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PROBABILISTIC CONDITIONAL INDEPENDENCE STRUCTURES AND MATROID THEORY: BACKGROUND [1]

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ABSTRACT.

Special conditional independence structures have been recognized to be matroids. This opens new possibilities for the application of matroid theory methods (duality, minors, expansions) to the study of conditional independence and, on the other hand, starts a new probabilistic branch of matroid representation theory. [2]

1. INTRODUCTION

Though of old origin, the concept of conditional stochastic independence has been reattracting attention of mathematicians during last decades. A new viewpoint, from Dawid (1979), consists in the simultaneous examination of all conditional independences (among triples of subsystems of a stochastic system) separately from the joint probability distribution. Subsequent development of this idea has been influenced by the logic of integrity constraints from databases (see Pearl(1988), Geiger, Pearl(1989), Studený(1992)) and is oriented toward searching for plausible conditional independence relations (Oliver, Smith(1990), Matúš(1992a)).

In the thirties similar approaches appeared in algebra. Namely, the recognition of distinguished dependence properties led to attempts to postulate them and then to treat dependence in a purely abstract manner. The resulting idea, matroid theory (Welsh(1976), White(1986), Aigner(1979)) need not be specially introduced.

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[2]*Key words and phrases:* Conditional independence, Dependence/Independence structures/relations, Matroids, Semimodular functions, Semimatroids, Expansions, Excluded Minors, Probabilistic representations.

The aim of this contribution is to introduce connections between conditional independence structures and matroid theory in the hope of developing a synergy between the fields. Our point of departure is the description of stochastic conditional independences among subsystems of random variables by Shannon entropy. This enables us to identify matroids as a special class of conditional independence structures (Section 3). At the same time, probabilistic structures are presented as semimatroids; this notion, introduced in Section 4., is a natural generalization of matroids and is closely related to them by expansions. However, neither all semimatroids nor all matroids are of the probabilistic type (Section 7.).

Random variables representing matroids in their conditional independence form are found to have a simple structure, namely, they are uniformly distributed. This revelation (in Section 5.) shifts the problem of probabilistically representable matroids towards lattice theory and makes it possibile to relate it with linear and algebraic matroid representations. Applying the algebra of minors we can, after the observation that minors of probabilistically representable matroids are of the same type, translate the famous characterization theorems for binary, ternary and regular matroids into the probabilistic language (Sections 8. and 9.).

2. PRELIMINARIES

Let N be a finite set. For brevity the union of two subsets I, J of N is written as IJ and elements i, j of N are not distinguished from singletons of N. Let $\mathcal{P}(N)$ be the family of all subsets of N and $\mathcal{S}(N)$ the family of all couples $(ij|K)$, where $K \subset N$ and $i, j \in N - K$; we partition $\mathcal{S}(N)$ into $\mathcal{Q}(N)$ containing the couples with $i = j$ and $\mathcal{R}(N)$ containing those with $i \neq j$.

If $\xi = (\xi_i)_{i \in N}$ is a system of random variables indexed by N, we shall denote by $\xi_K = (\xi_i)_{i \in K}$, $K \subset N$, its subsystems; $\xi_\emptyset$ is supposed to be a constant. All random variables considered here are assumed to take finite number of values. The statement "ξ_I is conditionally independent of ξ_J given ξ_K " we shorten to $\xi : I \perp J|K$, $I, J, K \subset N$. Our main interest will be focused on these conditional independence structures

$$[\![\xi]\!] = \{(ij|K) \in \mathcal{S}(N);\ \ \xi : i \perp j|K\}.$$

We have been using (cf. Matúš(1992a),(1992b)) the phrase $\mathcal{L} \subset \mathcal{S}(N)$ is *probabilistically* ($p-$)*representable* if it has just this form, i.e. if there exists ξ such that $\mathcal{L} = [\![\xi]\!]$. Note that $(i|K) \in [\![\xi]\!]$ means that ξ_i depends functionally on ξ_K (see Matúš(1991)).

A simple way for treating the conditional independence is to employ Shannon entropy. If we denote by $h_\xi(I)$, $I \subset N$, the Shannon entropy of the subsystem ξ_I then $\xi : I \perp J|K$ if and only if $\Delta h_\xi(I, J|K) = 0$, $I, J, K \subset N$, where we set

$$\Delta h(I, J|K) = h(IK) + h(JK) - h(IJK) - h(K)$$

for any function h on $\mathcal{P}(N)$. This claim is often referred as *entropy balance rule.*

Let us recall that a real function h on $\mathcal{P}(N)$ is normalized if $h(\emptyset) = 0$, locally nondecreasing if $\Delta h(i,i|K) \geq 0$, $(i|K) \in \mathcal{Q}(N)$, and locally semimodular if $\Delta h(i,j|K) \geq 0$, $(ij|K) \in \mathcal{R}(N)$. Any function h from the class $\mathbf{H}(N)$ of all functions with the above three properties is also globally nondecreasing ($\Delta h(I,I|K) \geq 0$, $I, K \subset N$) and globally semimodular ($\Delta h(I,J|K) \geq 0$, $I,J,K \subset N$ disjoint).

Owing to the fundamental properties of Shannon entropy the function h_ξ belongs always to $\mathbf{H}(N)$. With the notation $[\![h]\!] = \{(ij|K) \in \mathcal{S}(N);\ \Delta h(i,j|K) = 0\}, h \in \mathbf{H}(N)$, the entropy balance rule implies $[\![h_\xi]\!] = [\![\xi]\!]$.

We are going to point out that our restriction to the couples from $\mathcal{S}(N)$ is general enough to capture the conditional independences among all subsystems of the considered system ξ. To this end it should be noted that the equalitites

$$\begin{aligned}\Delta h(I,J|K) &= \Delta h(I-K, J-K|K) = \\ &= \Delta h(I, J-I|K) + \Delta h(I\cap J, I\cap J|(J-I)K)\end{aligned}$$

take place for all $I,J,K \subset N$. Further, if $I = \{i_1, \ldots, i_s\}$, $s \geq 1$, and K are disjoint subsets of the set N then

$$\Delta h(I,I|K) = \sum_{t=1}^{s} \Delta h(i_t, i_t|i_{t-1}\ldots i_1 K)$$

and if, moreover, $J = \{j_1, \ldots, j_v\} \subset N - IK$, $v \geq 1$, then

$$\Delta h(I,J|K) = \sum_{t=1}^{s}\sum_{u=1}^{v} \Delta h(i_t, j_u|i_{t-1}\ldots i_1 j_{u-1}\ldots j_1 K)$$

(cf. Matúš(1992b), Lemma 3). Adopting the notation $gl\ \mathcal{L}$ ($\mathcal{L} \subset \mathcal{S}(N)$ arbitrary) for the set of all triples $(I,J|K)$, $I,J,K \subset N$, defined by the condition

$$(\forall i \in I-K)(\forall j \in J-K)(\forall L \subset N)(K \subset L \subset IJK - ij \ \Rightarrow\ (ij|L) \in \mathcal{L})$$

we conclude that for functions h from $\mathbf{H}(N)$ the difference $\Delta h(I,J|K)$ equals zero if and only if $(I,J|K)$ belongs to $gl\ [\![h]\!]$. Finally, $\xi : I \perp J|K$ if and only if $(I,J|K) \in gl\ [\![\xi]\!]$ and thus there is no loss of generality in discussing the p-representability on the level of the family $\mathcal{S}(N)$.

3. MATROIDS

The entropy approach to the conditional independence structures is the very one discerning matroids among them. It suffices only to remind that a matroid with the basic set N can be specified by (or identified with) its rank functions r, i.e. by an integer-valued function $r \in \mathbf{H}(N)$ satisfying the cardinality bound ($r(I) \leq |I|$, $I \subset N$) . Immediately, we can accept the following terminology. A relation $\mathcal{L} \subset \mathcal{S}(N)$ is called *matroid* if and only if there exists a rank function $r \in \mathbf{H}(N)$ such that it determines $\mathcal{L}$, i.e. $\mathcal{L} = [\![r]\!]$.

The more convenient characterization of matroids is based on their cryptomorphic descriptions by independent sets or by dependence relations (see Welsh(1976)). Accordingly, a relation $\mathcal{L} \subset \mathcal{S}(N)$ is a matroid if and only if the function

$$r^{\mathcal{L}}(I) = \max\{|J|;\ J \subset I,\ \ \mathcal{Q}(J) \cap \mathcal{L} = \emptyset\},\ \ I \subset N,$$

is semimodular and determines $\mathcal{L}$. To see it, let us remind that the family of the independent sets $\mathcal{I}_r = \{I \subset N;\ r(I) = |I|\}$ of a rank function r is nonempty ($\emptyset \in \mathcal{I}$), hereditary ($K \subset L \in \mathcal{I} \Rightarrow K \in \mathcal{I}$) and augmentable ($(K, L \in \mathcal{I},\ |K| < |L|) \Rightarrow (\exists i \in L - K)(iK \in \mathcal{I})$) and that $r(I) = \max\{|J|;\ J \subset I, J \in \mathcal{I}_r\}$; the rank functions are thus completely specified by their independent sets and also vice versa. If we observe that $I \in \mathcal{I}_r$ is equivalent to $\mathcal{Q}(I) \cap [\![r]\!] = \emptyset$ then for matroids we get $r^{\mathcal{L}} = r$. On the other hand, if $\mathcal{L} = [\![r^{\mathcal{L}}]\!]$ with semimodular $r^{\mathcal{L}}$ then trivially $r^{\mathcal{L}}$ is a rank function and hence $\mathcal{L}$ a matroid.

We remark that a matroid $\mathcal{L}$ is uniquely given by its *dependence part* $\mathcal{L} \cap \mathcal{Q}(N)$ which is known to be transitive

$$[i \notin J, (i|K) \in \mathcal{L},\ (\forall k \in K - J)((k|J) \in \mathcal{L})] \Rightarrow (i|J) \in \mathcal{L}$$

and exchangable

$$[(i|K) \in \mathcal{Q}(N) - \mathcal{L},\ (i|jK) \in \mathcal{L}] \Rightarrow (j|iK) \in \mathcal{L}.$$

Accordingly, $\mathcal{L}$ is a matroid if and only if $\mathcal{L}$ is transitive, exchangable and $\mathcal{L} = [\![r^{\mathcal{L}}]\!]$.

Under a trivial restriction a matroid $\mathcal{L}$ is uniquely given also by its *conditional independence part* $\mathcal{L} \cap \mathcal{R}(N)$. Let us remind that an element i of N is called *loop* of a rank function r if $r(i) = 0$; this notion we shall extend to all functions of $\mathbf{H}(N)$. Equivalently, $i \in N$ is a loop of a matroid $\mathcal{L}$ if $(i|\emptyset) \in \mathcal{L}$ and this loop definition can be accepted for all relations $\mathcal{L}$. Then a loopless relation $\mathcal{L} \subset \mathcal{S}(N)$ is a matroid if and only if the function

$$r_{\mathcal{L}}(I) = \max\{|J|;\ J \subset I,\ \mathcal{R}(J) \subset \mathcal{L}\},\ \ I \subset N,$$

is semimodular and determines $\mathcal{L}$ (observe that for a loopless matroid $\mathcal{L}$ the set I is independent only if $\mathcal{R}(I) \subset \mathcal{L}$).

We can summarize that for a matroid $\mathcal{L}$ with the set of loops $\lambda(\mathcal{L})$ one have $r_{\mathcal{L}}(I) = r^{\mathcal{L}}(I) + |I \cap \lambda(\mathcal{L})|$, $I \subset N$, and whereas $r^{\mathcal{L}}$ is the rank function of $\mathcal{L}$, $r_{\mathcal{L}}$ is the rank function of a matroid obtained from $\mathcal{L}$ by the conversion of all its loops into isthmuses ($i \in N$ is an isthmus, or a coloop, if $r(i) = 1$ and $r(N) = r(i) + r(N - i)$).

4. SEMIMATROIDS

We shall say that a relation $\mathcal{L} \subset \mathcal{S}(N)$ is a *semimatroid* if there exists a function $h \in \mathbf{H}(N)$ determining it, $\mathcal{L} = [\![h]\!]$. All p-representable relations are, due to the entropy balance rule, semimatroids.

First, we observe that intersection of two semimatroids $\mathcal{L}_s = [\![h_s]\!]$, $s = 1, 2$, is again a semimatroid $\mathcal{L}_1 \cap \mathcal{L}_2 = [\![h_1 + h_2]\!]$. Since $\mathcal{R}(N)$ is a matroid (with the rank function given by $r(I) = |I|, I \subset N$) the conditional independence part of any semimatroid is a semimatroid. On the other hand, if $\mathcal{L} \subset \mathcal{R}(N)$ is a semimatroid then the class of all semimatroids $\mathcal{K} \subset \mathcal{S}(N)$ such that $\mathcal{K} \cap \mathcal{R}(N) = \mathcal{L}$ forms a lattice (with respect to the inclusion ordering) isomorphic to the Boolean algebra $\mathcal{P}(N)$. To see it, we remark that every semimatroid $\mathcal{L} = [\![h]\!]$, $h \in \mathbf{H}(N)$, is owing to

$$\Delta h(i, i|K) = \Delta h(i, N - iK|K) + \Delta h(i, i|N - i)\,, \quad (i|K) \in \mathcal{Q}(N),$$

uniquely given by its intersection with $\mathcal{R}(N) \cup \{(i|N - i) \in \mathcal{Q}(N);\ i \in N\}$. Next, the function

$$h^* = h + \sum_{i \in N} (h(N - i) - h(N))g_i\,,$$

where $g_i \in \mathbf{H}(N)$ is given by $g_i(I) = 0$ for $i \notin I$ and $g_i(I) = 1$ for $i \in I$ (note that $[\![g_i]\!] = \mathcal{R}(N) \cup \{(j|K) \in \mathcal{Q}(N);\ i \neq j\}$), belongs also to $\mathbf{H}(N)$. The desired isomorphism ψ takes the form as follows

$$\psi(I) = [\![h^* + \sum_{i \in N - I} g_i]\!], \ \ I \subset N.$$

It is not difficult to prove that every (dependence part of a) semimatroid $\mathcal{L}$ is transitive but not necessarily exchangable. Also, if $\mathcal{L} \subset \mathcal{Q}(N)$ is a transitive relation then there exists a system ξ such that $[\![\xi]\!] \cap \mathcal{Q}(N) = \mathcal{L}$ (see Matúš(1991)) and then there exists a semimatroid having his dependence part equal to $\mathcal{L}$.

Let $(M_i)_{i \in N}$ be a family of subsets of a finite set M and $M_I = \bigcup_{i \in I} M_i$, $I \subset N$. To every function $g \in \mathbf{H}(M)$ we construct the function $h \in \mathbf{H}(N)$ by the equality $h(I) = g(M_I)$, $I \subset N$. Correspondingly, from every semimatroid $\mathcal{K} = [\![g]\!]$ we construct the semimatroid $\mathcal{L} = [\![h]\!]$ directly by the following formula

$$\mathcal{L} = \{(ij|K) \in \mathcal{S}(N);\ (M_i, M_j|M_K) \in gl\ \mathcal{K}\}.$$

To comment this situation we shall say that g and $\mathcal{K}$ is an *expansion* of h and $\mathcal{L}$, respectively; if $\mathcal{K}$ is a matroid we speak shortly about a matroid expansion.

PROPOSITION 1. Every semimatroid $\mathcal{L} \subset \mathcal{S}(N)$ has a matroid expansion $\mathcal{K}$ on a finite set M partitioned into M_i, $i \in N$.

PROOF. The class $\mathbf{H}(N)$ considered for a subset of Euclidean space $R^{\mathcal{P}(N)}$ is a convex cone. Since the scalar product of any function $h \in \mathbf{H}(N)$ with the function identically equal to 1 is nonnegative and, moreover, positive if $h \neq 0$, the cone $\mathbf{H}(N)$ is pointed. Being defined by finite number of inequalities and one equality it has finite number of extreme rays. As the equality and the inequalities have integer-valued coefficients, there is an integer-valued function on every extremal ray of $\mathbf{H}(N)$ (for details see White(1986), Section 10.4.B).

Now we shall show that any semimatroid can be determined by an integer-valued function from $\mathbf{H}(N)$. If $\mathcal{L} = [\![h]\!]$, $h \in \mathbf{H}(N)$, then by Caratheodory theorem $h = \sum_{\alpha \in A} c_\alpha h_\alpha$

is a conical combination of integer-valued functions h_α from extreme rays of $\mathbf{H}(N)$; c_α are positive real numbers and the cardinality of A is at most $2^{|N|}$. In this way we get $\mathcal{L} = [\![h]\!] = \bigcap_{\alpha\in A} [\![h_\alpha]\!] = [\![\sum_{\alpha\in A} h_\alpha]\!]$.

Any integer-valued function h from $\mathbf{H}(N)$ has a matroid expansion r (White(1986), Section 10.2), i.e. there are finite disjoint sets M_i, $i \in N$ ($\sum_{i\in I} M_i = M_I$ for $I \subset N$, $M_N = M$ and $|M_i| = h(i)$, $i \in N$), together with the rank function $r \in \mathbf{H}(M)$ such that $h(I) = r(M_I)$, $I \subset N$. Thus, $\mathcal{K} = [\![r]\!]$ is a matroid expansion of $\mathcal{L}$. $\square$

It is obvious that if an expansion $\mathcal{K}$ of a semimatroid $\mathcal{L}$ is p-representable (e.g. by $\eta = (\eta_m)_{m\in M}$) then also $\mathcal{L}$ is p-representable (e.g. by $\xi = (\xi_i)_{i\in N}$ where $\xi_i = (\eta_m)_{m\in M_i}$). The fundamental open question sounds now whether every p-representable semimatroid has a p-representable matroid expansion.

5. P-REPRESENTABILITY

Before starting an analysis of p-representable semimatroids and especially matroids we shall elaborate the operation of direct sum and the related notion of connectedness for semimatroids.

The *direct sum* of two relations $\mathcal{L}_s \subset \mathcal{S}(N_s)$, $s = 1, 2$, on disjoint ground sets is the relation $\mathcal{L}$ on $N = N_1 + N_2$ given by

$$\begin{aligned} \mathcal{L} &= \{(ij|K) \in \mathcal{S}(N); i \in N_1, j \in N_2\} \\ &\cup \{(ij|K) \in \mathcal{S}(N); ij \subset N_1, (ij|K \cap N_1) \in \mathcal{L}_1\} \\ &\cup \{(ij|K) \in \mathcal{S}(N); ij \subset N_2, (ij|K \cap N_2) \in \mathcal{L}_2\}. \end{aligned}$$

Hence, the relation $\mathcal{L}|N_s = \{(ij|K \cap N_s) \in \mathcal{S}(N_s);\ (ij|K) \in \mathcal{L}\}$ equals $\mathcal{L}_s$, $s = 1, 2$, and $(N_1, N_2|\emptyset)$ belongs to $gl\ \mathcal{L}$ (also $\mathcal{L}$ is the greatest relation on N with these properties). If $\mathcal{L}_s$ is a matroid with the rank function r_s, $s = 1, 2$, then $\mathcal{L}$ is the matroid with the rank function $r(I) = r_1(I \cap N_1) + r_2(I \cap N_2)$, $I \subset N$, and our definition of the direct sum coincides with the standard one. The direct sum of semimatroids is a semimatroid.

Having a relation $\mathcal{L}$ on a set N and a partition $N_1 + N_2 = N$ the direct sum of the relations $\mathcal{L}|N_s, s = 1, 2$, always contains $\mathcal{L}$. For N_1 or N_2 empty it is even equal to $\mathcal{L}$. If this direct sum coincides with $\mathcal{L}$ for a nontrivial partition then the relation $\mathcal{L}$ is called *disconnected*. To every relation $\mathcal{L}$ there is a unique partition of N into nonempty blocks N_t such that $\mathcal{L}$ is the direct sum of the connected relations (components) $\mathcal{L}|N_t$, $t = 1, 2, \ldots, s$, $s \geq 1$. The connected components of matroids and semimatroids are matroids and semimatroids, respectively.

Our main result describes neat behaviour of p-representations of connected matroids.

THEOREM. Let $\xi = (\xi_i)_{i\in N}$ be a p-representation of a connected matroid $\mathcal{L}$ with the rank function r and $r(N) \geq 2$. Then there exists a unique integer $u_\xi \geq 2$, we shall call it *degree* of ξ, such that every subsystem ξ_K, $K \subset N$, takes just $u_\xi^{r(K)}$ different values with the same probabilities $u_\xi^{-r(K)}$.

PROOF. We shall assume that the random variable ξ_i, $i \in N$, has values in a finite nonempty set X_i and the probability $\Pr(\xi_i = x_i)$ of every outcome $x_i \in X_i$ is positive. Then the values x_K of a subsystem ξ_K, $K \subset N$, are elements of the Cartesian product $X_K = \prod_{k \in K} X_k$.

Let i and j be two distinct elements of N (note that $|N| \geq 3$). Since they are not loops we distinguish the following two cases. Either $r(ij) = 1$, i.e. they are parallel, or $r(ij) = 2$. If i, j are parallel then $(i|j), (j|i) \in \mathcal{L}$; ξ_i is a function of ξ_j and vice versa. Thus, $|X_i| = |X_j|$ and the distributions of ξ_i and ξ_j coincide up to a bijection between X_i and X_j.

In the second case the set ij is independent. Then, there exists a circuit $L \subset N$ (a minimal dependent set) of the matroid $\mathcal{L}$ containing ij (see Welsh(1976), Chapter 5); let us set $K = L - ij \neq \emptyset$. Since for every $x_i \in X_i$ and $x_j \in X_j$

$$(x_i x_j) = \Pr(\xi_i = x_i, \xi_j = x_j) = \Pr(\xi_i = x_i) \Pr(\xi_j = x_j) > 0$$

(we indicated here how we shall shorten these expressions), there exists $x_K \in X_K$ such that $(x_i x_j x_K) > 0$ and $(x_K) > 0$. According to $(i, K|\emptyset), (j, K|\emptyset) \in gl\ \mathcal{L}$ and $(i|jK), (j|iK) \in \mathcal{L}$ we arrive at

$$(x_i)(x_K) = (x_i x_K) = (x_i x_j x_K) = (x_j x_K) = (x_j)(x_K).$$

The random variables ξ_i and ξ_j are thus uniformly distributed and $|X_i| = |X_j|$.

This implies that all variables ξ_i, $i \in N$, are uniformly distributed over sets of the same cardinality $u_\xi \geq 2$. If I is nonempty independent set of the matroid $\mathcal{L}$ then all random variables of the system ξ_I are mutually independent and hence ξ_I is uniformly distributed on a set of cardinality $u_\xi^{|I|}$. Finally, for every nonempty set $K \subset N$ we choose one of its maximal independent subsets I ($r(I) = |I| = r(K)$) and deduce that systems ξ_I and ξ_K are equidistributed. $\square$

6. REMARKS

A disconnected semimatroid $\mathcal{L}$ is easily found to be p-representable if and only if its connected components are so (the subsystems ξ_{N_t} of a p-representation ξ of $\mathcal{L}$ represent the connected components $\mathcal{L}|N_t$, $t = 1, 2, \ldots, s$, $s \geq 1$, and are mutually independent).

Let ξ be a p-representation of a matroid $\mathcal{L}$. If the connected components of $\mathcal{L}$ with the ranks not less than two have the subsystem representations with the same degree u_ξ we can call this number also the *degree* of ξ. In fact, the remaining connected components are either loops (representable by constant variables) or connected matroids with the ranks equal to one. In the last case the corresponding variables of the system ξ can be redefined, conserving ξ to be a p-representation of $\mathcal{L}$, in such a way that they are also uniformly distributed on sets of cardinality u_ξ. For the p-representations of this kind we establish $h_\xi(I) = r(I)\log u_\xi$, $I \subset N$, where r is the rank function of $\mathcal{L}$.

These considerations motivate us to say that a matroid $\mathcal{L}$ is *strongly p-representable* if there exists ξ and a positive constant c_ξ such that $h_\xi = c_\xi r$ (see Matúš(1992a)). For a matroid $\mathcal{L}$ with the rank greater than one we define the *p-characteristic set* $\chi(\mathcal{L})$ as the set of all integers $u \geq 2$ such that there exists a strong p-representation ξ of $\mathcal{L}$ with the degree $u_\xi = u$ ($c_\xi = \log u_\xi$). For the remaining matroids we assume $\chi(\mathcal{L})$ to be equal to the set $\{u; u \geq 2\}$.

The p-characteristic set of a matroid is the intersection of the p-characteristic sets of all his connected components and thus a matroid is strongly p-representable if and only if the intersection is nonempty. Theorem implies that for connected matroids the strong p-representability coincides with the normal (weak) one; this fact follows also directly from results of White(1986), Section 10.4 (the rank functions of connected matroids are in extreme rays of the cone $\mathbf{H}(N)$). We do not have an example of a p-representable matroid which is not strongly p-representable, i.e. examples of two matroids with disjoint p-characteristic sets.

The p-characteristic set of every matroid $\mathcal{L}$ is infinite unless it is empty (see below). In fact, if $u, v \in \chi(\mathcal{L})$ and $\xi = (\xi_i)_{i\in N}$, $\eta = (\eta_i)_{i\in N}$ are the corresponding strong p-representations of $\mathcal{L}$ (with the degrees u and v, respectively) then we can assume ξ and η to be stochastically independent and then $\zeta = (\zeta_i)_{i\in N}$, where $\zeta_i = (\xi_i, \eta_i)$, $i \in N$, is due to $h_\zeta = h_\xi + h_\eta$ a strong p-representation of $\mathcal{L}$ with the degree uv. The similar argumentation has been used to show that intersection of p-representable semimatroids is p-representable.

7. VAMOS CUBE

In Matúš(1992a) we proved that any matroid $\mathcal{L}$ linearly representable over a finite field F (i.e. there are vectors $e_i, i \in N$, in a linear space over the field F such that the dimension of the linear span of $e_i, i \in K$, equals the rank of K, $K \subset N$) is strongly p-representable and $|F| \in \chi(\mathcal{L})$. As all matroids on N, $|N| \leq 7$, and all matroids on N, $|N| = 8$, of the rank not equal to 4 are linearly representable, they are also strongly p-representable. Vamos cube, the matroid $\mathcal{K}$ on the eight-element set $M = ijkli'j'k'l'$ with the rank function

$$\begin{array}{ll} r(I) = 3, & I \in \mathcal{A} = \{iki'k',\ jkj'k',\ ili'l',\ jlj'l',\ klk'l'\}, \\ r(I) = \min\{4, |I|\}, & I \in \mathcal{P}(M) - \mathcal{A}, \end{array}$$

is not linearly representable over any field and it is thus a natural candidate of a non-p-representable matroid.

Some time ago my colleague M. Studený formulated a problem (similar to the existence of non-p-representable semimatroids) during the solution of which I found the following auxiliary result.

LEMMA 1. If $\mathcal{L}$ is a p-representable semimatroid on a four-element set $N = ijkl$ then

$$[\{(kl|ij),(i|jkl),(j|ikl),(ij|k),(ij|l)\} \subset \mathcal{L}] \Rightarrow [\{(i|kl),(j|kl)\} \subset \mathcal{L}].$$

PROOF. Let $\xi = (\xi_i, \xi_j, \xi_k, \xi_l)$ be a p-representation of $\mathcal{L}$ meeting the antecedent assumption. We adopt the notation of the foregoing proof.

For $x_k \in X_k$ and $x_l \in X_l$ such that $(x_k x_l) > 0$ we have to prove that if both probabilities $(x_i x_j x_k x_l)$ and $(y_i y_j x_k x_l)$ are positive for some $x_i, y_i \in X_i$ and $x_j, y_j \in X_j$ then $x_i = y_i$ and $x_j = y_j$.

If $(x_i x_j x_k x_l)$ $(y_i y_j x_k x_l) > 0$ then $(x_i x_l)(x_j x_l)(y_i x_l)(y_j x_l)(x_l)^{-2} > 0$ and $(ij|l) \in \mathcal{L}$ implies $(x_i y_j x_l)(y_i x_j x_l) > 0$. Similarly $(x_i y_j x_k)(y_i x_j x_k) > 0$ by $(ij|k) \in \mathcal{L}$. Owing to $(kl|ij) \in \mathcal{L}$ we get $(x_i y_j x_k x_l)(y_i x_j x_k x_l) > 0$ and the assumptions $(i|jkl), (j|ikl) \in \mathcal{L}$ lead to the desirable equalities $x_i = y_i$ and $x_j = y_j$. □

By means of the partition $M = ii' + jj' + kk' + ll'$ we can define a semimatroid $\mathcal{L}$ on a four-element set $N = ijkl$ having Vamos cube $\mathcal{K}$ for its expansion. As this semimatroid fails Lemma 1, $\mathcal{K}$ and $\mathcal{L}$ are not p-representable.[3] This can be interpereted also as a new proof that Vamos cube is not linearly representable over the finite fields.

8. DUALITY AND MINORS

To any matroid $\mathcal{L} \subset \mathcal{S}(N)$ we denote (cf. Matúš(1992a))

$$\mathcal{L}^{\rceil} = \{(i|K) \in \mathcal{Q}(N); (i|N - iK) \notin \mathcal{L}\} \cup \{(ij|K) \in \mathcal{R}(N); (ij|N - ijK) \in \mathcal{L}\}$$

and observe that if r is the rank function of $\mathcal{L}$ then $\mathcal{L}^{\rceil}$ is the matroid with the rank function given by $r^{\rceil}(I) = |I| + r(N - I) - r(N)$, $I \subset N$. This is matroid *duality* (orthogonality) in our cryptomorphic description. One shows easily $\mathcal{L} = \mathcal{L}^{\rceil\rceil}$ for any $\mathcal{L} \subset \mathcal{S}(N)$.

The above formula cannot be applied directly to semimatroids as it does not send, generally, semimatroids to semimatroids (the defect is in its dependence part). We shall not need an extension of the duality for semimatroids here and so we only remark that if $\mathcal{L} = [\![h]\!]$, $h \in \mathbf{H}(N)$, is a semimatroid then $\mathcal{L}^{\rceil} \cap \mathcal{R}(N)$ is also a semimatroid, e.g. it descendens from the function $h^{\rceil} + \sum_{i \in N} g_i$ where

$$h^{\rceil}(I) = \sum_{i \in I} h(i) + h(N - I) - h(N), \quad I \subset N.$$

If $\mathcal{L}$ is a relation on N and $I \subset N$ then we shall denote by $\mathrm{re}_I(\mathcal{L}) = \mathcal{L} \cap \mathcal{S}(I)$ its *restriction* to I and by $\mathrm{co}_I(\mathcal{L}) = \{(ij|K) \in \mathcal{S}(I);\ (ij|K(N-I)) \in \mathcal{L}\}$ its *contraction* by $N - I$ (both considered for relations on I). Immediately, for $I \subset J \subset N$ we establish $\mathrm{re}_I = \mathrm{re}_I \mathrm{re}_J$, $\mathrm{co}_I = \mathrm{co}_I \mathrm{co}_J$ and $\mathrm{re}_I \mathrm{co}_J = \mathrm{co}_I \mathrm{re}_{I(N-J)}$. These two operations are dual in the

[3] Let us remark that the class of p-representable semimatroids on four-element set has not been determined up to now. Results of M. Studený and me in this direction are intended to be published in a common paper (see also Studený(1993)).

above sense $(\mathrm{re}_I(\mathcal{L}^\rceil))^\rceil = \mathrm{co}_I(\mathcal{L})$, $I \subset N$, $\mathcal{L} \subset \mathcal{S}(N)$. The restriction to I of a matroid with the rank function r is the (sub)matroid with rank function $r|_{\mathcal{P}(I)}$ and its contraction by $N-I$ is the matroid on I with the rank functioon having the values $r(J(N-I))-r(N-I)$, $J \subset I$. The restrictions and contractions of semimatroids are semimatroids.

A relation $\mathcal{K}$ on $I \subset N$ is called a *minor* of a relation $\mathcal{L}$ on N if there exists a set J such that $I \subset J \subset N$ and $\mathcal{K} = \mathrm{re}_I(\mathrm{co}_J(\mathcal{L}))$. The order of the operations is not substantial and thus minors of minors are minors.

LEMMA 2. Minors of p-representable semimatroids are p-representable. Minors of strongly p-representable matroids are strongly p-representable with the same degrees.

PROOF. Let $\xi = (\xi_i)_{i\in N}$ be a p-representation of a semimatroid $\mathcal{L}$. As $\mathrm{re}_I(\mathcal{L})$, $I \subset N$, and $\mathrm{co}_I(\mathcal{L})$, $N-I \subset \lambda(\mathcal{L})$ are trivially representable by ξ_I it rests to prove that contractions of $\mathcal{L}$ by nonloops are p-representable. The same is valid for matroids.

If $i \in N - \lambda(\mathcal{L})$ and $K = N - i$ we construct to every value x_i of ξ_i (our notation from previous proofs is still in action) a new system of random variables $\xi_K^{(x_i)} = (\xi_k^{(x_i)})_{k\in K}$ by restricting ξ_K to the event $\xi_i = x_i$, i.e.

$$\Pr(\xi_K^{(x_i)} = y_K) = \frac{(x_i y_K)}{(x_i)}, \quad y_K \in X_K.$$

All systems $\xi_K^{(x_i)}$, $x_i \in X_i$, are considered for mutually independent.

The contraction $\mathrm{co}_K(\mathcal{L})$ of the semimatroid $\mathcal{L} = [\![h_\xi]\!]$ is determined by the function

$$g(J) = h_\xi(iJ) - h_\xi(i) = -\sum_{x_{iJ}\in X_{iJ}} (x_{iJ})\log(x_{iJ}) + \sum_{x_i\in X_i}(x_i)\log(x_i) = -\sum_{x_{iJ}\in X_{iJ}} (x_{iJ})\log\frac{(x_{iJ})}{(x_i)},$$

where $J \subset K$; the value $g(J)$ is thus the *conditional entropy* of ξ_J given ξ_i. After the casting

$$g(J) = -\sum_{x_i\in X_i}(x_i)\sum_{y_J\in X_J}\frac{(x_i y_J)}{(x_i)}\log\frac{(x_i y_J)}{(x_i)} = \sum_{x_i\in X_i}(x_i) h_{\xi_K^{(x_i)}}(J), \quad J \subset K,$$

we can see that g is a convex combination of the functions $h_{\xi_K^{(x_i)}}$, $x_i \in X_i$. This implies $\mathrm{co}_K(\mathcal{L}) = [\![g]\!] = \bigcap_{x_i\in X_i}[\![h_{\xi_K^{(x_i)}}]\!]$ and thus $\eta = (\eta_k)_{k\in K}$ where $\eta_k = (\xi_k^{(x_i)})_{x_i\in X_i}$ is a p-representation of the relation $\mathrm{co}_K(\mathcal{L})$.

If ξ is a strong p-representation of a matroid $\mathcal{L}$ with the degree u then all systems $\xi_K^{(x_i)}$, $x_i \in X_i$, are equidistributed and their entropy functions coincide with g. They all are strong p-representations of the matroid $\mathrm{co}_K(\mathcal{L})$ with the degrees u. $\square$

9. EXCLUDED MINORS

We shall give here criteria for matroids to have the numbers two, three or all greater than one in their p-characteristic sets. The criteria correspond to the linear representation theory and are in the form of forbidden special small minors.

Let us recall that the uniform matroid of rank t on s-element set N is

$$\mathcal{U}_{t,s} = \{(ij|K) \in \mathcal{R}(N); |K| \neq t-1\} \cup \{(i|K) \in \mathcal{Q}(N);\ |K| \geq t\},\ s \geq t \geq 0,$$

with the rank function $r(I) = \min\{t, |I|\}, I \subset N$. From Theorem it is easy to see that the matroid $\mathcal{U}_{2,4}$ does not have a p-representation of degree two (the four random variables needed to represent $\mathcal{U}_{2,4}$ must be different, two-valued, uniformly distributed and defined on a four-element probabilistic space what is impossible). Similarly, there is no p-representation of the matroid $\mathcal{U}_{2,5}$ with the degree three.[4]

The Fano matroid $\mathcal{F}$ on the set $N = ijkli'j'k'$ having seven elements has the rank function

$$\begin{array}{ll} r(I) = 2, & I \in \mathcal{A} = \{i'jk, ij'k, ijk', ii'l, jj'l, kk'l, i'j'k'\}, \\ r(I) = \min\{3, |I|\}, & I \in \mathcal{P}(N) - \mathcal{A} \end{array}$$

and is known to be linearly representable only over fields with the characteristic two (see Aigner(1979)). We shall show that $3 \notin \chi(\mathcal{F})$. In fact, if ξ is a p-representation of $\mathcal{F}$ with the degree three then by Theorem the underlying probability space can be identified with the Cartesian product of three examplares of the Galois field $GF(3)$; $\Omega = GF(3)^3$. The restrictions $\mathrm{re}_I(\mathcal{F})$, $I \in \{i'jk, ij'k, ijk'\}$ are matroids $\mathcal{U}_{2,3}$ and the atoms of $\xi_{i'}(\xi_{j'}, \xi_{k'})$ are two-dimensional parallel planes in the linear space Ω. The restriction $\mathrm{re}_{ijkl}(\mathcal{F})$ is $\mathcal{U}_{3,4}$ and then there are only four choices of ξ_l all having for atoms planes. Finally, ξ is also a linear representation of $\mathcal{F}$ over $GF(3)$ (the seven random variables correspond to linear functionals on Ω) what is a contradiction. Analogically, we get $3 \notin \chi(\mathcal{U}_{3,5})$.

If ξ is a p-representation of $\mathcal{F}^{\rceil}$ with the degree three then we set $\Omega = GF(3)^4$ and handle $\xi_{i'}, \xi_{j'}, \xi_{k'}, \xi_l$ as projections. After the observation $\mathrm{re}_I(\mathcal{F}) = \mathcal{U}_{3,4}$, $I \in \{ij'k'l,\ i'jk'l,\ i'j'kl\}$, we proceed as above to establish that $3 \notin \chi(\mathcal{F}^{\rceil})$.

The following claims are elementary consequences of the preceding remarks, Lemma 2 and of the well-known characterizations theorems for binary, ternary and regular matroids.

PROPOSITION 2. A matroid has a strong p-representation of the degree two (three) if and only if it does not have a minor isomorphic to $\mathcal{U}_{2,4}$ ($\mathcal{U}_{2,5}, \mathcal{U}_{2,5}^{\rceil} = \mathcal{U}_{3,5}, \mathcal{F}, \mathcal{F}^{\rceil}$). A matroid has p-representations of any degree $u \geq 2$ if and only if its minors are not isomorphic to $\mathcal{U}_{2,4}$, $\mathcal{F}$ and $\mathcal{F}^{\rceil}$.

[4]Note that $\chi(\mathcal{U}_{2,3}) = \{u; u \geq 2\}$ and that all p-representations of this matroid correspond to the latin squares. The assertion $6 \notin \chi(\mathcal{U}_{2,4})$ is equivalent to the fact that the famous Euler problem on 36 officers does not have a solution. More generally, $u \in \chi(\mathcal{U}_{2,n})$, $n \geq 3$, if and only if there are $n-2$ orthogonal latin squares of the size u.

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