see immediately the claimed nonegativity.

Where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers and $x = (x_1, \ldots, x_n)$ is an n-tuple of complex numbers, we shall write $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$. The symmetric polynomials (called in [6] "Potenzproduktsummen")

$$p_{[\alpha]}(x_1, \ldots, x_n) = p_{[\alpha]}(x) = \sum_{\beta \in [\alpha]} x^\beta \;, \qquad x \in \mathbf{C}^n,$$

where β ranges over the set $[\alpha]$ of all permutations of α, include all the polynomials displayed above. For them α is to be taken as $k\iota_M$ where M is a nonempty subset of $N = \{1, 2, \ldots, n\}$ and ι_M is the indicator function of M. A natural question asks which of these polynomials must be supposed nonnegative in order to conclude that all the arguments are nonnegative. More formally, denoting by $\mathcal{A}$ the family of all our n-tuples α, find all $\mathcal{B} \subset \mathcal{A}$ such that $x \in \mathbf{C}^n$ and $p_{[\alpha]}(x) \geq 0$, $\alpha \in \mathcal{B}$, imply $x_i \geq 0$, $i \in N$.

The aim of this short note is first to show that any $\mathcal{B}$ with finite complement in $\mathcal{A}$ will suffice. Then we explore the same question replacing the complex numbers by square complex matrices with either the usual matrix product under commutativity assumption or the coordinatewise Hadamard product. At the same time the nonnegativity is considered either in all coodinates at once or as positive semidefiniteness.

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Theorem. *Let $x \in \mathbf{C}^N$ and let the values $p_{[\alpha]}(x)$ be nonnegative for all $\alpha \in \mathcal{A}$. Then necessarily all coordinates x_i, $i \in N$, are nonnegative.*

Proof. We proceed by induction on the cardinality n of the set N. The case $n = 1$ is trivial. Let $n \geq 2$ and $x \in \mathbf{C}^N$, having at least one nonzero coordinate, fulfil the assumptions of the theorem. If the support $M = \{i \in N; x_i \neq 0\} \neq \emptyset$ of x is a proper subset of N then the restriction $y = x|_M \in \mathbf{C}^M$ satisfies $p_{[\alpha|_M]}(y) = p_{[\alpha]}(x)$ for all $\alpha \in \mathcal{A}$ such that $\alpha|_{N-M} = 0$, and then the induction argument yields $x_i = y_i > 0$, $i \in M$. So we now suppose that $\Pi_{i \in N}\, x_i \neq 0$.

Let us suppose first that we can partition the set N into two nonempty subsets I and J such that $|x_i| > |x_j|$ for $i \in I$ and $j \in J$. Where b is an upper bound of the set $\{|x_i|; i \in N\}$ and an n-tuple α is zero on J, we can estimate the difference

$$\left| p_{[\alpha+k\iota_I]}(x) \prod_{i \in I} x_i^{-k} - p_{[\alpha|_I]}(x|_I) \right| \leq \sum \left\{ |x^\beta| \prod_{i \in I} |x_i|^{-k}; \beta \in [\alpha + k\iota_I], \exists j \in J : \beta_j > 0 \right\} \leq$$

$$\leq n! \quad b^{\sum_{i \in N} \alpha_i} \sum \left\{ \prod_{i' \in I'} |x_{i'}|^k \prod_{i \in I} |x_i|^{-k}; I' \subset N, I' \neq I, |I'| = |I| \right\}, \quad k \geq 1.$$

The point is, due to the choice of the partition, that

$$p_{[\alpha|_I]}(x|_I) = \lim_{k \to \infty} \frac{p_{[\alpha+k\iota_I]}(x)}{x^{k\iota_I}}.$$

However, x^{ι_I} is a positive number since the numbers x_i, $i \in N$, being roots of the polynomial equation $\sum_{l=0}^{n} (-1)^l c_l x^{n-l} = 0$ with the nonnegative coefficients $c_l = p_{[\alpha]}(x)$ (α's are indicators), can be either nonnegative or match into pairs of mutually adjoint complex numbers. This enables effective use of the induction argument to establish $x_i > 0$, $i \in I$. A little reflexion about the vector $y = (x_i^{-1})_{i \in N}$ and the polynomial values

$$p_{[\alpha]}(y) = \sum_{\beta \in [\alpha]} \frac{1}{x^\beta} = \sum_{\beta \in [k\iota_N - \alpha]} \frac{x^\beta}{x^{k\iota_N}} = \frac{p_{[k\iota_N - \alpha]}(x)}{x^{k\iota_N}} \geq 0$$

(k is taken sufficiently high) persuades us that the above reasoning, applied to y, gives $x_j > 0$, $j \in J$.

It remains to analyse the case $|x_i| = a$, $i \in N$. Due to homogeneity of the polynomials there is no loss of generality in assuming $a = 1$. Denoting by $\mathcal{A}_m$ the family $\{\alpha \in \mathcal{A}; \alpha_i \leq m, i \in N\}$ we can write

$$q_m(x) = \sum_{[\alpha] \subset \mathcal{A}_m} p_{[\alpha]}(x) = \sum_{[\alpha] \subset \mathcal{A}_m} \sum_{\beta \in [\alpha]} x^\beta = \sum_{\alpha \in \mathcal{A}_m} \prod_{i \in N} x_i^{\alpha_i} = \prod_{i \in N} \sum_{l=0}^{m} x_i^l, \qquad m \geq 0.$$

Now, no x_i, $i \in N$, can be a root of unity different from 1 for $x_i^m = 1$ would imply $q_{km-1}(x) = 0$, $k \geq 1$, and this yields $p_{[\alpha]}(x) = 0$ for every $\alpha \in \mathcal{A}$, contradicting $\Pi_{i \in N} x_i = 1$. If one of the coordinates x_i is, however, not a root of unity then an old number-theoretic fact says the sequence $(x_i^k, k \geq 1)$ is dense in the unit circle and, by compactness, for some numbers $k_1 < k_2 < \ldots$ every sequence $(x_j^{k_l}, l \geq 1)$ converges to some y_j, $j \in N$.

The limit point y_i can be adjusted to -1. The continuity of the polynomials implies that $y=(y_j)_{j\in N}$ satisfies the nonnegativity assumptions, which contradicts the presence of -1 among its coordinates. It follows that $x_i=1$, $i\in N$. $\square$

Remarks. 1. When the word "nonnegative" is replaced in the formulation of the theorem by "positive" the new claim becomes, obviously, valid. Let us mention that we encountered the need for these assertions during an analysis of special symmetric random arrays.

2. The assumptions of the theorem can be further weakened. For example, the nonnegativity of the polynomials can be required only for $\alpha\in\mathcal{B}=\mathcal{A}-\mathcal{A}_m$ (apply the theorem to $(x_i^{m+1})_{i\in N}$ and $(x_i^{m+2})_{i\in N}$, consecutively).

Corollary. *Let $X_1,X_2,\ldots,X_n$ be normal and pairwise commuting square complex matrices. The polynomials $p_{[\alpha]}(X_1,\ldots,X_n)$, $\alpha\in\mathcal{A}$, are positively semidefinite if and only if all the matrices $X_1,\ldots,X_n$ possess this property.*

Proof. By [2] (Ch. IX, Par. 15) there exists a unitary matrix U (UU^* is the identity matrix) such that all matrices UX_iU^*, $i\in N$, are diagonal. Then $p_{[\alpha]}(X_1,\ldots,X_n)\succeq 0$ (a self-explaining abbreviation for positive semidefiniteness), $\alpha\in\mathcal{A}$, if and only if

$$U\,p_{[\alpha]}(X_1,\ldots,X_n)\,U^*=p_{[\alpha]}(UX_1U^*,\ldots,UX_nU^*)\succeq 0\ ,\qquad \alpha\in\mathcal{A},$$

and by the theorem this is equivalent to the nonnegativity of all elements of the diagonal matrices UX_iU^*, $i\in N$, i.e. to $UX_iU^*\succeq 0$ and finally to $X_i\succeq 0$, $i\in N$. $\square$

Remarks. 1. The assumption of normality cannot be omitted as is easily seen from the example of the following two commuting matrices

$$X_1=\begin{bmatrix}1&1\\0&1\end{bmatrix}\quad\text{and}\quad X_2=\begin{bmatrix}1&-1\\0&1\end{bmatrix}\ .$$

Note that any symmetric polynomial of X_1 and X_2 is a multiple of the identity matrix.

2. If we want to drop the assumption of commutativity we must redefine the symmetric polynomials $p_{[\alpha]}$. No reasonable generalization of the corollary can be expected as there exist two matrices

$$X_1=\begin{bmatrix}2&0\\0&1\end{bmatrix}\succeq 0\qquad X_2=\begin{bmatrix}1&1\\1&1\end{bmatrix}\succeq 0\ .$$

such that the Hermitian matrix $X_1X_2+X_2X_1$ is not positively semidefinite (see[1]).

3. If the matrices of $X=(X_i)_{i\in N}$ are nonnegative (coordinatewisely) then trivially also all polynomials $p_{[\alpha]}(X)$ are nonnegative but the contrary is not valid as the following matrices

$$X_1=\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}\qquad X_2=\frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$$

testify (X_1 and X_2 are idempotent and they commute with $X_1X_2=0$).

4. Let us consider the symmetric polynomials $p^{\bullet}_{[\alpha]}(X_1,\ldots,X_n)$ built as above but with Hadamard product on the place of the usual one (see [4]). Since this product preserves positive semidefiniteness once matrices $X=(X_i)_{i\in N}$ have this property all their

polynomials $p^\bullet_{[\alpha]}(X)$ share it, too. The converse is, however, not the case: the polynomials $p^\bullet_{[\alpha]}(X)$ in the two matrices

$$X_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

are positively semidefinite.

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Institute of Information Theory and Automation
Academy of the Sciences of Czech Republic
Pod Vodárenskou věží 4, 182 08 Prague
Czech Republic
E-mail: matus@utia.cas.cz