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ON GAUSSIAN CONDITIONAL INDEPENDENCE STRUCTURES

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The simultaneous occurrence of conditional independences among subvectors of a regular Gaussian vector is examined. All configurations of the conditional independences within four jointly regular Gaussian variables are found and completely characterized in terms of implications involving conditional independence statements. The statements induced by the separation in any simple graph are shown to correspond to such a configuration within a regular Gaussian vector.

Keywords: multivariate Gaussian distribution, positive definite matrices, determinants, principal minors, conditional independence, probabilistic representability, semigraphoids, separation graphoids, gaussoids, covariance selection models, Markov perfectness

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1. INTRODUCTION

Conditional independence (*CI*) constraints on vectors of random variables naturally enter into many statistical models [9, 28]. Given some collection of *CI* constraints, a problem arises whether there exists a random vector that satisfies these constraints and violates the remaining ones [20, 24]. Such a vector of random variables, or its distribution, is considered for a *probabilistic representation* of the collection. The fine properties of *CI* and problem of representability are highly nontrivial and have been repeatedly studied, see [2, 3, 12, 22, 15, 18, 23, 24], even the case of four discrete random variables does not have a simple solution [19, 14, 16].

This paper focuses on the representability by the regular Gaussian (*rG*) vectors, studied first in [17]. This turns out to be a problem on existence of solutions of polynomial inequalities that involve determinants of submatrices of a symmetric matrix. The problem seems to be far from trivial because neither a finite axiomatic system [25] nor a finite set of forbidden minors [26] exist, similarly to its general version [22, 15]. For the algebraic relations among the principal minors of a symmetric matrix see [6]. The *CI* structures in singular vectors appeared in [26, 27].

The basic properties of *CI* constraints for *rG* vectors, listed in [17], are abstracted here to a new notion of *gaussoid*. The gaussoids belong to a special class of graphoids,

see Remark 1, and seem to provide a suitable working framework for the *rG*-representability. The main result of this paper is a description of all gaussoids that are representable by four jointly regular Gaussian variables, see Sections 4 and 5. It was previously announced in [11]. On the way, the *rG*-representability of the separability relation in simple graphs is examined, see Theorem 1 in Section 3. Among the consequences, there are a strengthening of [5, Theorem 2.3] and clarification of all *CI* constraints behind the covariance selection models [4], see Remark 3. In Section 4, all gaussoids over a four-element set are described. For each of them, either a suitable representation is found, by a covariance matrix witnessing the valid and violated conditional independences, or its nonexistence is established in Section 5. Corresponding new properties of *CI* in *rG* vectors are listed, analogously to the case of discrete variables [23]. These properties together with those defining the gaussoids characterize the *CI* structures of four regular Gaussian variables completely.

2. CONDITIONAL INDEPENDENCE IN GAUSSIAN DISTRIBUTIONS

Let $N = \{1, \dots, n\}$ be a finite set. The elements of N are not distinguished from singletons and a union of subsets of N is written as juxtaposition. Denote by $\mathcal{R}(N)$ the set of all couples $(ij|K)$ such that i and j are distinct singletons of N and $K \subseteq N\backslash ij$. Subsets of $\mathcal{R}(N)$ are also referred to as relations over N; $\emptyset$ and $\mathcal{R}(N)$ are the trivial ones.

If $\xi = (\xi_i)_{i\in N}$ is a random vector, its subvector $(\xi_i)_{i\in I}$, $I \subseteq N$, is denoted by ξ_I; in particular, $\xi_\emptyset$ is considered for a constant variable. Assuming $I, J, K \subseteq N$, the symbol $\xi\colon I \perp\!\!\!\perp J|K$ abbreviates the statement 'ξ_I and ξ_J are conditionally independent given ξ_K' [2, 3]. By [13, Lemma 3],

$$\xi\colon I \perp\!\!\!\perp J|K \iff \big[\,(\forall i \in I)(\forall j \in J)(\forall L \supseteq K)\; L \subseteq IJK\backslash ij \;\Rightarrow\; \xi\colon i \perp\!\!\!\perp j|L\,\big]. \qquad (1)$$

As a consequence, attention can be restricted to the conditional statements $\xi\colon i \perp\!\!\!\perp j|K$ where $i, j \in N$ are singletons and $K \subseteq N\backslash ij$.

A relation $\mathcal{L}$ over N is *probabilistically representable* if there exists a random vector such that

$$(ij|K) \in \mathcal{L} \iff \xi\colon i \perp\!\!\!\perp j|K \qquad \text{for all } (ij|K) \in \mathcal{R}(N). \qquad (2)$$

Here, i and j are distinct, which need not be assumed in general [19, 14, 16].

The random vector is regular Gaussian (rG) if its distribution has the density, with respect to the Lebesgue measure on $\mathbb{R}^N$, of the form

$$x \mapsto (2\pi)^{-\frac{n}{2}} |A|^{-\frac{1}{2}} \exp\big[-\tfrac{1}{2}\,(x-\mu)^T A^{-1}(x-\mu)\big], \quad x \in \mathbb{R}^N.$$

Here, μ is a (column) vector from $\mathbb{R}^N$ and $A = (a_{i,j})_{i,j\in N}$ a real positive definite matrix; they equal the mean and the covariance matrix of the vector, respectively. Further, $|A|$ denotes the determinant of A and ^{T} the transposition. A relation $\mathcal{L}$ over N is *rG-representable* if there exists an rG vector ξ such that (2) holds.

For a real matrix $A = (a_{i,j})_{i,j\in N}$ and nonempty subsets I, J of N denote by $A_{I,J}$ the submatrix $(a_{i,j})_{i\in I, j\in J}$. Let A_I abbreviate $A_{I,I}$. By Sylvester's criterion, cf. [21,

p. 96, Th. 19.2.1] or [8, p. 175], a real symmetric matrix A is positive definite if and only if the determinants $|A_{\{1,\dots,s\}}|$, $s = 1, \dots, n$, are positive. This implies $|A_I| > 0$ for all $I \subseteq N$ nonempty, the invertibility of A, and the positive definiteness of A^{-1}. Let $A^* = (a^*_{i,j})_{i,j\in N}$ denote the *classical adjoint* of A, with the element $a^*_{i,j}$ equal to $(-1)^{i+j}|A_{N\setminus\{j\},N\setminus\{i\}}|$. If A is invertible then

$$A^{-1} \cdot |A| = A^* \,, \tag{3}$$

see e.g. [21, p. 11] or [8, p. 95].

In an rG vector ξ with the covariance matrix A, for $i, j \in N$ distinct the statement $\xi\colon i \perp\!\!\!\perp j | N\setminus ij$ is equivalent to $(A^{-1})_{i,j} = 0$ [9, Proposition 5.2]. When $(ij|K) \in \mathcal{R}(N)$ this equivalence applies to the subvector indexed by ijK, which is rG and has the covariance matrix A_{ijK}, and takes the form

$$\xi\colon i \perp\!\!\!\perp j | K \iff [(A_{ijK})^{-1}]_{i,j} = 0 \,. \tag{4}$$

It follows from (3) that $[(A_{ijK})^{-1}]_{i,j}$ is equal to $|A_{iK,jK}|/|A_{ijK}|$ up to a sign, and therefore $\xi\colon i \perp\!\!\!\perp j | K$ is equivalent to $|A_{iK,jK}| = 0$. Hence, a relation $\mathcal{L} \subseteq \mathcal{R}(N)$ is rG-representable if and only if there exists a real positive definite matrix $A = (a_{i,j})_{i,j\in N}$ such that $\mathcal{L} = \langle\!\langle A \rangle\!\rangle$ where

$$\langle\!\langle A \rangle\!\rangle = \{(ij|K) \in \mathcal{R}(N)\colon\ |A_{iK,jK}| = 0\} \,, \tag{5}$$

following the notation of [17]. Such a matrix A is called an *rG-representation* of $\mathcal{L}$. The construction in (5) is of interest for general matrices, too.

The *dual couple* of $(ij|K) \in \mathcal{R}(N)$ is $(ij|N\setminus ijK)$ and the *dual* of a relation $\mathcal{L}$ over N is the relation $\mathcal{L}^\urcorner$ consisting of the dual couples of the elements of $\mathcal{L}$, thus

$$\mathcal{L}^\urcorner = \{(ij|N \setminus ijK)\colon (ij|K) \in \mathcal{L}\} \,.$$

Obviously, $(ij|K)$ is the dual couple of $(ij|N \setminus ijK)$ and $(\mathcal{L}^\urcorner)^\urcorner = \mathcal{L}$. Note that the definitions of both the dual couple and the dual of a relation, going back to [12], depend on the underlying set N, which is sometimes stressed by speaking about the duality with respect to N.

Lemma 1 *If $A = (a_{i,j})_{i,j\in N}$ is an invertible matrix then $\langle\!\langle A \rangle\!\rangle^\urcorner = \langle\!\langle A^{-1} \rangle\!\rangle$.*

The proof relies on the matrix identity

$$|(A^*)_{I,J}| = (-1)^c |A_{N\setminus J,N\setminus I}| |A|^{r-1} \tag{6}$$

following from [21, Theorem 2.5.2], where I, J are two subsets of N of the same cardinality r strictly between 0 and n and c is a suitable number, either 0 or 1.

Proof. By definitions, $(ij|K) \in \langle\!\langle A \rangle\!\rangle^\urcorner$ if and only if $(ij|N \setminus ijK) \in \langle\!\langle A \rangle\!\rangle$ which is equivalent to vanishing of $|A_{i(N\setminus ijK),j(N\setminus ijK)}|$, equal to $|A_{N\setminus jK,N\setminus iK}|$. Using (6) and (3), demanding the invertibility of A,

$$|A_{N\setminus jK,N\setminus iK}| = (-1)^c \frac{|(A^*)_{iK,jK}|}{|A|^{r-1}} = (-1)^c \cdot |(A^{-1})_{iK,jK}| \cdot |A|$$

where r is the cardinality of iK. Therefore, $|A_{N\setminus jK,N\setminus iK}|$ vanishes if and only if $|(A^{-1})_{iK,jK}|$ does, which is equivalent to $(ij|K) \in \langle\!\langle A^{-1} \rangle\!\rangle$. □

Corollary 1 *A relation is rG-representable by a matrix A if and only if its dual is rG-representable by A^{-1}.*

Thus, the duality provides a neat symmetry of rG-representability, not available for the probabilistic representability by discrete random variables [14, p. 416].

Definition 1 *A relation $\mathcal{L} \subseteq \mathcal{R}(N)$ is called a* gaussoid *over N if it satisfies*

$$\{(ij|L),(ik|jL)\} \subseteq \mathcal{L} \;\Rightarrow\; \{(ik|L),(ij|kL)\} \subseteq \mathcal{L}, \tag{7}$$

$$\{(ij|kL),(ik|jL)\} \subseteq \mathcal{L} \;\Rightarrow\; \{(ij|L),(ik|L)\} \subseteq \mathcal{L}, \tag{8}$$

$$\{(ij|L),(ik|L)\} \subseteq \mathcal{L} \;\Rightarrow\; \{(ij|kL),(ik|jL)\} \subseteq \mathcal{L}, \tag{9}$$

$$\{(ij|L),(ij|kL)\} \subseteq \mathcal{L} \;\Rightarrow\; (ik|L) \in \mathcal{L} \vee (jk|L) \in \mathcal{L} \tag{10}$$

whenever $i,j,k \in N$ are distinct and $L \subseteq N \setminus ijk$.

Remark 1 By [17, Corollary 1], the relation $\langle\!\langle A \rangle\!\rangle$ satisfies (7)–(10) whenever A is a positive definite matrix, and thus every rG-representable relation is a gaussoid. A relation over N is called a *semigraphoid* if it satisfies (7) and a *pseudographoid* [20, p. 84, (3.6e) and (3.6b)] if it satisfies (8). It is *weakly transitive* [20, p. 128, (3.34f)] if it satisfies (10). A semigraphoid which is a pseudographoid is called a *graphoid*. Thus, the gaussoids are the weakly transitive graphoids obeying (9).

Lemma 2 *The duals of gaussoids are gaussoids.*

Proof. Let $\mathcal{L} \subseteq \mathcal{R}(N)$ be a gaussoid. If $\{(ij|L),(ik|jL)\} \subseteq \mathcal{L}^\rceil$ then $\mathcal{L}$ contains $\{(ij|N\setminus ijL),(ik|N\setminus ijkL)\}$. Since $N\setminus ijL$ equals $k(N\setminus ijkL)$, an application of (7) implies $\{(ij|N\setminus ijkL),(ik|N\setminus ikL)\} \subseteq \mathcal{L}$, thus $\{(ij|kL),(ik|L)\} \subseteq \mathcal{L}^\rceil$. Hence, $\mathcal{L}^\rceil$ is a semigraphoid. Analogously, $\mathcal{L}^\rceil$ is a pseudographoid using (9), and $\mathcal{L}^\rceil$ obeys (9) using that $\mathcal{L}$ is a pseudographoid. For the weak transitivity, if $\{(ij|L),\ (ij|kL)\} \subseteq \mathcal{L}^\rceil$, an argument of the same kind applies once again and provides $(ik|jL) \in \mathcal{L}^\rceil$ or $(jk|iL) \in \mathcal{L}^\rceil$. Therefore, the weak transitivity of $\mathcal{L}^\rceil$ follows from the fact that $\mathcal{L}^\rceil$ is a semigraphoid, proved above. □

For a simple undirected graph G with the vertex set N, let

$$\langle G\rangle = \{(ij|K) \in \mathcal{R}(N)\colon\ \text{`}K \text{ separates } i \text{ and } j \text{ in } G\text{'}\}\,.$$

The separation above is defined even for $I,J,K \subseteq N$ disjoint: K separates I and J in G if every path in G connecting I and J intersects K. This means that every sequence $i_1,\ldots,i_t$, $t \geq 1$, of distinct vertices with $i_1 \in I$, $i_t \in J$, and i_r, i_{r+1} adjacent for $r = 1,\ldots,t-1$ contains a vertex from K.

Remark 2 The relations $\langle G\rangle$ are called *separation graphoids*. In [15, p. 108] it is shown that these graphoids are actually the pseudographoids that are ascending [12] and transitive, thus satisfy

$$(ij|L) \in \mathcal{L} \quad \Rightarrow \quad (ij|kL) \in \mathcal{L}, \tag{11}$$
$$(ij|L) \in \mathcal{L} \quad \Rightarrow \quad (ik|L) \in \mathcal{L} \vee (jk|L) \in \mathcal{L}, \tag{12}$$

respectively, for any disjoint singletons i, j, k of N and $L \subseteq N \setminus ijk$. For the transitivity axiom (12) see also [20, p. 94, (3.10e)]. It is easy to check that (8),(11) and (12) together imply (7)–(10) whence the separation graphoids are gaussoids.

Let $\mathcal{R}_*(N)$ denote the set of couples $(ij|K)$ from $\mathcal{R}(N)$ with $K = N \setminus ij$.

Lemma 3 *If $\mathcal{M} \subseteq \mathcal{R}_*(N)$ and G is a graph with the vertex set N having i and j adjacent if and only if $(ij|N \setminus ij) \notin \mathcal{M}$ then every pseudographoid $\mathcal{L}$ containing $\mathcal{M}$ contains $\langle G\rangle$.*

This assertion is simple and known, though not in this form. Since we have no satisfactory reference a short proof is included. For related results, see e.g. [9, p. 34, Theorem 3.7] and [20, p. 97, Theorem 3].

Proof. The descending induction on the cardinality of K is applied to prove that $(ij|K) \in \langle G\rangle$ implies $(ij|K) \in \mathcal{L}$. For the couples with $K = N \setminus ij$ this implication follows from the construction of G and $\mathcal{M} \subseteq \mathcal{L}$. Suppose that $(ij|K) \in \langle G\rangle$ with $K \neq N \setminus ij$ and validity of the implication for any other couple $(i'j'|K')$ with K' of greater cardinality than that of K. Since there exists an element k in $N \setminus ijK$ the transitivity (12) of $\langle G\rangle$ implies that either $(ik|K) \in \langle G\rangle$ or $(jk|K) \in \langle G\rangle$. Changing the roles of i and j, if necessary, let us assume that the former is true. Then, as $\langle G\rangle$ is ascending (11), $(ij|kK)$ and $(ik|jK)$ belong to $\langle G\rangle$. By the induction hypothesis they belong to $\mathcal{L}$. Since $\mathcal{L}$ is a pseudographoid it contains $(ij|K)$. □

Two relations $\mathcal{L}$ and $\mathcal{M}$ over N are *isomorphic* if there exists a permutation π on N such that $\mathcal{M} = \pi(\mathcal{L})$ where $\pi(\mathcal{L}) = \{(\pi(ij)|\pi(K))\colon (ij|K) \in \mathcal{L}\}$. Let A^π denote the matrix $(a_{\pi(i),\pi(j)})_{i,j\in N}$, obtained from $A = (a_{i,j})_{i,j\in N}$ by permuting the rows and columns simultaneously. It is not difficult to see that $\pi^{-1}\langle\!\langle A\rangle\!\rangle = \langle\!\langle A^\pi\rangle\!\rangle$, and thus an isomorphic relation to an rG-representable relation is rG-representable.

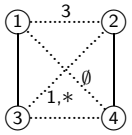

Figure 1

Following [15], a relation $\mathcal{L}$ is visualized by a diagram, as in Fig. 1. Every element i of N is associated with the circle $\textcircled{i}$. If $(ij|K)$ is in $\mathcal{L}$ for all $K \subseteq N\setminus ij$, the circles $\textcircled{i}$ and $\textcircled{j}$ are connected by a dashed line, like $\textcircled{3}$ and $\textcircled{4}$ above. If $(ij|K) \notin \mathcal{L}$

for any such K they are connected by a full line, like ① and ③. Otherwise, the dashed line joins the circles $\textcircled{i}$ and $\textcircled{j}$ and is labelled by the list of sets K such that $(ij|K) \in \mathcal{L}$. The symbol $*$ is used to abbreviate the set $N\backslash ij$ in a list. Thus, the diagram in Fig. 1 visualizes the relation

$$\{(12|3), (14|\emptyset), (23|1), (23|14), (34|\emptyset), (34|1), (34|2), (34|12)\}$$

over $N = \{1, 2, 3, 4\}$.

3. REPRESENTATIONS OF THE SEPARATION GRAPHOIDS

For a simple undirected graph G with the vertex set N and $\varepsilon \in \mathbb{C}$ let $A^{G,\varepsilon}$ be the symmetric matrix with the elements

$$a_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ \varepsilon & \text{if } i \neq j \text{ and } i, j \text{ are adjacent in } G, \\ 0 & \text{otherwise,} \end{cases}$$

where $i, j \in N$.

Theorem 1 *If G is a simple graph with the vertex set N then $\langle G\rangle^{\urcorner} = \langle\!\langle A^{G,\varepsilon} \rangle\!\rangle$ for all but finitely many $\varepsilon \in \mathbb{C}$.*

The proof of Theorem 1 is based on the following lemma, over the ring $\mathbb{C}[x]$ of polynomials in x with complex coefficients.

Lemma 4 *Let a matrix $B = (b_{i,j})_{i,j\in M}$ with $M = \{1, \ldots, m\}$ have its elements in $\{0, 1, x\} \subseteq \mathbb{C}[x]$ and $b_{i,j} = 1$ be equivalent to $i = j \neq 1$. Then the determinant $|B|$ equals the zero polynomial if and only if*

$$(b_{i_0,i_1} b_{i_1,i_2} \ldots b_{i_{t-1},i_t}) \cdot b_{i_t,i_0} = 0 \tag{13}$$

for any sequence $i_0 = 1, i_1, \ldots, i_t$, $0 \leqslant t < m$, of distinct elements of M.

Proof. For a permutation π on M let C_π be the cycle of π containing 1, thus the inclusion-smallest $C \subseteq M$ such that $\pi(C) = C$ and $1 \in C$. Let t_π denote the cardinality of $C_\pi \setminus \{1\}$. The elements of C_π can be ordered to $i_0 = 1, i_1, \ldots, i_{t_\pi}$ such that $\pi(i_r) = i_{r+1}$ for $0 \leqslant r < t_\pi$ and $\pi(i_{t_\pi}) = i_0$, provided $t_\pi > 0$. The product b_π of $b_{i,\pi(i)}$ over $i \in M$ factorizes by

$$b'_\pi = \textstyle\prod_{i\in C_\pi} b_{i,\pi(i)} = (b_{i_0,i_1} b_{i_1,i_2} \ldots b_{i_{t_\pi-1},i_{t_\pi}}) \cdot b_{i_{t_\pi},i_0}\,.$$

Hence, (13) holds with the above quantification if and only if $b'_\pi = 0$ for every permutation π on M. The latter obviously implies $|B| = 0$, using that $|B|$ equals $\sum_\pi \operatorname{sgn}(\pi)\, b_\pi$.

It suffices to prove that if $|B|$ vanishes then $b'_\pi = 0$ for all permutations π on M. The induction on $0 \leqslant t_\pi < m$ is employed, proving the case $t_\pi = 0$ and the induction step simultaneously. Suppose $0 \leqslant t < m$ and $b'_\pi = 0$ for π with $t_\pi < t$. Then $|B|$

equals the sum of $\mathrm{sgn}(\pi)\, b_\pi$ over the permutations with $t_\pi \geqslant t$. Those of them satisfying $t_\pi = t$ and $\pi(j) = j$, $j \in M \setminus C_\pi$, have the same sign and $b_\pi = b'_\pi$ equal to 0 or x^{t+1}. Each of the remaining permutations has either $t_\pi \geqslant t+1$ or a cycle C with r elements contained in $M \setminus C_\pi$. Hence, if nonzero, b_π has the factors $b'_\pi = x^{t_\pi+1}$ or $b'_\pi \prod_{i\in C} b_{i,\pi(i)} = x^{t_\pi+1+r}$, correspondingly. In both cases, each nonzero b_π has the degree at least $t+2$. It follows that if $|B| = 0$ then $b'_\pi = 0$ for π satisfying $t_\pi = t$ and $\pi(j) = j$, $j \in M \setminus C_\pi$. Then $b'_\pi = 0$ even for any π with $t_\pi = t$ because $b'_\pi = b'_\rho$ for the permutation ρ equal to π on C_π and to the identity on $M \setminus C_\pi$. □

Proof of Theorem 1. Let $A^{G,x} = (a_{i,j})_{i,j\in N}$ be the matrix with elements in $\mathbb{C}[x]$ obtained by replacing ε in $A^{G,\varepsilon}$ by the indeterminate x. Then $\langle G\rangle^\urcorner$ equals

$$\langle\!\langle A^{G,x}\rangle\!\rangle_{\mathbb{C}[x]} = \{(ij|K) \in \mathcal{R}(N)\colon\ |A^{G,x}_{iK,jK}| = 0 \text{ in } \mathbb{C}[x]\}\,.$$

In fact, transpose in $A^{G,x}_{iK,jK}$ the first and i-th rows and the first and j-th columns, if necessary. Lemma 4 implies that the determinant of this matrix is the zero polynomial if and only if

$$a_{i,k_1}a_{k_1,k_2}\dots a_{k_{t-1},k_t}a_{k_t,j} = 0 \quad \text{in } \mathbb{C}[x]$$

for any sequence $i = k_0, k_1, \dots, k_t, k_{t+1} = j$ of distinct elements of ijK. This product is zero if and only if $a_{k_r,k_{r+1}} = 0$ for some $0 \leqslant r \leqslant t$, thus k_r and k_{r+1} are not adjacent in G. This means that the sequence $i, k_1, \dots, k_t, j$ is not a path in G. Hence, the determinant of $A^{G,x}_{iK,jK}$ vanishes if and only if every path connecting i and j in G intersects $N \setminus ijK$, which is the separation of i and j by $N \setminus ijK$, or in symbols $(ij|K) \in \langle G\rangle^\urcorner$.

Since $A^{G,\varepsilon}$ obtains from $A^{G,x}$ by substitution, $\langle\!\langle A^{G,x}\rangle\!\rangle_{\mathbb{C}[x]}$ is contained in $\langle\!\langle A^{G,\varepsilon}\rangle\!\rangle$. The inclusion is strict if and only if there exists a couple $(ij|K)$ such that the determinant of $A^{G,x}_{iK,jK}$ is a nonzero polynomial with the root ε. Since the set of all roots of any such polynomial in $\mathbb{C}[x]$ is finite the assertion follows. □

Corollary 2 *The separation graphoids and their duals are rG-representable.*

Proof. The principal minors of $A^{G,x}$ are polynomials whose constant terms are equal to 1. Hence, $A^{G,\varepsilon}$ is real positive definite as soon as a real ε is sufficiently close to zero. By Theorem 1, there exists such an ε with $\langle G\rangle^\urcorner = \langle\!\langle A^{G,\varepsilon}\rangle\!\rangle$. Then, Corollary 1 implies the rG-representability of the separation graphoids. □

Corollary 3 *For any graph G there exists a regular Gaussian vector $\xi = (\xi_i)_{i\in N}$ such that*

$$(\forall I, J, K \subseteq N \text{ disjoint}) \quad \text{'}K \text{ separates } I \text{ and } J \text{ in } G\text{'} \iff \xi\colon I \perp\!\!\!\perp J|K. \tag{14}$$

Proof. It is not difficult to see that (1) remains true if the statements $\xi\colon I \perp\!\!\!\perp J|K$ and $\xi\colon i \perp\!\!\!\perp j|L$ are replaced by 'K separates I and J in G' and 'L separates i and j in G', respectively. This version of (1) and (1) itself imply that (14) is equivalent to

$$(\forall (ij|K) \in \mathcal{R}(N)) \quad \text{'}K \text{ separates } i \text{ and } j \text{ in } G\text{'} \iff \xi\colon i \perp\!\!\!\perp j|K, \tag{15}$$

which is the rG-representability of $\langle G\rangle$ by ξ, shown in Corollary 2. □

Remark 3 A random vector $\xi = (\xi_i)_{i\in N}$, or its distribution, is called *Markov w.r.t. G* if $\xi\colon I \perp\!\!\!\perp J|K$ whenever K separates I and J in G. The class $\mathcal{M}_G$ of the rG distributions that are Markov w.r.t. G is equivalently defined by requiring the (i,j)-element in the inverse of the covariance matrix of an rG distribution to vanish for i,j not adjacent in G, see the covariance selection models in [4, 9]. By Corollary 3, for disjoint $I,J,K \subseteq N$ such that K does not separate I and J in G there exists a vector ξ with its distribution in $\mathcal{M}_G$ that violates $\xi\colon I \perp\!\!\!\perp J|K$. This property of $\mathcal{M}_G$, called *Markov perfectness* of the class of distributions, was proved in [5, Theorem 2.3]. In contrary to analogous situations with discrete variables, it does not imply directly Corollary 3. M. Studený kindly drew our attention to [10, Theorem 6.1], describing a related result on AMP Markov perfectness; the methods of that work are completely different from ours. The matrix $A^{G,\varepsilon}$ with a chain G and $\varepsilon = -\frac{1}{2}$ occurred previously in [5, p. 794] and [7, p. 109].

4. THE GAUSSOIDS OVER A FOUR–ELEMENT SET

This section is devoted to classification of the isomorphism classes of the gaussoids, mostly when N is a four-element set. The main idea is to partition the gaussoids $\mathcal{L}$ according to their intersections with $\mathcal{R}_\emptyset(N) = \{(ij|\emptyset)\colon i,j \in N \text{ distinct}\}$ and $\mathcal{R}_*(N) = \mathcal{R}_\emptyset(N)^\urcorner$. By Remark 2, the separation graphoids are gaussoids, and then Lemma 3 implies that for any $\mathcal{M} \subseteq \mathcal{R}_*(N)$ the graph G having i and j not adjacent if and only if $(ij|N\setminus ij) \in \mathcal{M}$ induces the inclusion-smallest gaussoid $\langle G\rangle$ over N that intersects $\mathcal{R}_*(N)$ in $\mathcal{M}$. This is combined with the duality of Lemma 2 allowing for the restriction to the case when the cardinality of $\mathcal{L}\cap\mathcal{R}_*(N)$ is not smaller than that of $\mathcal{L}\cap\mathcal{R}_\emptyset(N)$. Thus, the separation graphoids and their duals are taken for a starting point.

Lemma 5 *If a separation graphoid is isomorphic to the dual of a separation graphoid then the underlying graphs are isomorphic to a graph whose connected components are cliques.*

Proof. Suppose $\langle G\rangle = \langle H\rangle^\urcorner$ for graphs G,H with the same vertex set N. Then i and j are not adjacent in G if and only if $(ij|N\setminus ij) \in \langle G\rangle$. This means $(ij|\emptyset) \in \langle H\rangle$ which is equivalent to nonexistence of a path between i and j in H. By analogous argumentation with $\langle H\rangle = \langle G\rangle^\urcorner$, it follows that for i and j not adjacent in G there is no path between i and j in G. The same holds in H. Hence, $G = H$, this graph has cliques for its connected components, and the assertion follows. □

From now on all gaussoids are over the four-element set $N = \{1,2,3,4\}$. There are obviously 11 isomorphism classes of the separation graphoids, see Figure 2 for the 9 nontrivial ones. By duality, there are also 11 isomorphism classes of the duals of separation graphoids. Lemma 5 implies that the overlap is in the isomorphism classes of the following five graphoids

The gaussoid can be uniquely given by its intersection with $\mathcal{R}_*(N)$ if the cardinality of this intersection is sufficiently high.

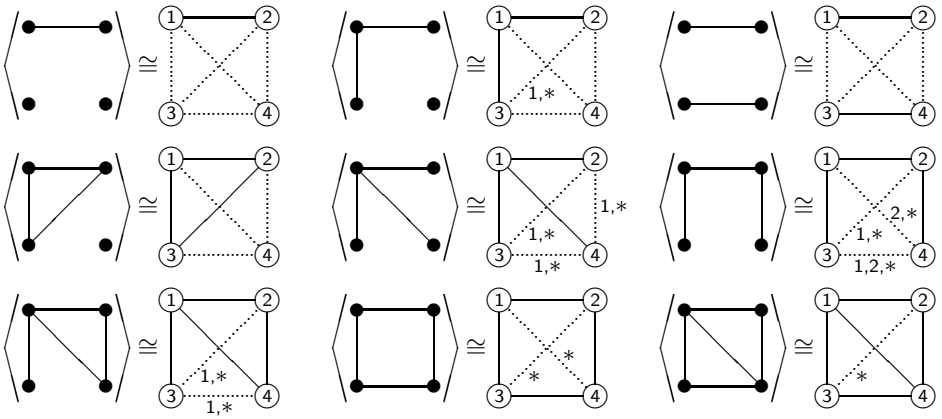

Figure 2

Lemma 6 *If a gaussoid intersects $\mathcal{R}_*(N)$ in a set of cardinality at least three, then it is isomorphic to one of the separation graphoids*

$$\langle \; \rangle,\ \langle \; \rangle,\ \langle \; \rangle,\ \langle \; \rangle,\ \langle \; \rangle^{\urcorner},\ \langle \; \rangle,\ \langle \; \rangle \ \textit{or}\ \ \langle \; \rangle. \tag{16}$$

Proof. The assertion is trivial if the intersection has the cardinality at least four and in the cases $\langle \; \rangle^{\urcorner} \cap \mathcal{R}_*(N)$ and $\langle \; \rangle \cap \mathcal{R}_*(N)$. The remaining two cases are also straightforward, looking at the diagrams of gaussoids in Figure 2. □

An element $(ij|K)$ of $\mathcal{R}(N)$ is called a *t-couple* if the cardinality of K is t.

Lemma 7 *A gaussoid $\mathcal{L}$ with two 2-couples and at most two 0-couples is isomorphic to $\langle \; \rangle$, $\langle \; \rangle$ or to $\mathcal{L}_1$–$\mathcal{L}_7$, see Figure 3.*

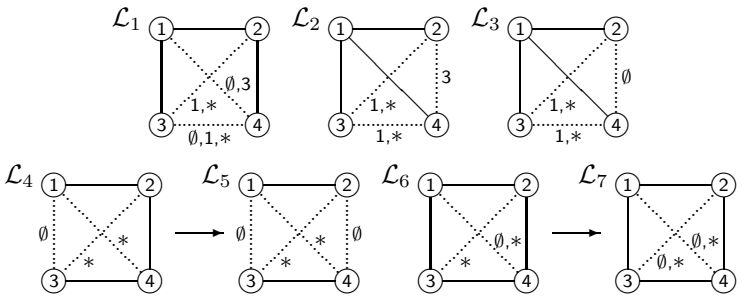

Figure 3

Proof. By Lemma 3, it suffices to assume that the gaussoid $\mathcal{L}$ contains $\langle \; \rangle$ or $\langle \; \rangle$ strictly. In the first case, if $\mathcal{L}$ contains a 1-couple not in $\langle \; \rangle$, then it is isomorphic to $\mathcal{L}_1$ or $\mathcal{L}_2$, otherwise to $\mathcal{L}_3$. In the second case, $\mathcal{L}$ cannot contain 1-couples and must be isomorphic to $\mathcal{L}_4$–$\mathcal{L}_7$. □

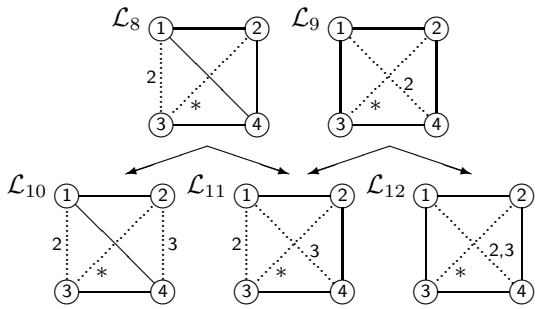

Figure 4

Lemma 8 *A gaussoid $\mathcal{L}$ with one* 2*-couple and no* 0*-couple is isomorphic to* $\langle \boxtimes \rangle$ *or $\mathcal{L}_8$–$\mathcal{L}_{12}$, see Figure 4. If $\mathcal{L}$ contains one* 0*-couple instead, then it is isomorphic to $\mathcal{L}_{13}$–$\mathcal{L}_{22}$, or $\mathcal{L}_{21}{}^{\urcorner}$, see Figure 5 and Figure 6.*

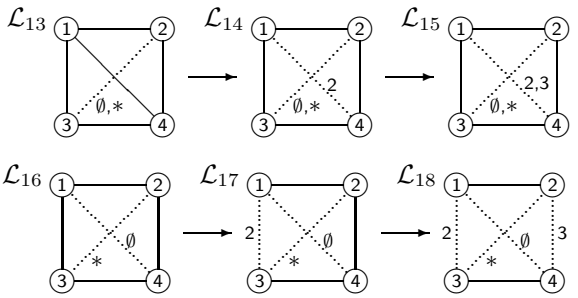

Figure 5

Proof. A gaussoid $\mathcal{L}$ containing only one 2-couple (23|14) is isomorphic to $\langle \boxtimes \rangle$ or, without 0-couples, contains $\mathcal{L}_8$ or $\mathcal{L}_9$. In the latter case, $\mathcal{L}$ is isomorphic to $\mathcal{L}_8$–$\mathcal{L}_{12}$. With a single 0-couple, $\mathcal{L}$ contains $\mathcal{L}_{13}$, $\mathcal{L}_{16}$ or $\mathcal{L}_{19}$. If the inclusion is strict the argumentation divides to three parts represented by Figure 5 and Figure 6. □

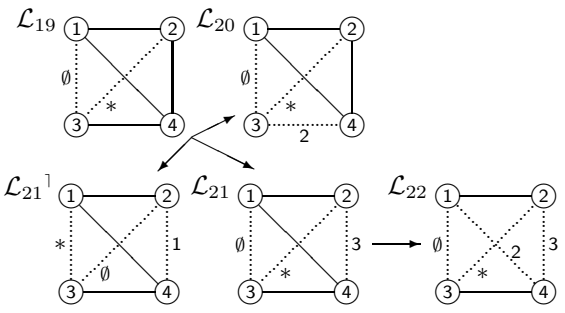

Figure 6

Lemma 9 *If a relation $\mathcal{L} \neq \emptyset$ consists of* 1*-couples only and satisfies (8) and (9) then it is isomorphic to one of the nine gaussoids of Figure 7.*

Proof. With one 1-couple, $\mathcal{L}$ is isomorphic to $\mathcal{L}_{23}$. With two 1-couples, $(ij|k)$ and $(i'j'|k')$, $\mathcal{L}$ is isomorphic to $\mathcal{L}_{24}$ when $ij = i'j'$, to $\mathcal{L}_{25}$ when ij and $i'j'$ are disjoint, or to $\mathcal{L}_{26}$ when ij and $i'j'$ intersect but differ, using (8) and (9). This reasoning is depicted in the first two lines of Figure 7. With three 1-couples one can assume that $\mathcal{L}$ contains $\mathcal{L}_{24}$, $\mathcal{L}_{25}$, or $\mathcal{L}_{26}$. In the first case, $\mathcal{L}$ is isomorphic to $\mathcal{L}_{27}$ and in the second one to $\mathcal{L}_{27}$ or $\mathcal{L}_{28}$ using (8) and (9). In the third case, $\mathcal{L}$ is isomorphic to $\mathcal{L}_{28}$ or $\mathcal{L}_{29}$. These assertions correspond to the arrows between the second and third line of Figure 7. The analogous argumentation shows that $\mathcal{L}$ can have at most four 1-couples, in which case it is isomorphic to $\mathcal{L}_{30}$ or $\mathcal{L}_{31}$. □

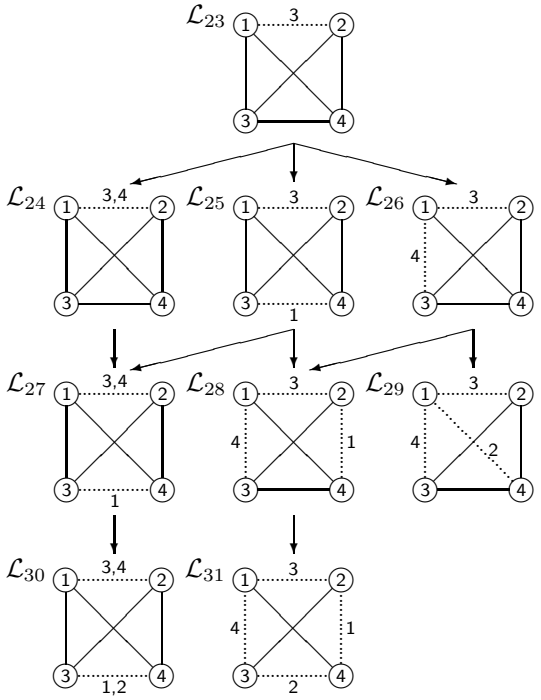

Figure 7

Theorem 2 *Over a set of cardinality four, the gaussoids partition into* 58 *isomorphism classes:* 17 *classes of the separation graphoids or their duals and the classes generated by* $\mathcal{L}_1$–$\mathcal{L}_{31}$, $\mathcal{L}_2{}^\rceil$–$\mathcal{L}_4{}^\rceil$, $\mathcal{L}_6{}^\rceil$, $\mathcal{L}_8{}^\rceil$–$\mathcal{L}_{12}{}^\rceil$ *and* $\mathcal{L}_{21}{}^\rceil$.

By duality and the above four lemmas a gaussoid or its dual is either a separation graphoid or an isomorphic copy of one of the relations $\mathcal{L}_t$ with $1 \leqslant t \leqslant 31$. After a verification that $\mathcal{L}_t$ is isomorphic to $\mathcal{L}_t{}^\rceil$ if and only if t is not within 2–4, 6, 8–12, 21, which can be done visually in the above diagrams, it follows that there are at most 58 isomorphism classes.

Proof. For a gaussoid $\mathcal{L}$ let r be the maximum of the cardinalities of $\mathcal{L} \cap \mathcal{R}_\emptyset(N)$ and $\mathcal{L} \cap \mathcal{R}_*(N)$. Partition the gaussoids into four classes according to whether $r \geq 3$

or r equals 2, 1 or 0. By Lemma 6, the gaussoids in the first class, are isomorphic to those listed in (16) or to their duals, and therefore, by Lemma 5, they give rise to 10 isomorphism classes. The gaussoids from the second class, see Lemma 7, are isomorphic to $\langle$ $\rangle$, $\langle$ $\rangle$, $\mathcal{L}_1$–$\mathcal{L}_7$ or their duals and partition into 15 isomorphism classes. In the third class, the gaussoids are isomorphic to $\langle$ $\rangle$ or $\mathcal{L}_8$–$\mathcal{L}_{22}$ or their duals, by Lemma 8, and therefore partition into 23 isomorphism classes. The gaussoids from the fourth class, see Lemma 9, are isomorphic to $\langle$ $\rangle$ or $\mathcal{L}_{23}$–$\mathcal{L}_{31}$ or their duals and partition into 10 isomorphism classes. □

5. REPRESENTABILITY OVER A FOUR–ELEMENT SET

In this section the rG-representable relations over a set of cardinality four are characterized by showing that five of the gaussoids of Theorem 2 are not rG-representable and by discussing rG-representations of the others.

Lemma 10 *Any rG-representable gaussoid $\mathcal{L}$ over N satisfies*

$$\{(ij|L),(kl|L),(ik|jlL),(jl|ikL)\}\subseteq\mathcal{L} \;\Rightarrow\; (ik|L)\in\mathcal{L}, \tag{17}$$
$$\{(ij|L),(kl|iL),(kl|jL),(ij|klL)\}\subseteq\mathcal{L} \;\Rightarrow\; (kl|L)\in\mathcal{L}, \tag{18}$$
$$\{(ij|L),(jl|kL),(kl|iL),(ik|jlL)\}\subseteq\mathcal{L} \;\Rightarrow\; (ik|L)\in\mathcal{L}, \tag{19}$$
$$\{(ij|kL),(ik|lL),(il|jL)\}\subseteq\mathcal{L} \;\Rightarrow\; (ij|L)\in\mathcal{L}, \tag{20}$$
$$\{(ij|kL),(ik|lL),(jl|iL),(kl|jL)\}\subseteq\mathcal{L} \;\Rightarrow\; (ij|L)\in\mathcal{L} \tag{21}$$

for all distinct singletons i,j,k,l of N and $L\subseteq N\setminus ijkl$.

Proof. It suffices to assume that $\mathcal{L}$ is over $ijklL\subseteq N$. The substitution of a couple by its dual with respect to $ijklL$ followed by the analogous substitution with respect to $ijkl$ remove L from the involved couples. For example, $(ij|L)\to(ij|kl)\to(ij|\emptyset)$. Therefore, Corollary 1 implies that the implications (17)–(21) hold with general L if and only if they hold with $L=\emptyset$. Thus, it suffices to consider only $N=\{1,2,3,4\}$.

Let $A=(a_{i,j})_{i,j\in N}$ be a real positive definite matrix. It can be supposed that A has the diagonal elements equal to 1,

$$A=\begin{pmatrix}1&a&b&c\\a&1&d&e\\b&d&1&f\\c&e&f&1\end{pmatrix},$$

otherwise the multiplications of each i-th row and each i-th column of A by the square root of $a_{i,i}^{-1}$ result in a positive definite matrix A' with $\langle\!\langle A'\rangle\!\rangle=\langle\!\langle A\rangle\!\rangle$ and the unit diagonal.

Suppose first $\mathcal{L}_5=\{(13|\emptyset),(24|\emptyset),(14|23),(23|14)\}\subseteq\langle\!\langle A\rangle\!\rangle$. This implies $b=0$, $e=0$, $adf-cd^2+c=0$ and $acf-c^2d+d=0$. Adding and subtracting the last two equations provide $(c+d)(af-cd+1)=0$ and $(c-d)(cd-af+1)=0$, respectively. Since $|A|$, equal to $(af-cd+1)^2-(c-d)^2-(a+f)^2$ and $(cd-af+1)^2-(c+d)^2-(a-f)^2$, is positive, both $c+d$ and $c-d$ are zero. Then, $c=0$, thus $(14|\emptyset)\in\langle\!\langle A\rangle\!\rangle$; proving (17).

If $\mathcal{L}_{15} = \{(23|\emptyset), (14|2), (14|3), (23|14)\}$ is a subset of $\langle\!\langle A \rangle\!\rangle$ then $d = 0$, $c = ae$, $c = bf$ and $acf + bce - ab - ef = 0$. Substituting ae and bf for the first and the second occurrence of c in the last equality, respectively, and multiplying the resulting equation by ef yield $c^2 = -e^2f^2\left(1 - a^2 - b^2\right)$ using $abef = c^2$. Since the principal minor $|A_{123}| = 1 - a^2 - b^2$ is positive, $c = 0$, thus $(14|\emptyset) \in \langle\!\langle A \rangle\!\rangle$; proving (18).

The assumption $\mathcal{L}_{22} = \{(13|\emptyset), (14|2), (24|3), (23|14)\} \subseteq \langle\!\langle A \rangle\!\rangle$ implies $b = 0$, $c = ae$, $e = df$ and $acf - c^2d - ef + d = 0$. Substituting for c and e in the last equation provides $d[a^2f^2(1 - d^2) + 1 - f^2] = 0$ and since the principal minors $|A_{23}| = 1 - d^2$ and $|A_{34}| = 1 - f^2$ of A are positive, $d = 0$, thus $(23|\emptyset) \in \langle\!\langle A \rangle\!\rangle$; proving (19).

Suppose that $\mathcal{L}_{29} = \{(12|3), (13|4), (14|2)\} \subseteq \langle\!\langle A \rangle\!\rangle$. This implies $a = bd$, $b = cf$ and $c = ae$. Then $a(1 - def) = 0$ by successive substitutions. Here, $1 - def > 0$ because d, e and f are in the absolute value less than one, by the positive definiteness of A. Hence, $a = 0$, thus $(12|\emptyset) \in \langle\!\langle A \rangle\!\rangle$; proving (20).

The assumption $\mathcal{L}_{31} = \{(12|3), (13|4), (24|1), (34|2)\} \subseteq \langle\!\langle A \rangle\!\rangle$ implies $a = bd$ $b = cf$, $e = ac$ and $f = de$, and consequently $a(1 - c^2d^2) = 0$. Using $1 - c^2d^2 > 0$, it follows that $a = 0$, thus $(12|\emptyset) \in \langle\!\langle A \rangle\!\rangle$; proving (21). □

Corollary 4 *The gaussoids $\mathcal{L}_5, \mathcal{L}_{15}, \mathcal{L}_{22}, \mathcal{L}_{29}$, and $\mathcal{L}_{31}$ are not rG-representable.*

Remark 4 The implications (17)–(21) can be proved also by information theoretical tools similarly to their discrete analogues, derived in [23]. For (20)–(21) see also [17, Corollary 2].

Theorem 3 *A relation over $N = \{1, 2, 3, 4\}$ is rG-representable if and only if it is a gaussoid not isomorphic to $\mathcal{L}_5$, $\mathcal{L}_{15}$, $\mathcal{L}_{22}$, $\mathcal{L}_{29}$, or $\mathcal{L}_{31}$.*

Proof. If a relation over N is rG-representable then it is a gaussoid by Remark 1, not isomorphic to any of the five gaussoids, by Corollary 4.

On the other hand, if a gaussoid over N is not isomorphic to one of the five listed gaussoids, then by Theorem 2 it is a separation graphoid or its dual, or it is isomorphic to one of the remaining gaussoids of Theorem 2. By Corollary 2, the separation graphoids and their duals are rG-representable. On account of Corollary 1, it suffices to find rG-representations for the duals of the gaussoids $\mathcal{L}_1$–$\mathcal{L}_{31}$, up to the five excluded cases. These representations are presented in Table 1 where ε is a sufficiently small positive number, $f_\varepsilon = \frac{2\varepsilon}{1+\varepsilon^2}$, $g_\varepsilon = \frac{1-\varepsilon^2}{1+\varepsilon^2}$ and δ is a real number sufficiently close to $\frac{3}{4}$. Then the matrices $A^{(i)}$ are positive definite and $\mathcal{L}_t{}^\urcorner = \langle\!\langle A^{(t)} \rangle\!\rangle$ for all appropriate indices t. □

Corollary 5 *A gaussoid over a four-element set is rG-representable if and only if it satisfies (17)–(21).*

Remark 5 There are 679 gaussoids over a four-element set, and 629 of them are rG-representable.

i	$A^{(i)}$	i	$A^{(i)}$	i	$A^{(i)}$	i	$A^{(i)}$
1	$\begin{pmatrix} 1 & \varepsilon & \varepsilon & \varepsilon^2 \\ \varepsilon & 1 & 0 & \varepsilon \\ \varepsilon & 0 & 1 & 0 \\ \varepsilon^2 & \varepsilon & 0 & 1 \end{pmatrix}$	2	$\begin{pmatrix} 1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & 0 & \varepsilon^2 \\ \varepsilon & 0 & 1 & 0 \\ \varepsilon & \varepsilon^2 & 0 & 1 \end{pmatrix}$	3	$\begin{pmatrix} 1 & \varepsilon & \varepsilon & 1\text{-}\varepsilon^2 \\ \varepsilon & 1 & 0 & \varepsilon \\ \varepsilon & 0 & 1 & 0 \\ 1\text{-}\varepsilon^2 & \varepsilon & 0 & 1 \end{pmatrix}$	4	$\begin{pmatrix} 1 & 1\text{-}\varepsilon^2 & \varepsilon^2 & 0 \\ 1\text{-}\varepsilon^2 & 1 & 0 & \varepsilon \\ \varepsilon^2 & 0 & 1 & -\varepsilon \\ 0 & \varepsilon & -\varepsilon & 1 \end{pmatrix}$
6	$\begin{pmatrix} 1 & \varepsilon^2 & \varepsilon^2 & 0 \\ \varepsilon^2 & 1 & 0 & \varepsilon \\ \varepsilon^2 & 0 & 1 & -\varepsilon \\ 0 & \varepsilon & -\varepsilon & 1 \end{pmatrix}$	7	$\begin{pmatrix} 1 & \varepsilon & \varepsilon & 0 \\ \varepsilon & 1 & 0 & \varepsilon \\ \varepsilon & 0 & 1 & -\varepsilon \\ 0 & \varepsilon & -\varepsilon & 1 \end{pmatrix}$	8	$\begin{pmatrix} 1 & \varepsilon & \varepsilon^2 & \varepsilon \\ \varepsilon & 1 & 0 & \varepsilon \\ \varepsilon^2 & 0 & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 \end{pmatrix}$	9	$\begin{pmatrix} 1 & \varepsilon & \varepsilon & \varepsilon^2 \\ \varepsilon & 1 & 0 & \varepsilon^2 \\ \varepsilon & 0 & 1 & \varepsilon \\ \varepsilon^2 & \varepsilon^2 & \varepsilon & 1 \end{pmatrix}$
10	$\begin{pmatrix} 1 & \varepsilon & \varepsilon^2 & \varepsilon \\ \varepsilon & 1 & 0 & \varepsilon^2 \\ \varepsilon^2 & 0 & 1 & \varepsilon \\ \varepsilon & \varepsilon^2 & \varepsilon & 1 \end{pmatrix}$	11	$\begin{pmatrix} 1 & \varepsilon & \varepsilon^3 & \varepsilon^2 \\ \varepsilon & 1 & 0 & \varepsilon \\ \varepsilon^3 & 0 & 1 & \varepsilon \\ \varepsilon^2 & \varepsilon & \varepsilon & 1 \end{pmatrix}$	12	$\begin{pmatrix} 1 & \varepsilon & \varepsilon & \varepsilon^2 \\ \varepsilon & 1 & 0 & \varepsilon \\ \varepsilon & 0 & 1 & \varepsilon \\ \varepsilon^2 & \varepsilon & \varepsilon & 1 \end{pmatrix}$	13	$\begin{pmatrix} 1 & -\varepsilon & \varepsilon & \varepsilon \\ -\varepsilon & 1 & 0 & \varepsilon \\ \varepsilon & 0 & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 \end{pmatrix}$
14	$\begin{pmatrix} 1 & -\varepsilon & \varepsilon & \varepsilon^2 \\ -\varepsilon & 1 & 0 & \varepsilon \\ \varepsilon & 0 & 1 & \varepsilon \\ \varepsilon^2 & \varepsilon & \varepsilon & 1 \end{pmatrix}$	16	$\begin{pmatrix} 1 & \varepsilon & \varepsilon & 2\varepsilon^2 \\ \varepsilon & 1 & 0 & \varepsilon \\ \varepsilon & 0 & 1 & \varepsilon \\ 2\varepsilon^2 & \varepsilon & \varepsilon & 1 \end{pmatrix}$	17	$\begin{pmatrix} 1 & 1\text{-}\varepsilon^2 & \varepsilon^2 & \varepsilon \\ 1\text{-}\varepsilon^2 & 1 & 0 & \varepsilon \\ \varepsilon^2 & 0 & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 \end{pmatrix}$	18	$\begin{pmatrix} 1 & g_\varepsilon & \varepsilon f_\varepsilon & \varepsilon \\ g_\varepsilon & 1 & 0 & \varepsilon g_\varepsilon \\ \varepsilon f_\varepsilon & 0 & 1 & f_\varepsilon \\ \varepsilon & \varepsilon g_\varepsilon & f_\varepsilon & 1 \end{pmatrix}$
19	$\begin{pmatrix} 1 & \varepsilon & \varepsilon^3 & \varepsilon^4 \\ \varepsilon & 1 & 0 & \varepsilon \\ \varepsilon^3 & 0 & 1 & -\varepsilon \\ \varepsilon^4 & \varepsilon & -\varepsilon & 1 \end{pmatrix}$	20	$\begin{pmatrix} 1 & 2\text{-}\delta^{-2} & \delta & \delta \\ 2\text{-}\delta^{-2} & 1 & 0 & \delta \\ \delta & 0 & 1 & \delta^2 \\ \delta & \delta & \delta^2 & 1 \end{pmatrix}$	21	$\begin{pmatrix} 1 & \varepsilon & \varepsilon f_\varepsilon & \varepsilon \\ \varepsilon & 1 & 0 & \varepsilon^2 \\ \varepsilon f_\varepsilon & 0 & 1 & 2\varepsilon \\ \varepsilon & \varepsilon^2 & 2\varepsilon & 1 \end{pmatrix}$	23	$\begin{pmatrix} 1 & \varepsilon^2 & \varepsilon & \varepsilon \\ \varepsilon^2 & 1 & \varepsilon & -\varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon \\ \varepsilon & -\varepsilon & \varepsilon & 1 \end{pmatrix}$
24	$\begin{pmatrix} 1 & \varepsilon^2 & \varepsilon & \varepsilon \\ \varepsilon^2 & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 \end{pmatrix}$	25	$\begin{pmatrix} 1 & \varepsilon^2 & \varepsilon & \varepsilon \\ \varepsilon^2 & 1 & \varepsilon & -\varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon^2 \\ \varepsilon & -\varepsilon & \varepsilon^2 & 1 \end{pmatrix}$	26	$\begin{pmatrix} 1 & \varepsilon^3 & \varepsilon^2 & \varepsilon \\ \varepsilon^3 & 1 & \varepsilon & \varepsilon \\ \varepsilon^2 & \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 \end{pmatrix}$	27	$\begin{pmatrix} 1 & \varepsilon^3 & \varepsilon & \varepsilon \\ \varepsilon^3 & 1 & \varepsilon^2 & \varepsilon^2 \\ \varepsilon & \varepsilon^2 & 1 & \varepsilon^2 \\ \varepsilon & \varepsilon^2 & \varepsilon^2 & 1 \end{pmatrix}$
28	$\begin{pmatrix} 1 & \varepsilon^3 & \varepsilon^2 & \varepsilon \\ \varepsilon^3 & 1 & \varepsilon & \varepsilon^4 \\ \varepsilon^2 & \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon^4 & \varepsilon & 1 \end{pmatrix}$	30	$\begin{pmatrix} 1 & \varepsilon^2 & \varepsilon & \varepsilon \\ \varepsilon^2 & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon^2 \\ \varepsilon & \varepsilon & \varepsilon^2 & 1 \end{pmatrix}$				

Table 1

ACKNOWLEDGEMENT

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