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Towards Classification of Semigraphoids

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Abstract. Semigraphoids are special sets of triples (I, J, K), I, J, K disjoint subsets of a finite set, that mimic conditional independences. New constructions on semigraphoids are introduced, the most crucial being factors and expansions. They are aimed at study of new classes of semigraphoids, that are constructed from semigraphoids of a trivial structure, e.g. from uniform semigraphoids, and at bringing each semigraphoid to a canonical form. Canonical semigraphoids are defined and each semigraphoid is constructed from a canonical one by means of a pure minor and an expansion.

Semigraphoid closure and generators are investigated. The case of two generators is analysed in detail. Invariants for semigraphoids based on relations among generators are introduced and the corresponding classes of semigraphoids are related to classes built from uniform semigraphoids.

Representability of semigraphoids by linear spaces and random variables is reexamined. The semigraphoids with at most two generators are proved to be linear and hence, by a simple lemma, probabilistic.

1. Introduction

A semigraphoid over a finite set N is a set of triples (I, J, K), $I, J, K \subseteq N$ disjoint, in other words, a ternary relation on the power set of N, enjoying a few simple properties. Equivalently, one can study sets of couples $(I \cup J, K)$ having both I and J singletons that obey a single axiom. Semigraphoids are discrete structures behind the conditional independence modeling in philosophy and artificial intelligence: the incidence of (I, J, K) to a semigraphoid is interpreted as 'conditional independence of I and J given K'. Semigraphoids underpin also manipulations with the conditional independence constraints of numerous models of multivariate and Bayesian statistics. Yet, semigraphoids are believed to underlie amalgamation-like constructions on algebraic and combinatorial structures. Introduced by Paz and Pearl, see [20], but implicit yet in [4, 21], the notion of semigraphoid has appeared in several dozens of papers; for references see [2, 5, 6, 7, 9, 14, 23].

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In spite of a substantial progress on many special subclasses of semigraphoids, especially those related to graphs [1, 3, 8, 24], only a few deeper results on the very notion of semigraphoid have appeared. Notably the pioneering work [23] of Studený deals with semigraphoids generated by two triples and their representations through random variables.

This paper outlines approaches to and makes first steps towards a classification of semigraphoids. Classification means in algebra a study of invariants together with 'easy-to-understand' classes of algebraic structures which are obtained from 'building stones' by means of constructions. A starting point for our invariants is that of semigraphoid closure, generators, and a mutual position among generators. As for the building stones we confine ourselves to a narrow class of uniform semigraphoids. The more work is then to be done on constructions: new techniques involve pure minors, factors and expansions. They are combined with several other natural constructions developed earlier and extended here; especially with intersections, minors [14], majors, duality [10], loops and direct sums [12], and parallel extensions. A few classes of semigraphoids with prescribed invariants are shown to be contained in classes of semigraphoids constructed from 'building stones', see Section 5.

Section 7 is devoted to the factors and expansions which seem to be crucial when trying to bring semigraphoids to canonical forms. Expansions necessitate analysis of a notion of dominance in semigraphoids that is presented in Section 8. A discussion on morphisms between semigraphoids entails. The main result is presented in Section 10 where we define the canonical semigraphoids and reduce general semigraphoids to the canonical ones, see Theorem 1. This is done by means of special factors that are identified as pure minors, i.e. 'nice' subconfigurations of canonical semigraphoids. In the reverse direction, each semigraphoid can be constructed as an expansion of a pure minor of a canonical semigraphoid.

If starting from generators the first nontrivial case is that of two generators. Brought to the canonical form, only a single semigraphoid with two generators over a set N of twelve elements is of interest. This special semigraphoid is examined in a visual, formula-free way in Sections 3, 6 and 9 and illustrates general methods under consideration.

The last section is devoted to representations of semigraphoids in linear spaces by linear subspaces. The main result, Theorem 2, claims that the semigraphoids with at most two generators are representable in this way. This theorem stands on top of quite a number of previous results, namely, on assertions describing structure of these semigraphoids and on lemmas ensuring representability of duals, direct sums, majors, parallel extensions, factors and expansions. More general representations in probability spaces through random variables are discussed in Remark 14 of Section 11.

2. Basic definitions and facts

Let N be a finite set with elements $i, j, k, l \in N$ and subsets $I, J, K, L \subseteq N$. Elements of N are not distinguished from singletons and the sign for union of subsets of N is omitted for simplicity. The set of all couples $(ij|K)$ where i, j are different, $i \notin K$ and $j \notin K$ will be denoted by $\mathcal{R}(N)$. A subset $\mathcal{L}$ of $\mathcal{R}(N)$ is termed also a relation over N. Such a relation is called *semigraphoid* over N if it satisfies

$$\{(ij|kL), (ik|L)\} \subseteq \mathcal{L} \iff \{(ik|jL), (ij|L)\} \subseteq \mathcal{L}\,.$$

Since intersections of semigraphoids are semigraphoids intersection of all semigraphoids $\mathcal{K} \subseteq \mathcal{R}(N)$ containing a given relation $\mathcal{L}$ over N is the smallest semigraphoid over N

containing $\mathcal{L}$. This semigraphoid is denoted by $\mathfrak{c}(\mathcal{L})$ and called the *semigraphoid closure* of $\mathcal{L}$. In a constructive way, let $\mathcal{L}^{(0)} = \mathcal{L}$ and for $r \geqslant 0$

$$\mathcal{L}^{(r+1)} = \mathcal{L}^{(r)} \cup \bigcup \left\{ \{(ik|jL), (ij|L)\} \, ; \; \{(ij|kL), (ik|L)\} \subseteq \mathcal{L}^{(r)} \right\} .$$

Obviously, $\mathcal{L}^{(1)} = \mathcal{L}^{(0)}$ if and only if $\mathcal{L}$ is a semigraphoid. The sequence $\mathcal{L}^{(r)}$, $r \geqslant 0$, is nondecreasing and once $\mathcal{L}^{(s+1)} = \mathcal{L}^{(s)}$ this equality holds also for all $r \geqslant s$. By finiteness of N, the smallest $s \geqslant 0$ satisfying the equality exists, and will be denoted by $s_{\mathcal{L}}$. It is straightforward that $\mathfrak{c}(\mathcal{L}) = \mathcal{L}^{(s_{\mathcal{L}})}$. The numbers $s_{\mathcal{L}_n}$ can grow exponentially in n for a sequence $\mathcal{L}_n$ over $\{1, \dots, n\}$, see [16].

Occasionally, it is possible to decompose the semigraphoid closure into union of closures.

Lemma 1. *If $l \in N$ and $\mathcal{L}_1, \mathcal{L}_2$ are relations over N such that $l \notin ijK$ for every $(ij|K) \in \mathcal{L}_1$ and $l \in K$ for every $(ij|K) \in \mathcal{L}_2$ then $\mathfrak{c}(\mathcal{L}_1 \cup \mathcal{L}_2)$ equals $\mathfrak{c}(\mathcal{L}_1) \cup \mathfrak{c}(\mathcal{L}_2)$.*

Proof. Neither $(ij|kL) \in \mathcal{L}_1$ and $(ik|L) \in \mathcal{L}_2$ nor $(ij|kL) \in \mathcal{L}_2$ and $(ik|L) \in \mathcal{L}_1$ can occur, and hence $(\mathcal{L}_1 \cup \mathcal{L}_2)^{(1)}$ equals $\mathcal{L}_1^{(1)} \cup \mathcal{L}_2^{(1)}$. In addition, $l \notin ijK$ for $(ij|K) \in \mathcal{L}_1^{(1)}$ and $l \in K$ for $(ij|K) \in \mathcal{L}_2^{(1)}$. By induction, $(\mathcal{L}_1 \cup \mathcal{L}_2)^{(r)}$ equals $\mathcal{L}_1^{(r)} \cup \mathcal{L}_2^{(r)}$, $r \geqslant 0$, and the assertion follows. $\square$

Where I, J, K are three disjoint subsets of N, let

$$(I, J|K)_\star = \left\{ (ij|L) \in \mathcal{R}(N) \, ; \; i \in I, j \in J \text{ and } K \subseteq L \subseteq IJK - ij \right\} .$$

Obviously $(I, J|K)_\star = (J, I|K)_\star$ is a semigraphoid which is nonempty if and only if both I and J are nonempty. We call these semigraphoids *elementary*. The inclusion $(I, J|K)_\star \subseteq \mathcal{L}$ is commented as the conditional independence of I and J given K in $\mathcal{L}$. By induction on the cardinality of IJ, any semigraphoid $\mathcal{L}$ satisfies

$$(I, J|KL)_\star \cup (I, K|L)_\star \subseteq \mathcal{L} \iff (I, JK|L)_\star \subseteq \mathcal{L}$$

for arbitrary disjoint $I, J, K, L \subseteq N$, cf. Lemma 3 of [11, p. 747]. We refer to this property as to the *extended semigraphoid axiom.* Let us remark that in the most common definition of semigraphoids, cf. [2, 7, 22], sets of ordered triples (I, J, K), or ternary relations, are postulated to have the above property and the symmetry between I and J.

A semigraphoid $\mathcal{L}$ over N is said to have $r \geqslant 0$ *generators* if $\mathcal{L} = \mathfrak{c}(\mathcal{K})$ where $\mathcal{K}$ is union of r elementary semigraphoids. The empty semigraphoid has zero generators and any nonempty semigraphoid has one generator if and only if it is elementary. A substantial part of this paper is devoted to the case of two generators.

Given a relation $\mathcal{L}$ over N, let $\mathcal{G}_{\mathcal{L}}$ be the graph with the vertex set N having the edge ij if and only if $(ij|K) \in \mathcal{L}$ for at least one $K \subseteq N$. Since $\mathcal{G}_{\mathcal{L}} = \mathcal{G}_{\mathcal{L}^{(1)}}$ a simple induction argument implies $\mathcal{G}_{\mathcal{L}} = \mathcal{G}_{\mathfrak{c}(\mathcal{L})}$.

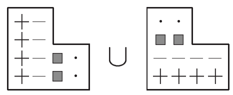
The symmetric group on N acts in a natural way on $\mathcal{R}(N)$ so that a permutation τ of N transforms a couple $(ij|K)$ to $(\tau(i)\tau(j)|\tau(K))$. If $\mathcal{L} \subseteq \mathcal{R}(N)$ is invariant to the action of a permutation τ, $\mathcal{L} = \tau(\mathcal{L})$, then also $\mathcal{L}^{(1)}$ is invariant to the action. By induction, $\mathfrak{c}(\mathcal{L})$ is also invariant to the action.

Where $\mathcal{L}$ is a relation over N and $\mathcal{K}$ is a relation over M, there is a straightforward notion of isomorphism between $\mathcal{K}$ and $\mathcal{L}$. For a bijection $f \colon N \to M$ the relations $\mathcal{L}$ and $\mathcal{K}$ are *isomorphic* once $(ij|K) \in \mathcal{L}$ is equivalent to $(f(i)f(j)|f(K)) \in \mathcal{K}$. All classes of relations, especially in Section 5, are considered to be closed to the isomorphisms. Definitions of morphisms are more subtle, see Remarks 11 and 12 in Section 8.

3. Example

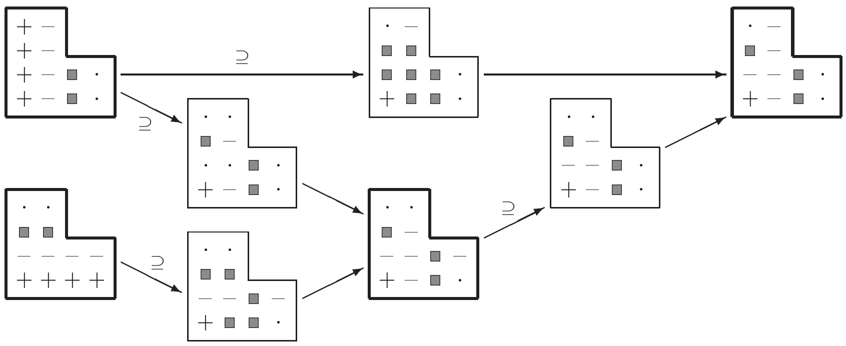
Let N be a set of cardinality 12 and the elements of N be organized as in the pictograph. This section and Sections 6 and 9 examine the relation

$$\mathcal{L}_{\square} =$$



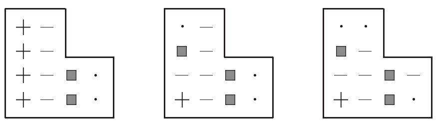
over N and its semigraphoid closure. Here, the left pictograph represents the elementary semigraphoid $(I, J|K)_\star$ with the elements of I marked by $+$, the elements of J by $-$ and the elements of K by the square ■. Analogous pictographs are used for all elementary semigraphoids over N. To take into account symmetries of $\mathcal{L}_{\square}$, let G be the group of permutations of N generated by the transposition of the left two columns, the transposition of the bottom two lines, and the reflection of $N =$ along the diagonal. It is easy to see that G is isomorphic to the group of isometries of a square and consists of eight permutations.

The following scheme illustrates applications of the extended semigraphoid axiom.



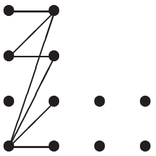
Arrows labeled by $\supseteq$ indicate the inclusion between elementary semigraphoids. Two pairs of head-by-head meeting arrows mean applications of the axiom.

Since $\mathcal{L}_{\square}$ is invariant to the action of the permutations from G also $\mathfrak{c}(\mathcal{L}_{\square})$, a semigraphoid with two generators, is invariant to the action. In particular, $\mathfrak{c}(\mathcal{L}_{\square})$ contains $(\tau(I), \tau(J)|\tau(K))_\star$ whenever it contains $(I, J|K)_\star$ and $\tau \in G$. On account of the highlighted frames of the above scheme, the semigraphoid closure of $\mathcal{L}_{\square}$ contains the following three elementary semigraphoids

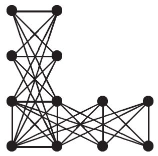


and, then also the elementary semigraphoids obtained from them by the action of G. Denoting by $\mathcal{K}_{\square}$ union of the 18 elementary semigraphoids obtained in this way, $\mathcal{K}_{\square} \subseteq \mathfrak{c}(\mathcal{L}_{\square})$. Actually, it will be later shown that $\mathcal{K}_{\square} = \mathfrak{c}(\mathcal{L}_{\square})$. For a proof of the opposite inclusion $\mathcal{K}_{\square} \supseteq \mathfrak{c}(\mathcal{L}_{\square})$ it suffices to verify that $\mathcal{K}_{\square}$ is a semigraphoid; we have, however, no simple direct argument. Instead, it will be shown that $\mathcal{K}_{\square}$ can be expressed as intersection of special semigraphoids, see the proof of Proposition 4 in Section 6.

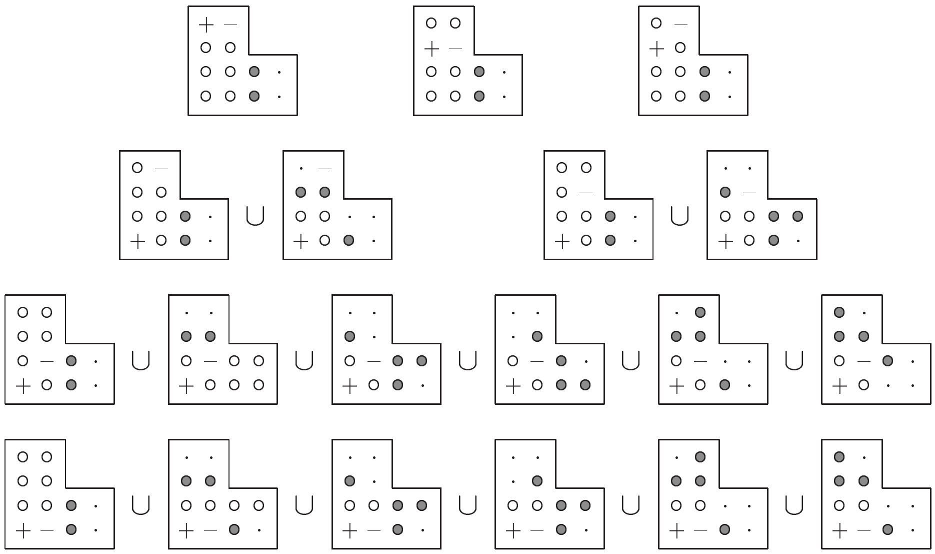
When working with relations invariant to the action of G in forthcoming sections, symmetries are heavily employed. For example, the relation $\mathcal{K}_{\square}$ is partitioned into the sets $\mathcal{K}^{ij}_{\square} = \{(ij|K)\ ;\ (ij|K) \in \mathcal{K}_{\square}\}$ where ij runs over the edges of the graph $\mathcal{G}_{\mathcal{K}_{\square}}$. This graph equals $\mathcal{G}_{\mathcal{L}_{\square}}$ because $\mathcal{L}_{\square} \subseteq \mathcal{K}_{\square} \subseteq \mathfrak{c}(\mathcal{L}_{\square})$ and $\mathcal{G}_{\mathcal{L}_{\square}} = \mathcal{G}_{\mathfrak{c}(\mathcal{L}_{\square})}$. It is easy to see that the edges of this graph partition under the action of G into seven orbits and the following seven edges



of $\quad \mathcal{G}_{\mathcal{K}_{\Box}} = \mathcal{G}_{\mathcal{L}_{\Box}} =$



belong to distinct orbits. Since $\mathcal{K}_{\Box}$ is invariant to the action of G it obtains by permutations from the sets $\mathcal{K}_{\Box}^{ij}$ where ij runs over the set of the seven edges. These seven sets are written as unions of sets $\{(ij|K)\,;\ L_1 \subseteq K \subseteq L_2\}$ and it is not difficult to encode them in the form



where $+$ and $-$ indicate an edge ij, L_1 consists of the elements marked by $\bullet$, and $L_2 - L_1$ consists of the elements marked by $\circ$.

4. Basic constructions

The *restriction* of a relation $\mathcal{L}$ over N to $I \subseteq N$ is $\mathsf{re}_I\,\mathcal{L} = \mathcal{L} \cap \mathcal{R}(I)$ and the *contraction* of $\mathcal{L}$ to I is $\mathsf{co}_I\,\mathcal{L} = \{(ij|K) \in \mathcal{R}(I)\,;\ (ij|K(N-I)) \in \mathcal{L}\}$, both being relations over I. It easy to see that the restriction to L of the elementary semigraphoid $(I,J|K)_\star$ equals $(I \cap L, J \cap L|K)_\star$ if $K \subseteq L$ and $\emptyset$ otherwise, and its contraction to L equals $(I \cap L, J \cap L|K \cap L)_\star$ if $IJK \supseteq N - L$ and $\emptyset$ otherwise.

A relation $\mathcal{K}$ over $I \subseteq N$ is called *minor* of $\mathcal{L}$, see [12, 14], if for some $I \subseteq J \subseteq N$

$$\mathcal{K} = \mathsf{co}_I\,\mathsf{re}_J\,\mathcal{L} = \{(ij|K) \in \mathcal{R}(I)\,;\ (ij|K(J-I)) \in \mathcal{L}\}\ .$$

Definition 1. A minor $\mathcal{K}$ over I of a relation $\mathcal{L}$ over N is called *pure* if $ij \subseteq I$ and $(ij|K) \in \mathcal{L}$ imply $(ij|K \cap I) \in \mathcal{K}$.

Remark 1. This implication is equivalent to the equality between $\mathcal{K}$ and the relation $\{(ij|K \cap I);\ ij \subseteq I\,, (ij|K) \in \mathcal{L}\}$ where the inclusion $\subseteq$ follows from the fact that $\mathcal{K}$ is a minor of $\mathcal{L}$. Note that in the situation $N = \{1,2,3,4\}$ and $\mathcal{L} = \{(1,2|4),(1,2|\emptyset),(1,3|\emptyset)\}$ the relation $\mathcal{K} = \{(1,2|\emptyset)\}$ over $I = \{1,2\}$ is the minor $\mathsf{co}_I\,\mathsf{re}_I\,\mathcal{L}$ of $\mathcal{L}$ which is pure. However, the same relation taken over $I = \{1,2,3\}$ is the minor $\mathsf{co}_I\,\mathsf{re}_N\,\mathcal{L}$ of $\mathcal{L}$ which is not pure. Thus, when discussing pure minors one has to keep track over the sets they live on.

Minors of semigraphoids are obviously semigraphoids. Hence, the minor $\mathsf{co}_I\,\mathsf{re}_J\,\mathfrak{c}(\mathcal{L})$ includes $\mathfrak{c}(\mathsf{co}_I\,\mathsf{re}_J\,\mathcal{L})$. A sufficient condition for equality in this inclusion is supplied by Definition 1.

Lemma 2. *If $\mathcal{K}$ equals $\mathsf{co}_I\,\mathsf{re}_J\,\mathcal{L}$ for some $I \subseteq J \subseteq N$ and $\mathcal{K}$ over I is a pure minor of $\mathcal{L}$ over N then $\mathfrak{c}(\mathcal{K}) = \mathsf{co}_I\,\mathsf{re}_J\,\mathfrak{c}(\mathcal{L})$ and $\mathfrak{c}(\mathcal{K})$ is a pure minor of $\mathfrak{c}(\mathcal{L})$.*

Proof. This implication is a consequence of the assertions that $\mathcal{K}^{(r)}$ equals $\mathsf{co}_I\,\mathsf{re}_J\,\mathcal{L}^{(r)}$ and $\mathcal{K}^{(r)}$ is a pure minor of $\mathcal{L}^{(r)}$ for $r \geqslant 0$. This follows by an induction argument on r from the assertions that $\mathcal{K}^{(1)}$ equals $\mathsf{co}_I\,\mathsf{re}_J\,\mathcal{L}^{(1)}$ and $\mathcal{K}^{(1)}$ is a pure minor of $\mathcal{L}^{(1)}$.

Since $\mathcal{K}$ is a minor of $\mathcal{L}$ the incidences $(ij|kK) \in \mathcal{K}$ and $(ik|K) \in \mathcal{K}$ hold if and only if $(ij|kK(J-I)) \in \mathcal{L}$ and $(ik|K(J-I)) \in \mathcal{L}$. Then $(ik|jK(J-I))$ and $(ij|K(J-I))$ belong to $\mathcal{L}^{(1)}$, and thus $(ik|jK)$ and $(ij|K)$ belong to $\mathsf{co}_I\,\mathsf{re}_J\,\mathcal{L}^{(1)}$. Hence, $\mathcal{K}^{(1)}$ is a subset of $\mathsf{co}_I\,\mathsf{re}_J\,\mathcal{L}^{(1)}$.

The opposite inclusion is proved as soon as $\mathsf{co}_I\,\mathsf{re}_J\,\{(ik|jL),(ij|L)\}$ is contained in $\mathcal{K}^{(1)}$ whenever $\{(ij|kL),(ik|L)\} \subseteq \mathcal{L}$. This is true trivially when $i \notin I$ or $L \nsubseteq J$. Otherwise, it remains to show that $\mathsf{co}_I\,\{(ik|jL)\}$ is contained in $\mathcal{K}^{(1)}$ if $k \in I$, $j \in J$, $J-I \subseteq jL$, and $\mathsf{co}_I\,\{(ij|L)\}$ is contained in $\mathcal{K}^{(1)}$ if $j \in I$ and $J-I \subseteq L$. In the former case, the contraction contains $(ik|L\cap I)$ when $j \notin I$ and this couple belongs to $\mathcal{K}$ because $(ik|L) \in \mathcal{L}$ and $\mathcal{K}$ is pure. In the latter case, the contraction contains $(ij|L \cap I)$ when $k \notin I$ and this couple belongs to $\mathcal{K}$ because $(ij|L) \in \mathcal{L}$ and $\mathcal{K}$ is pure. Otherwise, $jk \subseteq I$ and then $\{(ij|kL),(ik|L)\} \subseteq \mathcal{L}$ implies that $(ij|(kL)\cap I)$ and $(ik|L\cap I)$ belong to $\mathcal{K}$ since $\mathcal{K}$ is pure. Now, the former contraction contains the unique couple $(ik|(jL)\cap I)$ and the latter contraction contains $(ij|L\cap I)$, both obviously belonging to $\mathcal{K}^{(1)}$.

To verify that $\mathcal{K}^{(1)}$ is a pure minor of $\mathcal{L}^{(1)}$, one has to show that $ij \subseteq I$ and $(ij|K) \in \mathcal{L}^{(1)}$ imply $(ij|K\cap I) \in \mathcal{K}^{(1)}$. This is obvious when $(ij|K) \in \mathcal{L}$ because $\mathcal{K}$ is pure. Otherwise, $(ij|kK)$ and $(ik|K)$ belong to $\mathcal{L}$ for some $k \in N - ijK$ or $(ik|j(K-k))$ and $(ij|K-k)$ belong to $\mathcal{L}$ for some $k \in K$. In the first case, $(ij|(kK)\cap I)$ belongs to $\mathcal{K}$, and if $k \notin I$ then $(ij|K\cap I) \in \mathcal{K}$, and thus $(ij|K\cap I) \in \mathcal{K}^{(1)}$. For $k \in I$ also $(ik|K\cap I)$ belongs to $\mathcal{K}$, and then $(ij|K\cap I) \in \mathcal{K}^{(1)}$. In the second case, $(ij|(K-k)\cap I)$ belongs to $\mathcal{K}$, and if $k \in K-I$ then $(ij|K\cap I) \in \mathcal{K}$, and thus $(ij|K\cap I) \in \mathcal{K}^{(1)}$. For $k \in K\cap I$ also $(ik|(j(K-k))\cap I)$ belongs to $\mathcal{K}$, and then $(ij|K\cap I) \in \mathcal{K}^{(1)}$. ◻

The equation $\mathcal{K} = \mathsf{co}_I\,\mathsf{re}_J\,\mathcal{L}$ holds for $\mathcal{K}$ and $\mathcal{L}$ of the same cardinality if and only if $ijK \subseteq J$ and $J-I \subseteq K$ for all $(ij|K) \in \mathcal{L}$. In this situation we say that $\mathcal{L}$ is a *major* of $\mathcal{K}$. Obviously, $\mathcal{K}$ is a pure minor of its major $\mathcal{L}$ and Lemma 2 applies. Moreover, $\mathcal{L}^{(r)}$ majorizes $\mathcal{K}^{(r)} = \mathsf{co}_I\,\mathsf{re}_J\,\mathcal{L}^{(r)}$, $r \geqslant 0$, and $\mathfrak{c}(\mathcal{L})$ majorizes $\mathfrak{c}(\mathcal{K})$.

Having two relations $\mathcal{L}_1 \subseteq \mathcal{R}(N_1)$ and $\mathcal{L}_2 \subseteq \mathcal{R}(N_2)$ with N_1 and N_2 disjoint, the *direct sum* is defined as

$$\begin{aligned}\mathcal{L}_1 \oplus \mathcal{L}_2 = &\big\{(ij|K) \in \mathcal{R}(N_1 \cup N_2)\,;\; i \in N_1\,,\; j \in N_2\big\}\\ &\cup \big\{(ij|KL)\,;\; (ij|K) \in \mathcal{L}_1\,,\; L \subseteq N_2\big\}\\ &\cup \big\{(ij|KL)\,;\; (ij|K) \in \mathcal{L}_2\,,\; L \subseteq N_1\big\}\,.\end{aligned}$$

Direct sums of semigraphoids are semigraphoids which follows from this assertion.

Lemma 3. $\mathfrak{c}(\mathcal{L}_1 \oplus \mathcal{L}_2) = \mathfrak{c}(\mathcal{L}_1) \oplus \mathfrak{c}(\mathcal{L}_2)$

An element i of N is a *loop* of a relation $\mathcal{L} \subseteq \mathcal{R}(N)$ if $\mathcal{L} = \mathcal{R}(i) \oplus \mathsf{re}_{N-i}\,\mathcal{L}$. If $I \subseteq N$ is a set of loops of $\mathcal{L}$ then $\mathcal{L} = \mathcal{R}(I) \oplus \mathsf{re}_{N-I}\,\mathcal{L}$.

There is a natural notion of *duality*

$$\mathcal{L}^{\rceil} = \big\{(ij|K) \in \mathcal{R}(N)\,;\ (ij|N-ijK) \in \mathcal{L}\big\}$$

applying to any relation $\mathcal{L} \subseteq \mathcal{R}(N)$. Obviously, $\mathcal{L} = (\mathcal{L}^{\rceil})^{\rceil}$. It is easy to see that duals of semigraphoids are semigraphoids and

Lemma 4. $\mathfrak{c}(\mathcal{L}^{\rceil}) = \mathfrak{c}(\mathcal{L})^{\rceil}$

For $\mathcal{L} \subseteq \mathcal{R}(N)$, $k \in N$ and $l \notin N$, the *parallel extension* $\mathcal{L}_{k\|l} \subseteq \mathcal{R}(lN)$ *of* $\mathcal{L}$ *at* k *by* l is defined as follows:

1. $\mathcal{L}_{k\|l}$ is an extension of $\mathcal{L}$, i.e. $\mathcal{L}_{k\|l} \cap \mathcal{R}(N) = \mathcal{L}$,
2. $\mathcal{L}_{k\|l}$ is invariant to the permutation τ on lN that transposes k and l,
3. $\mathcal{L}_{k\|l}$ involves the set $\mathcal{J}^{lN}_{k,l}$ of couples $(ij|K) \in \mathcal{R}(lN)$ such that ij contains k and K contains l, or ij contains l and K contains k,
4. $\mathcal{L}_{k\|l}$ contains no couple $(kl|K)$,
5. $(ij|klK) \in \mathcal{L}_{k\|l}$ if and only if $(ij|kK) \in \mathcal{L}$, $K \subseteq N - ij$.

Lemma 5. $\mathfrak{c}(\mathcal{L}_{k\|l}) = \mathfrak{c}(\mathcal{L})_{k\|l}$

Proof. By the definition,

$$\mathcal{L}_{k\|l} = \mathcal{L} \cup \tau(\mathcal{L}) \cup \mathcal{J}^{lN}_{k,l} \cup \mathit{co}^{-1}_{+kl}\,(\mathit{co}_{N-k}\,\mathcal{L})$$

where co^{-1}_{+kl} is the mapping that maps $(ij|K)$ from $\mathit{co}_{N-k}\,\mathcal{L}$ to $(ij|klK)$. Obviously, the semigraphoid closure $\mathfrak{c}(\mathcal{L}_{k\|l})$ contains $\mathfrak{c}(\mathcal{L})$, $\tau(\mathfrak{c}(\mathcal{L})) = \mathfrak{c}(\tau(\mathcal{L}))$ and $\mathcal{J}^{lN}_{k,l}$. If $(ij|K)$ belongs to $\mathit{co}^{-1}_{+kl}\,(\mathit{co}_{N-k}\,\mathfrak{c}(\mathcal{L}))$ then $(ij|K-l) \in \mathfrak{c}(\mathcal{L})$ where $k \in K-l$, and thus $(ij|K-l)$ belongs to $\mathfrak{c}(\mathcal{L}_{k\|l})$. Since $(il|j(K-l))$ belongs to $\mathcal{J}^{lN}_{k,l}$, contained in $\mathfrak{c}(\mathcal{L}_{k\|l})$, the couple $(ij|K)$ is in $\mathfrak{c}(\mathcal{L}_{k\|l})$. Therefore

$$\mathfrak{c}(\mathcal{L}_{k\|l}) \supseteq \mathfrak{c}(\mathcal{L}) \cup \tau(\mathfrak{c}(\mathcal{L})) \cup \mathcal{J}^{lN}_{k,l} \cup \mathit{co}^{-1}_{+kl}\,(\mathit{co}_{N-k}\,\mathfrak{c}(\mathcal{L}))\,.$$

Due to the above identity, $\mathfrak{c}(\mathcal{L}_{k\|l}) \supseteq \mathfrak{c}(\mathcal{L})_{k\|l}$. The opposite inclusion follows from $\mathcal{L}_{k\|l} \subseteq \mathfrak{c}(\mathcal{L})_{k\|l}$ if any parallel extension of a semigraphoid is a semigraphoid. This fact is established in two steps.

Let $\mathcal{L}$ be a semigraphoid for the rest of this proof. First, the aim is to show that $\mathcal{K} = \mathcal{L} \cup \tau(\mathcal{L})$ is a semigraphoid. Let $\{(ij|mL), (im|L)\}$ be contained in $\mathcal{K}$. Since $kl \nsubseteq ijmL$ and $\mathcal{K}$ is τ-invariant one can restrict to the case $l \notin ijmL$. Then the two couples belong to $\mathcal{L}$ and thus $(i, jm|L)_\star \subseteq \mathcal{L} \subseteq \mathcal{K}$ because $\mathcal{L}$ is a semigraphoid. Second, $\mathcal{M} = \mathit{co}^{-1}_{+kl}\,(\mathit{co}_{N-k}\,\mathcal{L})$ is a semigraphoid since it is a major of a contraction of the semigraphoid $\mathcal{L}$. The unions $\mathcal{L} \cup \mathcal{M}$ and $\tau(\mathcal{L}) \cup \mathcal{M}$ are semigraphoids by Lemma 1. Therefore, $\mathcal{K} \cup \mathcal{M}$ is a semigraphoid, too.

Starting to prove that $\mathcal{L}_{k\|l}$ is a semigraphoid, let $\mathcal{L}_{k\|l}$ contain $(ij|mL)$ and $(im|L)$. Since $\mathcal{J}^{lN}_{k,l}$ is obviously a semigraphoid one has $(i, jm|L)_\star \subseteq \mathcal{L}_{k\|l}$ whenever the two couples belong to $\mathcal{J}^{lN}_{k,l}$. The same happens when they belong to the semigraphoid $\mathcal{K} \cup \mathcal{M}$. Otherwise, using the τ-invariance of $\mathcal{L}_{k\|l}$, one of the couples is $(\bar{\imath}k|l\bar{L})$ for $\bar{\imath} \in N-k$ and $\bar{L} \subseteq N - \bar{\imath}k$, and the second couple is 1. $(\bar{\imath}l|\bar{L})$, or 2. $(\bar{\imath}\bar{\jmath}|l\bar{L}-\bar{\jmath})$ for $\bar{\jmath} \in \bar{L}$, or 3. $(\bar{\imath}\bar{\jmath}|kl\bar{L})$ for $\bar{\jmath} \in N - \bar{\imath}k\bar{L}$, or 4. $(k\bar{\jmath}|l\bar{L}-\bar{\jmath})$ for $\bar{\jmath} \in \bar{L}$, or 5. $(k\bar{\jmath}|\bar{\imath}l\bar{L})$ for $\bar{\jmath} \in N - \bar{\imath}k\bar{L}$. It is straightforward from the definition of $\mathcal{L}_{k\|l}$ that, correspondingly, 1. $(\bar{\imath}, kl|\bar{L})_\star$ 2. $(\bar{\imath}, \bar{\jmath}k|l\bar{L}-\bar{\jmath})_\star$ 3. $(\bar{\imath}, \bar{\jmath}k|l\bar{L})_\star$ 4. $(k, \bar{\imath}\bar{\jmath}|l\bar{L}-\bar{\jmath})_\star$ or 5. $(k, \bar{\imath}\bar{\jmath}|l\bar{L})_\star$ is contained in $\mathcal{L}_{k\|l}$. Thus $\mathcal{L}_{k\|l}$ is a semigraphoid. □

It is easy to verify that parallel extensions can be taken in any order, that is, $(\mathcal{L}_{k_1\|l_1})_{k_2\|l_2}$ equals $(\mathcal{L}_{k_2\|l_2})_{k_1\|l_1}$ for $k_1, k_2 \in N$ and $l_1 \neq l_2$ out of N.

5. Semigraphoids via invariants and constructions

One way to define a class of semigraphoids is to prescribe invariants, that is, constraints concerning the whole isomorphism classes. Examples of invariants are the minimal number of generators and the minimal cardinality of a dominant family, see Section 8. Two other invariants, based on the way how generators live together, are introduced below.

Definition 2. A semigraphoid $\mathcal{L}$ has *solitary* generators if $\mathcal{L} = \mathfrak{c}(\mathcal{K})$ for union $\mathcal{K}$ of $r \geqslant 0$ elementary semigraphoids $(I_t, J_t|K_t)_\star$, $1 \leqslant t \leqslant r$, which satisfy $K_s \nsubseteq I_tJ_tK_t$ and $K_t \nsubseteq I_sJ_sK_s$ for all $1 \leqslant s < t \leqslant r$. The generators of $\mathcal{L} = \mathfrak{c}(\mathcal{K})$ are *inferenceless* if $K_s \nsubseteq I_tJ_tK_t$ or $K_t \nsubseteq I_sJ_sK_s$ for all $s \neq t$ between 1 and r.

Remark 2. Obviously, an elementary semigraphoid has zero or one solitary generator according to it is empty or not, and if a semigraphoid has solitary generators then the generators are inferenceless. If $\mathcal{L} = \mathfrak{c}(\mathcal{K})$ has inferenceless generators as above then

$$\mathcal{K}^{(1)} \subseteq \bigcup_{1 \leqslant s,t \leqslant r} \mathfrak{c}((I_s, J_s|K_s)_\star \cup (I_t, J_t|K_t)_\star)$$

where the closures can be omitted by Lemma 1. Therefore, $\mathcal{K}^{(1)} \subseteq \mathcal{K}$ implies $\mathcal{L} = \mathcal{K}$, and thus the closures in Definition 2 are superfluous.

Another way to specify a class of semigraphoids is to prescribe a starting class of 'easy-to-understand' semigraphoids, building blocks, and to close it to some constructions. Natural building blocks seem to be among the *uniform* semigraphoids, cf. [14],

$$\mathcal{U}_{r,n} = \{(ij|K) \in \mathcal{R}(N)\ ;\ |K| \neq r-1\}$$

where n is the cardinality of N and $0 < r \leqslant n$ is called the rank of $\mathcal{U}_{r,n}$; for technical reasons, $\mathcal{U}_{1,0} = \emptyset$ and $\mathcal{U}_{1,1} = \emptyset$ are also admitted. Constructions will be in this paper exclusively the duality ($\rceil$), intersection ($\cap$), direct sum ($\oplus$), parallel extension ($\|$) and, later, factors and expansions, see Section 7. We adopt the notation $\langle\, \boldsymbol{C} \mid \mathit{list}\, \rangle$ for the smallest class of semigraphoids containing a class of semigraphoids $\boldsymbol{C}$ and closed to (isomorphisms and) each construction contained in a *list* of constructions. The class $\langle\, \boldsymbol{C} \mid \mathit{list}\, \rangle$ can be also defined recurrently by applying the constructions from the *list* to the semigraphoids from $\boldsymbol{C}$ and to the semigraphoids obtained in previous steps.

To outline the approach, a result of Matúš [10] is evoked to describe the class built from the uniform semigraphoids of rank 1 by means of intersections and direct sums.

Proposition 1. *The class $\langle\, \mathcal{U}_{1,n} \text{ for } n \geqslant 0 \mid \cap, \oplus\, \rangle$ coincides with the class of all semigraphoids which are ascending in the sense*

$$(ij|K) \in \mathcal{L}\,,\ \ L \subseteq N - ijK \implies (ij|KL) \in \mathcal{L}$$

and satisfy

$$\{(ij|kL), (ij|lL), (kl|L)\} \subseteq \mathcal{L} \implies (ij|L) \in \mathcal{L}\,.$$

Proof. It is not difficult to see that the class $\boldsymbol{A}$ of ascending semigraphoids satisfying the latter implication is closed to intersections and direct sums. Since $\boldsymbol{A}$ contains obviously the uniform semigraphoids $\mathcal{U}_{1,n}$ it follows by minimality that the class $\langle\, \mathcal{U}_{1,n} \text{ for } n \geqslant 0 \mid \cap, \oplus\, \rangle$ is contained in $\boldsymbol{A}$. In [10], cf. the proof of Theorem 3, each semigraphoid $\mathcal{L}$ over N belonging to $\boldsymbol{A}$ was expressed as intersection of the semigraphoids $I_N^* = \{(ij|K) \in \mathcal{R}(N)\ ;\ ijK \cap I \neq ij\}$ over I in a family $\mathcal{V} \subseteq 2^N$. Now, I_N^* is the direct sum of $\mathit{re}_I\, I_N^*$, isomorphic to $U_{1,|I|}$, and $\mathcal{R}(N-I)$ where $\mathcal{R}(N-I)$ is the direct sum of semigraphoids isomorphic to $U_{1,1}$. This implies

$$\begin{aligned} \boldsymbol{A} &\subseteq \langle\, U_{1,n} \oplus \underbrace{U_{1,1} \oplus \ldots \oplus U_{1,1}}_{t} \text{ for } n \geqslant 0 \text{ and } t \geqslant 0 \mid \cap \,\rangle \\ &\subseteq \langle\, U_{1,n} \text{ for } n \geqslant 0 \mid \cap, \oplus \,\rangle\,. \end{aligned}$$

and the assertion follows. □

Let us remark that the class $\boldsymbol{A} = \langle\, U_{1,n}$ for $n \geqslant 0 \mid \cap, \oplus \,\rangle$ defined in the previous proof contains the semigraphoids underlying the global, local and pairwise Markovness over undirected graphs, cf. [11, 9].

If the operation of duality is added to the list $\{\cap, \oplus\}$ of Proposition 1 then the semigraphoids with solitary generators can be captured.

Proposition 2. *Every semigraphoid with solitary generators belongs to the class*

$$\langle\, U_{1,n} \text{ for } n \geqslant 0 \parallel \rceil, \cap, \oplus \,\rangle\,.$$

Proof. The inclusion

$$\langle\, U_{1,n} \text{ for } n \geqslant 0 \mid \rceil, \cap, \oplus \,\rangle \supseteq \langle\, I_N^*, (I_N^*)^{\rceil} \text{ for } I \subseteq N \text{ finite } \mid \cap \,\rangle$$

is a consequence of the simple fact that the class on the left is closed to duality and direct sums. Hence, it suffices to show that each semigraphoid $\mathcal{L}$ equal to union of its solitary generators $(I_t, J_t|K_t)_\star$, see Remark 2, coincides with the intersection of all semigraphoids I_N^* and their duals that contain $\mathcal{L}$.

Let $\mathcal{V} \subseteq 2^N$ contain $I \subseteq N$ if and only if

$$I \cap I_t = \emptyset \quad\text{or}\quad I \cap J_t = \emptyset \quad\text{or}\quad I \cap K_t \neq \emptyset$$

for all $1 \leqslant t \leqslant r$. When $I \in \mathcal{V}$ one has $I_N^* \supseteq (I_t, J_t|L)_\star$ once $K_t \subseteq L \subseteq N - I_tJ_t$. Therefore, the intersection $\bigcap_{I\in\mathcal{V}} I_N^*$, denoted by $\mathfrak{a}(\mathcal{L})$, contains

$$\bigcup_{t=1}^{r} \bigcup\nolimits_{K_t \subseteq L_t \subseteq N - I_tJ_t} (I_t, J_t|L_t)_\star\,.$$

Actually, $\mathfrak{a}(\mathcal{L})$ equals this double union. In fact, the graph $\mathcal{G}_{\mathcal{L}} = \mathcal{G}_{\mathfrak{a}(\mathcal{L})} = (N, E)$ is union of r bipartite graphs, $E = \bigcup_{t=1}^r E_t$ where $E_t = \{ij;\ i \in I_t,\ j \in J_t\}$. Let ij be a two-element subset of N. One can suppose, up to a renumbering of generators, that ij belongs to $E_1, \ldots, E_t$ and does not belong to $E_{t+1}, \ldots, E_r$ for some $0 \leqslant t \leqslant r$. If $(ij|K)$ is not in the double union then $K \not\supseteq K_1, \ldots, K \not\supseteq K_t$, and then for $l_1 \in K - K_1, \ldots, l_t \in K - K_t$ the set $ijl_1 \ldots l_t$ belongs to $\mathcal{V}$. Hence $(ij|K)$ is not in $(ijl_1 \ldots l_t)_N^* \supseteq \mathfrak{a}(\mathcal{L})$.

The dual semigraphoid $\mathcal{L}^{\rceil}$ is the union of $(I_t, J_t|N - I_tJ_tK_t)_\star$ over $1 \leqslant t \leqslant r$. Since $\mathcal{L} \subseteq \mathfrak{a}(\mathcal{L})$ and

$$\mathcal{L}^{\rceil} \subseteq \mathfrak{a}(\mathcal{L}^{\rceil}) = \bigcup_{t=1}^{r} \bigcup\nolimits_{N - I_tJ_tK_t \subseteq L_t \subseteq N - I_tJ_t} (I_t, J_t|L_t)_\star$$

it follows

$$\mathcal{L} \subseteq \mathfrak{a}(\mathcal{L}^{\rceil})^{\rceil} = \bigcup_{s=1}^{r} \bigcup\nolimits_{L_s \subseteq K_s} (I_s, J_s|L_s)_\star\,.$$

Then $\mathcal{L}$ is contained in $\mathfrak{a}(\mathcal{L}) \cap \mathfrak{a}(\mathcal{L}^{\rceil})^{\rceil}$ which is equal to the union of

$$\Big(\bigcup\nolimits_{K_t \subseteq L_t \subseteq N - I_t J_t} (I_t, J_t | L_t)_\star\Big) \cap \Big(\bigcup\nolimits_{L_s \subseteq K_s} (I_s, J_s | L_s)_\star\Big)$$

over t, s between 1 and r. Since the generators are solitary these intersections

$$\bigcup\nolimits_{ij \in E_t \cap E_s} \big\{(ij|L)\ ;\ K_t \subseteq L \subseteq I_s J_s K_s - ij\big\}$$

are empty when $s \neq t$, and are equal to $(I_t, J_t | K_t)_\star$ otherwise. Finally, $\mathcal{L}$ is equal to $\mathfrak{a}(\mathcal{L}) \cap \mathfrak{a}(\mathcal{L}^{\rceil})^{\rceil}$, and thus belongs to the class $\langle\!\langle\, \mathsf{U}_{1,n}$ for $n \geqslant 0 \,|\, \rceil, \cap, \oplus \,\rangle\!\rangle$ by construction. $\square$

Remark 3. The proof of Proposition 2 gives a construction of any semigraphoid with zero or one generators by means of uniform semigraphoids. For semigraphoids with two inferenceless generators the construction does not work. For example, if $\mathcal{L} = (\mathit{14}, \mathit{2}|\mathit{3})_\star \cup (\mathit{1}, \mathit{2}|\emptyset)_\star$ over $N = \{\mathit{1}, \mathit{2}, \mathit{3}, \mathit{4}\}$ then $\mathfrak{a}(\mathcal{L}) \cap \mathfrak{a}(\mathcal{L}^{\rceil})^{\rceil}$ equals $\mathcal{L} \cup (\mathit{1}, \mathit{2}|\mathit{4})_\star$, and thus $\mathcal{L}$ is a semigraphoid with two inferenceless generators out of the class $\langle\!\langle I_N^*, (I_N^*)^{\rceil}$ for $I \subseteq N$ finite $\,|\, \cap \rangle\!\rangle$.

We have failed to include the semigraphoids with inferenceless generators to a class constructed from uniform semigraphoids in general. Nevertheless, the case of two such generators is covered by the following assertion. One new building stone is added, namely a factor of $\mathsf{U}_{2,5}$ to a four-element set, see Definition 3 in Section 7. These factors are all isomorphic and denoted by $(\mathsf{U}_{2,5})_4$.

Proposition 3. *Every semigraphoid $\mathcal{L}$ with two inferenceless generators belongs to the class $\langle\!\langle\, (\mathsf{U}_{2,5})_4,\ \mathsf{U}_{1,n}$ for $n \geqslant 0 \,|\, \rceil, \cap, \oplus \,\rangle\!\rangle$.*

Proof. By Remark 2, $\mathcal{L} = (I_1, J_1 | K_1)_\star \cup (I_2, J_2 | K_2)_\star$ and if the two generators are even solitary Proposition 2 applies. Otherwise, $K_1 \not\subseteq I_2 J_2 K_2$ and $K_2 \subseteq I_1 J_1 K_1$ up to a renumbering. By the previous proof, $\mathfrak{a}(\mathcal{L}) \cap \mathfrak{a}(\mathcal{L}^{\rceil})^{\rceil}$ equals the union of $\mathcal{L}$ and

$$\bigcup_{ij \in E_1 \cap E_2} \big\{(ij|L);\ K_2 \subseteq L \subseteq I_1 J_1 K_1 - ij \text{ and } K_1 \not\subseteq L \not\subseteq I_2 J_2 K_2 - ij\big\}\,.$$

Where $i, j, k, l \in N$ are different, let $\mathcal{M}_{ijk,l} \subseteq \mathcal{R}(N)$ have $N - ijkl$ as its set of loops and let $\mathsf{re}_{ijkl}\, \mathcal{M}_{ijk,l}$ be the factor $(\mathsf{U}_{2,5})_4$ with l corresponding to a block of size two. The dual of $\mathcal{M}_{ijk,l}$ has the same set of loops and $\mathsf{re}_{ijkl}\, \mathcal{M}_{ijk,l}^{\rceil}$ is the factor of $\mathsf{U}_{3,5}$ with l corresponding to a block of size two as well.

Let $(ij|L)$ belong to the above union over $E_1 \cap E_2$ and $l \in L - I_2 J_2 K_2$. If $l \in I_1 J_1$ then one can take $k \in K_1 - I_2 J_2 K_2$ and observe that $(ij|L) \notin \mathcal{M}_{ijl,k}$ whereas $\mathcal{L}$ is contained in $\mathcal{M}_{ijl,k}$. If $l \in K_1$ then one can take $k \in K_1 - L$ and observe that $(ij|L) \notin \mathcal{M}_{ijk,l}^{\rceil}$ whereas $\mathcal{L}$ is contained in $\mathcal{M}_{ijk,l}^{\rceil}$.

Hence, $\mathcal{L}$ is intersection of $\mathfrak{a}(\mathcal{L}) \cap \mathfrak{a}(\mathcal{L}^{\rceil})^{\rceil}$ with a family of semigraphoids each one being a copy of $\mathcal{M}_{ijk,l}$ or $\mathcal{M}_{ijk,l}^{\rceil}$. $\square$

6. Example: constructions on uniform semigraphoids

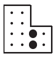
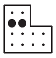
This section is devoted to a proof of the following proposition which relates the relations $\mathcal{L}$ and $\mathcal{K}$ over $N =$ from Section 3. The proof applies basic constructions from Section 4 and methods of the proof of Proposition 2.

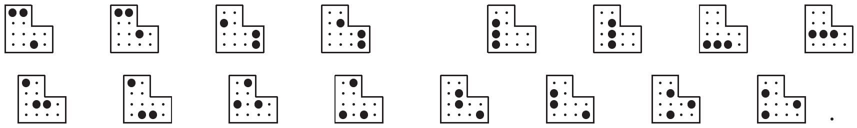
Proposition 4. *The semigraphoid closure of $\mathcal{L}$ equals $\mathcal{K}$ and belongs to the class*

$$\langle\!\langle\, \mathsf{U}_{2,3}, \mathsf{U}_{1,n} \text{ for } n \geqslant 0 \,|\, \rceil, \cap, \oplus, \| \,\rangle\!\rangle\,.$$

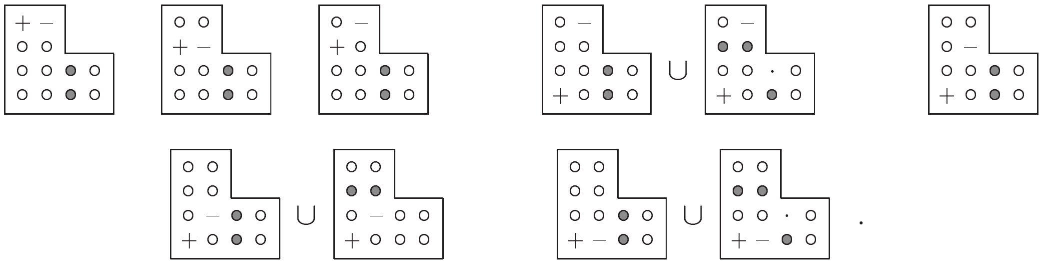
Proof. The idea is to express the relation $\mathcal{K}_{\square}$ as intersection of semigraphoids from this class. Then $\mathcal{K}_{\square}$ is a semigraphoid and $\mathcal{K}_{\square} \supseteq \mathfrak{c}(\mathcal{L}_{\square})$ follows from $\mathcal{K}_{\square} \supseteq \mathcal{L}_{\square}$. The inclusion $\mathcal{K}_{\square} \subseteq \mathfrak{c}(\mathcal{L}_{\square})$ was obtained by applying the extended semigraphoid axiom in Section 3.

Let $\mathcal{V} \subseteq 2^N$ be the family of subsets of N containing

1. every set $I \subseteq N$ which intersects  and 
2. the two-element sets ij which are not edges of $\mathcal{G}_{\mathcal{L}_{\square}}$
3. the following 16 tree-element sets



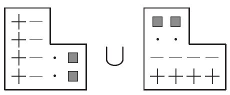
The family $\mathcal{V}$ is obviously invariant under the action of G on 2^N; note that the 16 sets exhaust three orbits. Using symmetries, it is easy to see that $\mathcal{L}_{\square} \subseteq I^*_{\square}$ for $I \in \mathcal{V}$, and hence $\mathcal{L}_{\square}$ is contained in the intersection $\mathcal{M}_{\square}$ of these semigraphoids. This semigraphoid over N contains $\mathfrak{c}(\mathcal{L}_{\square})$, and thus also $\mathcal{K}_{\square}$. Further, it is invariant under the action of G because $\mathcal{V}$ is invariant. Since $\mathcal{G}_{\mathcal{L}_{\square}} = \mathcal{G}_{\mathcal{K}_{\square}} = \mathcal{G}_{\mathcal{M}_{\square}}$ the semigraphoid $\mathcal{M}_{\square}$ can be encoded, analogously to $\mathcal{K}_{\square}$, as



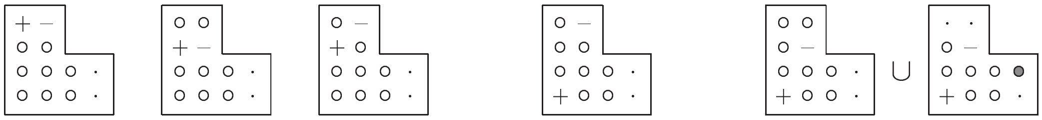
Let us remark in passing that the semigraphoid $\mathcal{M}_{\square}$ belongs to the class $\boldsymbol{A}$ and can be obtained by enlargement of the conditioning sets in $\mathcal{K}_{\square}$

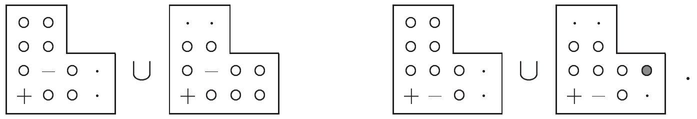
$$\mathcal{M}_{\square} = \{(ij|KL) \in \mathcal{R}(N)\,;\ (ij|K) \in \mathcal{K}_{\square} \text{ and } L \subseteq N - ijK\}\,.$$

The next step is to verify the inclusion $\mathcal{K}_{\square} \subseteq \varrho(\mathcal{M}^{\rceil}_{\square})$ where ϱ is the permutation that switches the top two lines and the right two columns of N simultaneously. To this end, observe that

$$\mathcal{L}^{\rceil}_{\square} \;=$$ 

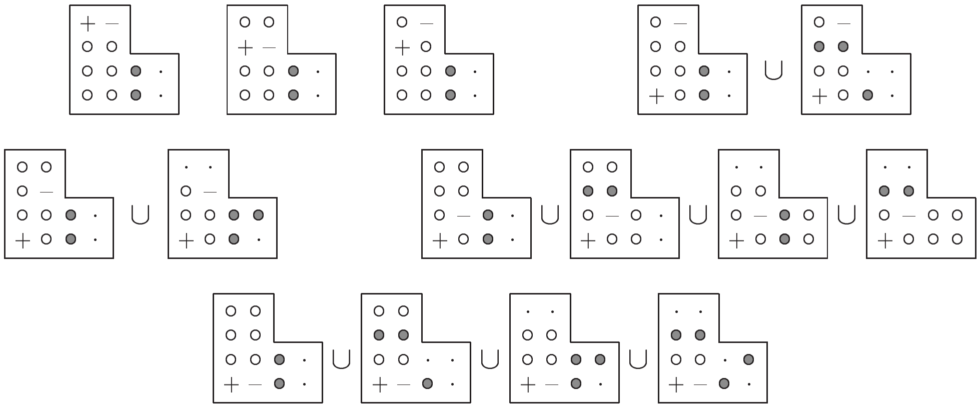
obtains by interchange of the square ■ and the dot on all respective places of these pictographs. Hence, $\mathcal{L}_{\square} = \varrho(\mathcal{L}^{\rceil}_{\square})$. It follows $\mathcal{L}_{\square} \subseteq \varrho(\mathcal{M}^{\rceil}_{\square})$ due to $\mathcal{L}_{\square} \subseteq \mathcal{M}_{\square}$. Obviously, $\varrho(\mathcal{M}^{\rceil}_{\square})$ is a semigraphoid, it contains $\mathfrak{c}(\mathcal{L}_{\square})$, and thus also $\mathcal{K}_{\square}$. Since ϱ and G commute, $\varrho\pi = \pi\varrho$ for all $\pi \in G$, the semigraphoid $\varrho(\mathcal{M}^{\rceil}_{\square})$ is invariant to the action of G. Now, the graphs of $\mathcal{L}_{\square}$ and $\varrho(\mathcal{M}^{\rceil}_{\square})$ are identical and the latter semigraphoid can be easily encoded as





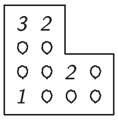
Note that the dual of a semigraphoid encoded by such a pictograph obtains by interchange of $\bullet$, and the dot.

At the moment, $\mathcal{K}_{\square}$ is contained in the intersection $\mathcal{M}^{\Diamond}_{\square} = \mathcal{M}_{\square} \cap \varrho(\mathcal{M}^{\rceil}_{\square})$ which belongs to the class $\langle\!\langle\, U_{1,n}$ for $n \geqslant 0 \parallel \rceil, \cap, \oplus \,\rangle\!\rangle$. The intersection can be encoded as

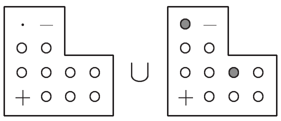
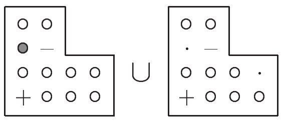


which exhibits that the inclusion $\mathcal{K}_{\square} \subset \mathcal{M}^{\Diamond}_{\square}$ is strict.

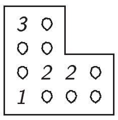
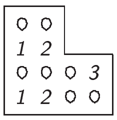
Let $\mathcal{N}_1$ be the semigraphoid over $N = \square$ described by the pictograph

$\mathcal{N}_1 =$  .

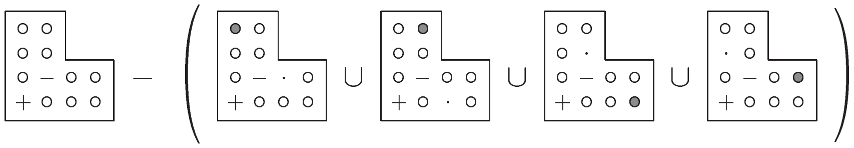
Here, each symbol ○ indicates a loop. When $\mathcal{N}_1$ is restricted to the set of non-loops then it equals the uniform semigraphoid $U_{1,3} = \{(12|\emptyset), (13|\emptyset), (23|\emptyset)\}$ with an element in parallel; the two parallel elements are indicated by 2. It is straightforward that $\mathcal{L}_{\square}$ is contained in the semigraphoid $\mathcal{N}_1$, cf. Lemma 5, and therefore also $\mathcal{K}_{\square} \subseteq \mathcal{N}_1$. In addition, $\mathcal{K}_{\square}$, being G-invariant, is contained in ${}_G\mathcal{N}_1 = \bigcap_{\pi \in G} \pi(\mathcal{N}_1)$. A look at the encoding of $\mathcal{K}_{\square}$ in Section 3 reveals that $\mathcal{K}_{\square} = \varrho(\mathcal{K}^{\rceil}_{\square})$ and hence $\mathcal{K}_{\square}$ is a subset of ${}_G\mathcal{N}_1 \cap \varrho({}_G\mathcal{N}_1^{\rceil}) = {}_G\mathcal{N}_1^{\Diamond}$. The semigraphoid ${}_G\mathcal{N}_1^{\Diamond}$ is G-invariant because both ${}_G\mathcal{N}_1$ and its dual ${}_G\mathcal{N}_1^{\rceil}$ are G-invariant and ϱ commutes with G. Especially, one has

 in $\mathcal{N}_1$ and  in $\varrho(\mathcal{N}_1^{\rceil})$.

Along the same lines, starting with

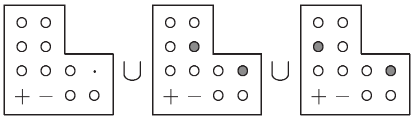
$\mathcal{N}_2 =$  and $\mathcal{N}_3 =$ 

one observes that



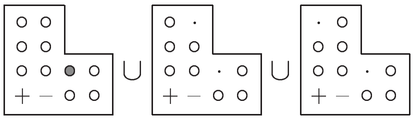
in ${}_G\mathcal{N}_2$,

and



in ${}_G\mathcal{N}_3$,

and



in $\varrho({}_G\mathcal{N}_3^{\rceil})$.

Now, it is not difficult to verify that $\mathcal{K}_{\boxplus} = \mathcal{M}_{\boxplus}^{\Diamond} \cap {}_G\mathcal{N}_1^{\Diamond} \cap {}_G\mathcal{N}_2 \cap {}_G\mathcal{N}_3^{\Diamond}$, and thus $\mathcal{K}_{\boxplus}$ equals the semigraphoid closure of $\mathcal{L}_{\boxplus}$. Further, $\mathcal{K}_{\boxplus}$ is intersection of a semigraphoid from the class $\langle\!\langle\, \mathsf{U}_{1,n} \text{ for } n \geqslant 0 \mid \rceil, \cap, \oplus \,\rangle\!\rangle$ with a semigraphoid from the class $\langle\!\langle\, \mathsf{U}_{2,3}, \mathsf{U}_{1,1} \mid \rceil, \cap, \oplus, \| \,\rangle\!\rangle$. Therefore, $\mathcal{K}_{\boxplus}$ belongs to the class $\langle\!\langle\, \mathsf{U}_{2,3}, \mathsf{U}_{1,n} \text{ for } n \geqslant 0 \mid \rceil, \cap, \oplus, \| \,\rangle\!\rangle$. $\square$

7. Factors and expansions

Let N be a factorset of M and $\rho\colon 2^N \to 2^M$ the natural mapping sending a set $K \subseteq N$ of blocks, nonempty disjoint subsets of M, to their union $K^\rho \subseteq M$.

Definition 3. Given a relation $\mathcal{K}$ over M the relation

$$\mathcal{K}_\rho = \{(ij|K) \in \mathcal{R}(N)\,;\ (i^\rho, j^\rho|K^\rho)_\star \subseteq \mathcal{K}\}$$

over N is called *factor of $\mathcal{K}$ to N by ρ.*

Remark 4. The factor $\mathcal{K}_\rho$ can be defined equivalently as union of the elementary semigraphoids $(I, J|K)_\star$ having $(I^\rho, J^\rho|K^\rho)_\star$ contained in $\mathcal{K}$. By the extended semigraphoid axiom, all factors of semigraphoids are semigraphoids and $\mathfrak{c}(\mathcal{K}_\rho) \subseteq \mathfrak{c}(\mathcal{K})_\rho$ where the inclusion can be obviously strict.

Definition 4. Given a relation $\mathcal{L}$ over N the relation

$$\mathcal{L}^\rho = \bigcup\nolimits_{(I,J|K)_\star \subseteq \mathcal{L}} (I^\rho, J^\rho|K^\rho)_\star$$

over M is called *expansion of $\mathcal{L}$ to M by ρ.*

Let us stress that the ex*pan*sions are completely different from the parallel ex*ten*sions of Section 4. For example, $\mathcal{L}^\rho = \emptyset$ for $\mathcal{L}$ empty but $\mathcal{L}_{k\|l}$ is nonempty for empty $\mathcal{L}$ over a set N of cardinality more than one.

Remark 5. Another meaningful notion of expansion might seem to be

$$\mathcal{L}^{\rho *} = \bigcup\nolimits_{(ij|K) \in \mathcal{L}} (i^\rho, j^\rho|K^\rho)_\star\ .$$

Obviously $\mathcal{L}^{\rho *} \subseteq \mathcal{L}^\rho$. However, for $M = \{\mathit{1}, \mathit{2}, \mathit{3}, \mathit{4}\}$ and $N = \{\mathit{1}, \mathit{2}, \mathit{5}\}$ with $\mathit{5} = \mathit{34}$ the semigraphoid $\mathcal{L} = (\mathit{1}, \mathit{25}|\emptyset)_\star$ expands to $\mathcal{L}^\rho = (\mathit{1}, \mathit{234}|\emptyset)_\star$ containing $(\mathit{12}|\mathit{3})$ whereby $(\mathit{12}|\mathit{3}) \notin \mathcal{L}^{\rho *}$. Note that $\mathcal{L}^\rho$ is and $\mathcal{L}^{\rho *}$ is not a semigraphoid.

Proposition 5. *All expansions of a semigraphoid are semigraphoids.*

Proof. Let $\mathcal{L}$ be a semigraphoid over N and $(i_\diamond k_\diamond|L_\diamond)$, $(i_\diamond j_\diamond|k_\diamond L_\diamond)$ two elements of $\mathcal{L}^\rho$. The aim is to show that $(i_\diamond, j_\diamond k_\diamond|L_\diamond)_\star$ is contained in $\mathcal{L}^\rho$.

If $i_\diamond \in i^\rho$, $j_\diamond \in j^\rho$ and $k_\diamond \in k^\rho$ for some i, j, k in N then $i \neq k$ and $i \neq j$.

Knowing that $(i_\diamond k_\diamond|L_\diamond)$ belongs to $(I_1^\rho, K_1^\rho|L^\rho)_\star$ for some $(I_1, K_1|L)_\star \subseteq \mathcal{L}$, sets I_1, K_1 and L are chosen appropriately. Obviously, $i_\diamond \in I_1^\rho$, $k_\diamond \in K_1^\rho$ and $L^\rho \subseteq L_\diamond \subseteq (I_1K_1L)^\rho - i_\diamond k_\diamond$ but one can have also

$$l^\rho \subseteq L_\diamond \;\Leftrightarrow\; l \in L \qquad \text{and} \qquad i_\diamond k_\diamond L_\diamond \cap l^\rho \neq \emptyset \;\Leftrightarrow\; l \in I_1K_1L\ .$$

In addition, I_1 is supposed to have the minimal possible cardinality.

Along the same guidelines, $(i_\diamond j_\diamond|k_\diamond L_\diamond) \in (I_2^\rho, J^\rho|K_2^\rho)_\star$ for some $(I_2, J|K_2)_\star \subseteq \mathcal{L}$, i.e. $i_\diamond \in I_2^\rho$, $j_\diamond \in J^\rho$, $K_2^\rho \subseteq k_\diamond L_\diamond \subseteq (I_2JK_2)^\rho - i_\diamond j_\diamond$,

$$l^\rho \subseteq k_\diamond L_\diamond \;\Leftrightarrow\; l \in K_2 \qquad \text{and} \qquad i_\diamond j_\diamond k_\diamond L_\diamond \cap l^\rho \neq \emptyset \;\Leftrightarrow\; l \in I_2JK_2\ .$$

The minimality of the cardinality of I_2 will be needed, too.

It is obvious that $j_\diamond \notin L_\diamond \supseteq L^\rho$, and then $j \notin L$; similarly $k_\diamond \in (I_2JK_2)^\rho$, and then $k \in I_2JK_2$. Let us single out two trivial cases: $j \in K_1$ leads to $(i_\diamond, j_\diamond k_\diamond|L_\diamond)_\star \subseteq (I_1^\rho, K_1^\rho|L^\rho)_\star$ and $k \in J$ implies $(i_\diamond, j_\diamond k_\diamond|L_\diamond)_\star \subseteq (I_2^\rho, J^\rho|K_2^\rho)_\star$, in both situations having the desirable inclusion $(i_\diamond, j_\diamond k_\diamond|L_\diamond)_\star \subseteq \mathcal{L}^\rho$. One can proceed under the assumptions $j \in N - K_1L$ and $k \in I_2K_2$.

When $j \in I_1$ and $k \in I_2$ then $(i_\diamond k_\diamond|j_\diamond L_\diamond)$ belongs to $(I_1^\rho, K_1^\rho|L^\rho)_\star$ and $(i_\diamond j_\diamond|L_\diamond)$ to $(I_2^\rho, J^\rho|K_2^\rho)_\star$. Clearly, $(i_\diamond, j_\diamond k_\diamond|L_\diamond)_\star \subseteq \mathcal{L}^\rho$ as needed. Let $j \in I_1$ and $k \in K_2$. From the displayed equivalences one deduces $L = K_2 - k$ and $I_2J = I_1(K_1 - k)$. Since $(I_1, K_1 - k|kL)_\star$ is a subset of $(I_1, K_1|L)_\star$, and $(I_1, K_1|L)_\star$ and $(I_2, J|kL)_\star$ are contained in $\mathcal{L}$ also $(I_1 \cap I_2, J(I_2 - I_1)|K_2)_\star$ is contained in $\mathcal{L}$. Due to the minimality of I_2 one has $I_1 \supseteq I_2$. But then $(I_2, k|L)_\star \subseteq \mathcal{L}$ and $(I_2, J|kL)_\star \subseteq \mathcal{L}$ imply $(I_2, kJ|L)_\star \subseteq \mathcal{L}$, and therefore $(i_\diamond, j_\diamond k_\diamond|L_\diamond)_\star \subseteq \mathcal{L}^\rho$.

At the moment one can assume $j \in N - I_1K_1L$, and even $jI_1K_1L = I_2JK_2$. Let $k \in K_2$, and thus $L = K_2 - k$ and $I_1(K_1 - k) = I_2(J - j)$. Since $(I_1, K_1 - k|kL)_\star$ and $(I_2, J - j|kL)_\star$ are subsets of $\mathcal{L}$ also $(I_1 \cap I_2, (J - j)(I_2 - I_1)|kL)_\star$ is contained in $\mathcal{L}$. This fact combined with $(I_1 \cap I_2, j|(J - j)(I_2 - I_1)kL)_\star \subseteq (I_2, J|K_2)_\star \subseteq \mathcal{L}$ implies that $(I_1 \cap I_2, J(I_2 - I_1)|K_2)_\star$ is a subset $\mathcal{L}$. Due to the minimality of I_2 one has $I_1 \supseteq I_2$. Now $(I_2, k|L)_\star$ and $(I_2, J|kL)_\star$ are subsets of $\mathcal{L}$, and thus $(I_2, kJ|L)_\star \subseteq \mathcal{L}$. This implies the desirable $(i_\diamond, j_\diamond k_\diamond|L_\diamond)_\star \subseteq \mathcal{L}^\rho$.

It remains to examine the case $j \in N - I_1K_1L$ and $k \in I_2$ where $K_2 = L$ and $I_1K_1 = I_2(J - j)$. From $(I_2, J - j|L)_\star \cup (I_1, K_1|L)_\star \subseteq \mathcal{L}$ one observes that the elementary semigraphoid $(I_1 \cap I_2, K_1(I_1 - I_2)|L)_\star$ is contained in $\mathcal{L}$. Further, $I_1 \subseteq I_2$ by the minimality of I_1. Since $K_1 = (J - j)(I_2 - I_1)$ one has $(I_1, j|K_1L)_\star \subseteq (I_2, J|L)_\star \subseteq \mathcal{L}$, and then $(I_1, jK_1|L)_\star \subseteq \mathcal{L}$. Finally, $(i_\diamond, j_\diamond k_\diamond|L_\diamond)_\star \subseteq \mathcal{L}^\rho$ as desired. □

Corollary 1. $\mathfrak{c}(\mathcal{L})^\rho = \mathfrak{c}(\mathcal{L}^\rho) = \mathfrak{c}(\mathcal{L}^{\rho *})$

Proof. The inclusion $\mathfrak{c}(\mathcal{L}) \supseteq \mathcal{L}$ implies $\mathfrak{c}(\mathcal{L})^\rho \supseteq \mathcal{L}^\rho$. By Proposition 5, $\mathfrak{c}(\mathcal{L})^\rho$ is a semigraphoid and therefore $\mathfrak{c}(\mathcal{L})^\rho \supseteq \mathfrak{c}(\mathcal{L}^\rho)$. Trivially, $\mathfrak{c}(\mathcal{L}^\rho) \supseteq \mathfrak{c}(\mathcal{L}^{\rho *})$.

On the other hand, if $(ij|kL)$ and $(ik|L)$ belong to $\mathcal{L}$ then $(i^\rho, j^\rho|k^\rho L^\rho)_\star$ and $(i^\rho, k^\rho|L^\rho)_\star$ are contained in $\mathcal{L}^{\rho *}$, and then $(i^\rho, k^\rho|j^\rho L^\rho)_\star$ and $(i^\rho, j^\rho|L^\rho)_\star$ are contained in $\mathfrak{c}(\mathcal{L}^{\rho *})$ by the extended semigraphoid axiom. Since $(\mathcal{L}_1 \cup \mathcal{L}_2)^{\rho *}$ equals $\mathcal{L}_1^{\rho *} \cup \mathcal{L}_2^{\rho *}$ (note that this is not true if the asterisks are omitted) one has the inclusion $\mathfrak{c}(\mathcal{L}^{\rho *}) \supseteq (\mathcal{L}^{(1)})^{\rho *}$, and by induction

$\mathfrak{c}(\mathcal{L}^{\rho *}) \supseteq (\mathcal{L}^{(r)})^{\rho *}$, $r \geqslant 0$. Thus $\mathfrak{c}(\mathcal{L}^{\rho *})$ contains $(\mathfrak{c}(\mathcal{L}))^{\rho *}$ and

$$\mathfrak{c}(\mathcal{L}^{\rho *}) \supseteq \mathfrak{c}\Big(\bigcup_{(ij|K)\in\mathfrak{c}(\mathcal{L})} (i^\rho, j^\rho|K^\rho)_\star\Big) \supseteq \bigcup_{(I,J|K)_\star\subseteq\mathfrak{c}(\mathcal{L})} (I^\rho, J^\rho|K^\rho)_\star = \mathfrak{c}(\mathcal{L})^\rho$$

because

$$\mathfrak{c}\Big(\bigcup_{(ij|L)\in(I,J|K)_\star} (i^\rho, j^\rho|L^\rho)_\star\Big) = (I^\rho, J^\rho|K^\rho)_\star \ .$$

In fact, this equality can be proved by induction on the cardinality of IJ similarly as the extended semigraphoid axiom was established. □

Expansions of factors and factors of expansions are of interest.

Remark 6. The inclusion $(\mathcal{K}_\rho)^\rho \supseteq \mathcal{K}$ can be obviously violated even for semigraphoids. The reverse one, $(\mathcal{K}_\rho)^\rho \subseteq \mathcal{K}$, need not take place, too. For example, for M, N as in Remark 5 and $\mathcal{K}$ equal to the union of $(1,2|\emptyset)_\star$, $(1,2|34)_\star$, $(1,34|\emptyset)_\star$ and $(1,34|2)_\star$ one has $\mathcal{K}_\rho = (1,25|\emptyset)_\star$. Then $(\mathcal{K}_\rho)^\rho$ equals $(1,234|\emptyset)_\star$ and thus $(12|3)$ belongs to $(\mathcal{K}_\rho)^\rho - \mathcal{K}$. However, $\mathcal{K}$ is not a semigraphoid because $(12|34)$ and $(13|4)$ are $\mathcal{K}$ and $(12|4) \notin \mathcal{K}$.

For semigraphoids the inclusion $(\mathcal{K}_\rho)^\rho \subseteq \mathcal{K}$ holds due to the following lemma.

Lemma 6. $(\mathcal{K}_\rho)^\rho \subseteq \mathfrak{c}(\mathcal{K})$

Proof. If $(i_\diamond j_\diamond|L_\diamond) \in (\mathcal{K}_\rho)^\rho$ then $(i_\diamond j_\diamond|L_\diamond) \in (I^\rho, J^\rho|K^\rho)_\star$ for some $(I,J|K)_\star$ contained in $\mathcal{K}_\rho$. For each $i \in I$, $j \in J$ and $K \subseteq L \subseteq IJK - ij$ one has $(ij|L) \in \mathcal{K}_\rho$, and hence $(i^\rho, j^\rho|L^\rho)_\star$ is contained in $\mathcal{K}$. By the displayed identity of the proof of Corollary 1, the closure of union of these $(i^\rho, j^\rho|L^\rho)_\star$ is equal to $(I^\rho, J^\rho|L^\rho)_\star$. Therefore, $(I^\rho, J^\rho|K^\rho)_\star$ is a subset of $\mathfrak{c}(\mathcal{K})$ and, in turn, $(i_\diamond j_\diamond|L_\diamond)$ belongs to $\mathfrak{c}(\mathcal{K})$. □

Corollary 2. $(\mathcal{K}_\rho)^\rho = \mathcal{K}$ *implies* $(\mathfrak{c}(\mathcal{K})_\rho)^\rho = \mathfrak{c}(\mathcal{K})$.

Proof. The inclusion $\subseteq$ follows from Lemma 6. On the other hand, $(\mathfrak{c}(\mathcal{K})_\rho)^\rho$ contains $(\mathcal{K}_\rho)^\rho = \mathcal{K}$ and as $\mathfrak{c}(\mathcal{K})_\rho$ is a obviously a semigraphoid and $(\mathfrak{c}(\mathcal{K})_\rho)^\rho$ is a semigraphoid by Proposition 5 one has $(\mathfrak{c}(\mathcal{K})_\rho)^\rho \supseteq \mathfrak{c}(\mathcal{K})$. □

Lemma 7. $\mathcal{L} = (\mathcal{L}^\rho)_\rho$

Proof. The inclusion $\subseteq$ is straightforward. Let $(ij|L) \in (\mathcal{L}^\rho)_\rho$ and thus $(i^\rho, j^\rho|L^\rho)_\star$ is a subset of $\mathcal{L}^\rho$. Then $(i_\diamond j_\diamond|L_\diamond) \in \mathcal{L}^\rho$ where $i_\diamond \in i^\rho$, $j_\diamond \in j^\rho$ and $L_\diamond$ equals $(ijK)^\rho - i_\diamond j_\diamond$. Therefore $(i_\diamond j_\diamond|L_\diamond)$ belongs to $(I^\rho, J^\rho|K^\rho)_\star$ for some $(I,J|K)_\star$ contained in $\mathcal{L}$. One can deduce that $i \in I$, $j \in J$ and $K \subseteq L \subseteq IJK - ij$ what means $(ij|L) \in \mathcal{L}$. □

Remark 7. The expansion $\mathcal{L}^\rho$ over M of a semigraphoid $\mathcal{L}$ over N can be also equivalently defined as the inclusion minimal semigraphoid over M having $\mathcal{L}$ for its factor by ρ. In fact, $\mathcal{L}^\rho$ is a semigraphoid by Proposition 5 and $\mathcal{L} = (\mathcal{L}^\rho)_\rho$ by Lemma 7; moreover, if $\mathcal{K}$ is a semigraphoid and $\mathcal{K}_\rho = \mathcal{L}$ then $\mathcal{L}^\rho = (\mathcal{K}_\rho)^\rho$ is contained in $\mathcal{K}$ by Lemma 6.

8. Dominant families

When working with expansions it has been necessary to keep systematically track on all elementary semigraphoids contained in a relation $\mathcal{L}$.

Definition 5. A family $\{(I_t, J_t|K_t)_\star\,;\ 1 \leqslant t \leqslant r\}$, $r \geqslant 0$, of elementary semigraphoids is *dominant in* $\mathcal{L}$ if all $(I_t, J_t|K_t)_\star$ are contained in $\mathcal{L}$ and each $(I,J|K)_\star \subseteq \mathcal{L}$ is contained in some $(I_t, J_t|K_t)_\star$.

Remark 8. For $I \neq \emptyset$ and $J \neq \emptyset$ the inclusion $(I, J|K)_\star \subseteq (I', J'|K')_\star$ takes place if and only if $I \subseteq I'$, $J \subseteq J'$ and $K' \subseteq K \subseteq I'J'K'$, up to the order of I and J. The above definition corresponds to Definition 7, [23, p. 76] where semigraphoids are treated as sets of ordered triples.

The inclusion maximal elementary semigraphoids $(I, J|K)_\star$ contained in $\mathcal{L}$, to be called the *dominators* of $\mathcal{L}$, are the most interesting ones as they must be present in each family that is dominant in $\mathcal{L}$. Obviously, $\mathcal{L}$ equals the union of any family dominant in $\mathcal{L}$. However, if $\mathcal{L}$ is union of a family of elementary semigraphoids then the family need not be dominant in $\mathcal{L}$, see the nontrivial examples of dominance in $\mathcal{L}_{\llcorner}$ and $\mathcal{K}_{\llcorner}$ in Section 9. In this section, dominance in pure minors, factors and expansions is examined, and morphisms between semigraphoids are discussed.

Lemma 8. *If* $\{(I_t, J_t|K_t)_\star\,;\ 1 \leqslant t \leqslant r\}$ *is dominant in* $\mathcal{L}$ *and* $\mathcal{K} = \mathsf{co}_I\, \mathsf{re}_J\, \mathcal{L}$ *is a pure minor of* $\mathcal{L}$ *then* $\{(I_t \cap I, J_t \cap I|K_t \cap I)_\star\,;\ 1 \leqslant t \leqslant r\}$ *is dominant in* $\mathcal{K}$.

Proof. The semigraphoids of the latter family are contained in $\mathcal{K}$ because $(I', J'|K')_\star \subseteq \mathcal{L}$ implies that $(I' \cap I, J' \cap I|K' \cap I)_\star$ is contained in $\mathcal{K}$. In fact, for $i \in I' \cap I$, $j \in J' \cap I$ one has $(ij|L) \in \mathcal{L}$ whenever $K' \subseteq L \subseteq I'J'K' - ij$. Then $(ij|L \cap I) \in \mathcal{K}$ because $\mathcal{K}$ is pure, and thus $(ij|K) \in \mathcal{K}$ once

$$K' \cap I \subseteq K \subseteq (I' \cap I)(J' \cap I)(K' \cap I) - ij\ .$$

If $(I', J'|K')_\star \subseteq \mathcal{K}$ then $(I', J'|K'(J - I))_\star \subseteq \mathcal{L}$, and this elementary semigraphoid must be contained in some $(I_t, J_t|K_t)_\star$. This means $I' \subseteq I_t$, $J' \subseteq J_t$ and $K_t \subseteq K'(J - I) \subseteq I_tJ_tK_t$, up to the order of I' and J'. One can conclude $I' \subseteq I_t \cap I$, $J' \subseteq J_t \cap I$ and $K_t \cap I \subseteq K' \subseteq I_tJ_tK_t \cap I$, in other words, $(I', J'|K')_\star$ is contained in $(I_t \cap I, J_t \cap I|K_t \cap I)_\star$. $\square$

Let $\mathcal{M} = (I_\diamond, J_\diamond|K_\diamond)_\star$ be an elementary semigraphoid over M, N a factorset of M and $\rho\colon 2^N \to 2^M$ as in Section 7. Let us denote by $I_{\mathcal{M}}$, $J_{\mathcal{M}}$ and $L_{\mathcal{M}}$ the inclusion maximal subsets of N satisfying $I_{\mathcal{M}}^\rho \subseteq I_\diamond$, $J_{\mathcal{M}}^\rho \subseteq J_\diamond$ and $L_{\mathcal{M}}^\rho \subseteq I_\diamond J_\diamond K_\diamond$, respectively, and by $K_{\mathcal{M}}$ the inclusion minimal subset of N satisfying $K_{\mathcal{M}}^\rho \supseteq K_\diamond$. Obviously, $I_{\mathcal{M}}$, $J_{\mathcal{M}}$ and $K_{\mathcal{M}}$ are disjoint and $I_{\mathcal{M}}J_{\mathcal{M}} \subseteq L_{\mathcal{M}}$. It is easy to see that $(i^\rho, j^\rho|A^\rho)_\star \subseteq \mathcal{M}$ if an only if $i \in I_{\mathcal{M}}$, $j \in J_{\mathcal{M}}$, up to the order of i and j, and $K_{\mathcal{M}} \subseteq A \subseteq L_{\mathcal{M}} - ij$ for $(ij|K) \in \mathcal{R}(N)$. Therefore, the factor $\mathcal{M}_\rho$ of $\mathcal{M}$ is union of the elementary semigraphoids $(I_{\mathcal{M}}, J_{\mathcal{M}}|A)_\star$ over the conditioning set A restricted by $K_{\mathcal{M}} \subseteq A \subseteq L_{\mathcal{M}} - I_{\mathcal{M}}J_{\mathcal{M}}$. For example, in the situation of Remark 5 the elementary semigraphoid $\mathcal{M} = (\mathit{13}, \mathit{24}|\emptyset)_\star$ has the factor $\mathcal{M}_\rho = \{(\mathit{12}|\emptyset), (\mathit{12}|\mathit{5})\}$.

Lemma 9. *If* $\mathcal{D}$ *is a dominant family in a semigraphoid* $\mathcal{K}$ *then*

$$\{(I_{\mathcal{M}}, J_{\mathcal{M}}|A)_\star\,;\ K_{\mathcal{M}} \subseteq A \subseteq L_{\mathcal{M}} - I_{\mathcal{M}}J_{\mathcal{M}}\,,\ \mathcal{M} \in \mathcal{D}\}$$

is dominant in the factor $\mathcal{K}_\rho$.

Proof. If $(I, J|K)_\star \subseteq \mathcal{K}_\rho$ then $(I^\rho, J^\rho|K^\rho)_\star \subseteq (\mathcal{K}_\rho)^\rho \subseteq \mathcal{K}$ using Lemma 6. Since $\mathcal{D}$ is dominant $(I^\rho, J^\rho|K^\rho)_\star$ is contained in $\mathcal{M} = (I_\diamond, J_\diamond|K_\diamond)_\star$ for some $\mathcal{M} \in \mathcal{D}$. Then, obviously, $I \subseteq I_{\mathcal{M}}$, $J \subseteq J_{\mathcal{M}}$, $IJK \subseteq L_{\mathcal{M}}$ and $K \supseteq K_{\mathcal{M}}$, up to the order of I and J. Consequently, $(I, J|K)_\star$ is contained in $(I_{\mathcal{M}}, J_{\mathcal{M}}|A)_\star$ where $A = K - I_{\mathcal{M}}J_{\mathcal{M}}$. $\square$

Remark 9. If $\mathcal{K}$ is not a semigraphoid then the assertion of Lemma 9 can fail. In Remark 6, $\mathcal{K}$ is written as union of its four dominators. The factors of the dominators are $(\mathit{1}, \mathit{2}|\emptyset)_\star$, $(\mathit{1}, \mathit{2}|\mathit{5})_\star$, $(\mathit{1}, \mathit{5}|\emptyset)_\star$ and $(\mathit{1}, \mathit{5}|\mathit{2})_\star$, respectively. However they do not form a dominant family of $\mathcal{K}_\rho$ because the dominator $(\mathit{1}, \mathit{25}|\emptyset)_\star$ of $\mathcal{K}_\rho$ is absent.

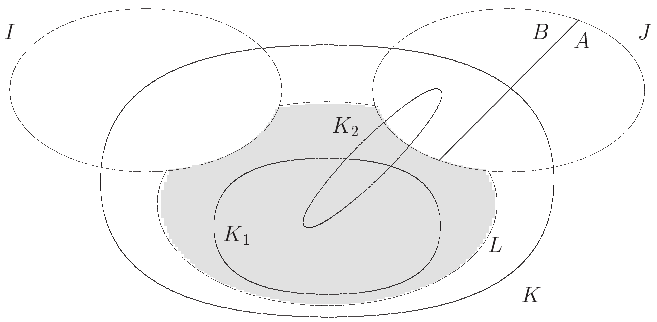
Proposition 6. *If* $\{(I_t, J_t|K_t)_\star\,;\ 1 \leqslant t \leqslant r\}$ *is a dominant family in a semigraphoid* $\mathcal{L}$ *then the family* $\{(I_t^\rho, J_t^\rho|K_t^\rho)\,;\ 1 \leqslant t \leqslant r\}$ *is dominant in the expansion* $\mathcal{L}^\rho$.

Proof. This follows from the assertion that if $(I_\diamond, J_\diamond|K_\diamond)_\star$ is contained in $\mathcal{L}^\rho$ then it is contained in the expansion $(I^\rho, J^\rho|K^\rho)_\star$ of some $(I, J|K)_\star$ contained in $\mathcal{L}$. Induction on the cardinality of $I_\diamond J_\diamond$ is applied. The cases $I_\diamond$ or $J_\diamond$ empty are trivial. If $I_\diamond$ and $J_\diamond$ are both singletons the assertion is true by the definition of $\mathcal{L}^\rho$.

Let $I = \{l \in N\,;\ l^\rho \cap I_\diamond \neq \emptyset\}$ and similarly with J and K constructed from $J_\diamond$ and $K_\diamond$, respectively. Moreover, let $L = \{l \in N\,;\ l^\rho \subseteq K_\diamond\}$. If $I_\diamond \cap l^\rho$ has more than one element then $(I_\diamond - i_\diamond, J_\diamond|K_\diamond)_\star \subseteq (I_\diamond, J_\diamond|K_\diamond)_\star \subseteq \mathcal{L}^\rho$, $i_\diamond \in I_\diamond \cap l^\rho$, and by induction $(I_\diamond - i_\diamond, J_\diamond|K_\diamond)_\star$ is a subset of $(I^\rho, J^\rho|K^\rho)_\star$ for some $(I, J|K)_\star$ contained in $\mathcal{L}$. Since $I_\diamond \subseteq I^\rho$ the elementary semigraphoid $(I_\diamond, J_\diamond|K_\diamond)_\star$ is contained in $(I^\rho, J^\rho|K^\rho)_\star$, and the induction step is over. Thus, one can assume that all nonempty intersections $I_\diamond \cap l^\rho$ are singletons. The same applies to $J_\diamond \cap l^\rho$.

By symmetry, one can suppose that $J_\diamond$ has at least two elements. It can be therefore partitioned into nonempty sets $A_\diamond$ and $B_\diamond$. Correspondingly, J partitions into $A = \{l \in N\,;\ l^\rho \cap A_\diamond \neq \emptyset\}$ and $B = \{l \in N\,;\ l^\rho \cap B_\diamond \neq \emptyset\}$.

Let $(I_\diamond, J_\diamond|K_\diamond)_\star \subseteq \mathcal{L}^\rho$. Then $(I_\diamond, B_\diamond|K_\diamond)_\star$ and $(I_\diamond, A_\diamond|B_\diamond K_\diamond)_\star$ are contained in $\mathcal{L}^\rho$. By induction, $(I_\diamond, B_\diamond|K_\diamond)_\star \subseteq (I_1^\rho, J_1^\rho|K_1^\rho)$ and $(I_\diamond, A_\diamond|B_\diamond K_\diamond)_\star \subseteq (I_2^\rho, J_2^\rho|K_2^\rho)$ for two elementary semigraphoids $(I_1, J_1|K_1)_\star$ and $(I_2, J_2|K_2)_\star$ contained in $\mathcal{L}$. Then $I \subseteq I_1$, $B \subseteq J_1$, and $I \subseteq I_2$, $A \subseteq J_2$, and therefore I and J are disjoint. Note also that $IJ \cap L = \emptyset$. From $K_1^\rho \subseteq K_\diamond \subseteq (I_1 J_1 K_1)^\rho$ one deduces $K_1 \subseteq L \subseteq K \subseteq I_1 J_1 K_1$, and from $K_2^\rho \subseteq B_\diamond K_\diamond \subseteq (I_2 J_2 K_2)^\rho$ analogously $K_2 \subseteq BL \subseteq BK \subseteq I_2 J_2 K_2$. Relying on the figure below (the shaded area corresponds to L)



one observes that $(IC, BD|L)_\star \subseteq \mathcal{L}$ for two disjoint sets C, D such that $CD = K - IJL$. Similarly, $(IE, AF|BL)_\star \subseteq \mathcal{L}$ for disjoint E, F satisfying $EF = K - IJL$.

The extended semigraphoid axiom is now used several times. From

$$(I(C \cap E), B|L)_\star \cup (I(C \cap E), AF|BL)_\star \subseteq \mathcal{L}$$

one has $(I(C \cap E), JF|L)_\star \subseteq \mathcal{L}$ and thus $(I(C \cap E), C \cap F|L)_\star \subseteq \mathcal{L}$. This combined with $(I(C \cap E), BD|(C \cap F)L)_\star \subseteq \mathcal{L}$ gives $(I(C \cap E), B(C \cap F)D|L)_\star \subseteq \mathcal{L}$ where $(C \cap F)D = (D \cap E)F$. From

$$(I(C \cap E), B(D \cap E)F|L)_\star \cup (I(C \cap E), AF|B(D \cap E)L)_\star \subseteq \mathcal{L}$$

one derives $(I(C \cap E), JF(D \cap E)|L)_\star \subseteq \mathcal{L}$. If $I_\imath = I(C \cap E)$, $J_\imath = JF(D \cap E)$ and $K_\imath = L$ then the elementary semigraphoid $(I_\imath, J_\imath|K_\imath)$ is contained in $\mathcal{L}$ and $I_\diamond \subseteq I^\rho \subseteq I_\imath^\rho$, $J_\diamond \subseteq J^\rho \subseteq J_\imath^\rho$, $L^\rho \subseteq K_\diamond$, and $K_\diamond \subseteq (IJK)^\rho \subseteq (I_\imath J_\imath K_\imath)^\rho$. This means that $(I_\diamond, J_\diamond|K_\diamond)_\star$ is contained in $(I_\imath^\rho, J_\imath^\rho|K_\imath^\rho)_\star$. $\square$

Remark 10. If $\mathcal{L}$ is not a semigraphoid then the assertion of Proposition 6 can fail. Let $M = \{1,2,3,4,5\}$, $N = \{1,2,3,6\}$, $6 = 45$, and $\mathcal{L}$ be the complement in $\mathcal{R}(N)$ of $\{(16|23),(36|\emptyset)\}$. Note that $\mathcal{L}$ is not a semigraphoid because $(12|63)$ and $(16|3)$ belong to $\mathcal{L}$ whereas $(16|23) \notin \mathcal{L}$. The elementary semigraphoids $(16,2|\emptyset)_\star$, $(16,2|3)_\star$, $(16,3|2)_\star$, and $(1,36|\emptyset)_\star$ are contained in $\mathcal{L}$. Hence

$$(145,2|\emptyset)_\star \cup (145,2|3)_\star \cup (145,3|2)_\star \cup (1,345|\emptyset)_\star \subseteq \mathcal{L}^\rho$$

and thus $(1,23|5)_\star = \{(12|5),(12|35),(13|25),(13|5)\}$ is a subset of $\mathcal{L}^\rho$. If $(1,23|5)_\star$ were a subset of $(I^\rho, J^\rho|K^\rho)_\star$ for some $(I,J|K)_\star \subseteq \mathcal{L}$ then $K = \emptyset$, $1 \subseteq I$, $23 \subseteq J$, up to the order of I and J, and then $6 \subseteq IJ$. However, this is impossible as none of the elementary semigraphoids $(16,23|\emptyset)_\star$ and $(1,236|\emptyset)_\star$ is contained in $\mathcal{L}$. To conclude, for the family of all dominators of $\mathcal{L}$, which is obviously dominant in $\mathcal{L}$, Proposition 6 does not hold.

Corollary 3. *If $\mathcal{L}^\rho$ for $\rho\colon 2^N \to 2^M$ is an expansion of a semigraphoid $\mathcal{L}$ over N, and $\mathcal{K} = (\mathcal{L}^\rho)^\sigma$ for $\sigma\colon 2^M \to 2^L$ is an expansion of $\mathcal{L}^\rho$ over M then $\mathcal{K}$ is the expansion $\mathcal{L}^{\sigma\rho}$ of $\mathcal{L}$ by the composition $\sigma\rho\colon 2^N \to 2^L$.*

Proof. The aim is to show

$$\bigcup_{(I_\diamond,J_\diamond|K_\diamond)_\star \subseteq \mathcal{L}^\rho} (I_\diamond^\sigma, J_\diamond^\sigma|K_\diamond^\sigma)_\star = \bigcup_{(I,J|K)_\star \subseteq \mathcal{L}} ((I^\rho)^\sigma, (J^\rho)^\sigma|(K^\rho)^\sigma)_\star = \mathcal{L}^{\sigma\rho}$$

because $(\mathcal{L}^\rho)^\sigma$ equals the left-hand side. By Proposition 6, the left union can run over $(I^\rho, J^\rho|K^\rho)_\star$ for $(I,J|K)_\star$ contained in $\mathcal{L}$. ☐

Remark 11. Let $f\colon M \to N$ be a surjective mapping and $\rho\colon 2^N \to 2^M$ be given for $I \subseteq N$ by $I^\rho = \{l \in M\,;\ f(l) \in I\}$. It is natural to say that f is a *strong morphism* between two semigraphoids $\mathcal{K} \subseteq \mathcal{R}(M)$ and $\mathcal{L} \subseteq \mathcal{R}(N)$ if $\mathcal{L}^\rho = \mathcal{K}$. By Corollary 3, the composition of morphisms is a morphism, considered within the class of semigraphoids. This is no longer true in general because Corollary 3 does not hold when the assumption that $\mathcal{L}$ is a semigraphoid is relaxed. To see this, one can continue with the example from Remark 10. Let $L = \{7,8,1,3,4,5\}$ and σ be given by $2 = 78$. From $(1,23|5)_\star \subseteq \mathcal{L}^\rho$ one deduces $(1,378|5)_\star \subseteq (\mathcal{L}^\rho)^\sigma$. However, $(13|57)$ is not an element of $\mathcal{L}^{\sigma\rho}$. In fact, in the opposite case $(13|57)$ would belong to $((I^\rho)^\sigma, (J^\rho)^\sigma|(K^\rho)^\sigma)_\star$ for some $(I,J|K)_\star \subseteq \mathcal{L}$, and one could argue $K = \emptyset$, $1 \subseteq I$, $3 \subseteq J$, up to the order of I and J, and $26 \subseteq IJ$, contradicting that none of the semigraphoids $(126,3|\emptyset)_\star$, $(1,236|\emptyset)_\star$, $(12,36|\emptyset)_\star$, and $(16,23|\emptyset)_\star$ is contained in $\mathcal{L}$.

Remark 12. With f, $\mathcal{K}$ and $\mathcal{L}$ as in the previous remark one can say that f is a *weak morphism* between $\mathcal{K}$ and $\mathcal{L}$ if $\mathcal{K}_\rho = \mathcal{L}$. The composition of weak morphisms is a week morphism in general, due to the following assertion parallel to Corollary 3. If $\mathcal{K}_\rho$ for $\rho\colon 2^N \to 2^M$ is a factor of a relation $\mathcal{K}$ over M, and $\mathcal{L} = (\mathcal{K}_\rho)_\sigma$ for $\sigma\colon 2^L \to 2^N$ is a factor of $\mathcal{K}_\rho$ then $\mathcal{L}$ is the factor $\mathcal{K}_{\rho\sigma}$ of $\mathcal{K}$ by the composition $\rho\sigma\colon 2^L \to 2^M$. This implication follows immediately from

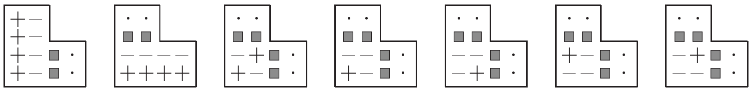
$$\mathcal{K}_{\rho\sigma} = \bigcup_{((I^\sigma)^\rho,(J^\sigma)^\rho|(K^\sigma)^\rho)_\star \subseteq \mathcal{K}} (I,J|K)_\star = \bigcup_{(I^\sigma,J^\sigma|K^\sigma)_\star \subseteq \mathcal{K}_\rho} (I,J|K)_\star = (\mathcal{K}_\rho)_\sigma$$

on account of Remark 4. Lemma 7 guarantees that within the class of semigraphoids each strong morphism is weak.

9. Example: dominance

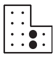
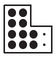
In this section dominant families in $\mathcal{L}$ and $\mathcal{K}$ are presented.

Lemma 10. *The family $\mathcal{L}$ has these seven dominators*



Proof. These elementary semigraphoids are obviously contained in $\mathcal{L}$ and there is no inclusion among them. Hence, it suffices to show that if $(I, J|K)_\star$ is contained in $\mathcal{L}$ then it is contained in one of the seven semigraphoids. If I or J is empty or the both sets are singletons then $(I, J|K)_\star$ is trivially included in the first or second of the seven semigraphoids.

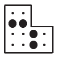
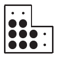
Let the right element of the top row of $N =$, to be denoted by j, belong to J. From structure of the graph $\mathcal{G}_{\mathcal{L}}$ one can see that I is a nonempty subset of the left column. Since $(ij|K) \in \mathcal{L}$ and $(ij|(I-i)(J-j)K) \in \mathcal{L}$ for some $i \in I$ one has

$K \supseteq$  and $IJK \subseteq$  , respectively.

If I has two different elements then one deduces from $\mathcal{G}_{\mathcal{L}}$ that J is contained in the second column (from left) of and therefore $(I, J|K)_\star$ is contained in the first of the seven semigraphoids. Otherwise, $I = i$ and J has at least two elements. If $k \in J - j$ then $(ik|(J-k)K) \in \mathcal{L}$ implies k belongs to the second column. Again, $(I, J|K)_\star$ is contained in the first semigraphoid.

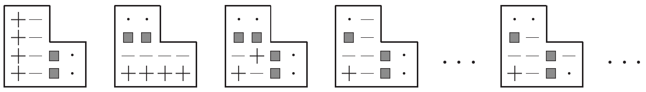
If J contains the right element of the top but one row of $N =$ one can argue analogously.

Now, one can assume by symmetry that $I \cup J$ is a subset of the left bottom 2×2 corner of . If $i \in I$ and $j_1, j_2 \in J$ such that ij_1 are in a row and ij_2 are in a column then similarly as above

$K \supseteq$  and $IJK \subseteq$  ,

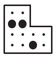
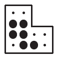
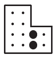
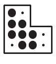
and $(I, J|K)_\star$ is contained in one of the seven elementary semigraphoids. The remaining cases, e.g. I contained in a row and J in another row, are trivial. $\square$

Lemma 11. *The family*

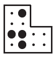
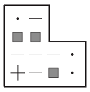


consists of the dominators of $\mathcal{K}$. Here, 7+7 missing elementary semigraphoids obtained by the action of G are omitted.

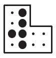
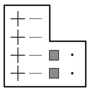
Proof. Along the same lines as in the previous proof, let $(I, J|K)_\star \subseteq \mathcal{K}$ and j, the right element of the top row, belong to J. The set I is contained in the left column. If the top or top but one element of the left column belongs to I then one argues as before. If the both bottom elements, say i, k, of the left column belong to I then J must be contained in the second column. Owing to $(ij|K) \in \mathcal{K}$ and $(kj|K) \in \mathcal{K}$ the set K must contain the third column and be contained in the left three columns. In this case $(I, J|K)_\star$ is a subset of the first of the 19 semigraphoids. Otherwise, one can suppose by symmetry that $I = i$ is the left bottom singleton of and J has at least two elements. From $(ij|K) \in \mathcal{K}$ one deduces

 $\subseteq K \subseteq$  or  $\subseteq K \subseteq$ .

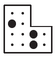
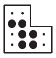
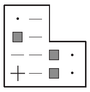
In the first case, $(ij|(J-j)K) \in \mathcal{K}$ implies

$J \subseteq$  and thus $(I,J|K)_\star \subseteq$  .

In the second case,

$J \subseteq$  and hence $(I,J|K)_\star \subseteq$ 

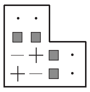
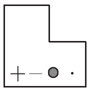
provided the bottom but one element of the left column, say k, is not in J; otherwise $(ik|K) \in \mathcal{K}$ implies

 $\subseteq K \subseteq$  and hence $(I,J|K)_\star \subseteq$  .

If J contains the right element of the second row an argument is analogous.

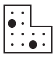
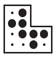
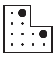
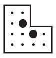
One can now assume by symmetry that $I \cup J$ is a subset of the left bottom 2×2 corner $\begin{smallmatrix} k & l \\ i & j \end{smallmatrix}$ of . Let $i \in I$ and J contain at least two elements.

If $I = il$ and $J = jk$ then

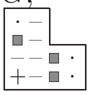
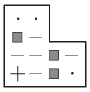
$(I,J|K)_\star \subseteq$  due to the row 

in $\mathcal{K}$ and symmetry.

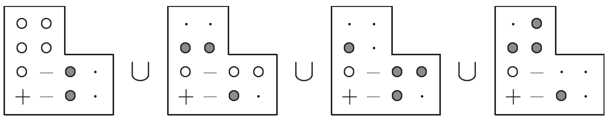
If $i \in I$, $l \notin I$ and $jk \subseteq J$ then

 $\subseteq K \subseteq$  , $K \not\supseteq$  and $K \cap$  $\neq \emptyset$

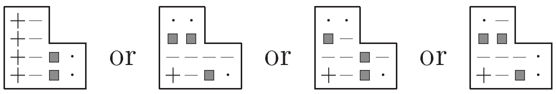
which implies, up to the action of G,

$(I,J|K)_\star \subseteq$  or $(I,J|K)_\star \subseteq$  .

In the remaining case, $i \in I$ and $jl \subseteq J$, it is not difficult but a bit laborious to find that $(i, jl|K)_\star \subseteq \mathcal{K}$ if and only if $(ij|K)$ belongs to



Then $(I,J|K)_\star$ is a subset of



respectively. $\square$

10. Canonical semigraphoids

Let O be the set $\{+,-,\blacksquare,\cdot\}$ having four elements and O^r_{t+} be the set of all r-tuples from O^r having $+$ in the t-th coordinate, $1 \leqslant t \leqslant r$. Analogously, O^r_{t-} and $O^r_{t\blacksquare}$ are defined replacing above $+$ by $-$ or $\blacksquare$, respectively. Given a subset N of O^r, let

$$\bar{\mathcal{L}}_N = \bigcup\nolimits_{t=1}^{r} (N \cap O^r_{t+}, N \cap O^r_{t-} | N \cap O^r_{t\blacksquare})_\star \,.$$

Definition 6. The semigraphoid $\mathfrak{c}(\bar{\mathcal{L}}_N)$ over $N \subseteq O^r$ is called r-*canonical* if N contains the set $P_r = \{+,-,\blacksquare\}^r \cup \{+,-,\cdot\}^r$, $r \geqslant 1$. The semigraphoid $\emptyset$ over $N = \emptyset$ is termed 0-*canonical*. A semigraphoid is *canonical* if it is r-canonical for some $r \geqslant 0$.

Obviously, r-canonical semigraphoids have r generators. The elementary semigraphoid $(+,-|\blacksquare)_\star$ over O is the unique 1-canonical semigraphoid. There are three non-isomorphic 2-canonical semigraphoids. Namely, for $N = O^2$ and $N = O^2 - \{(\blacksquare,\cdot)\}$ one obtains $\mathfrak{c}(\bar{\mathcal{L}}_N) = \bar{\mathcal{L}}_N$ by Lemma 1. The only nontrivial 2-canonical semigraphoid is $\mathfrak{c}(\bar{\mathcal{L}}_N)$ for $N = P_2$. Now, since $(\blacksquare,\blacksquare)$ and $(\cdot,\cdot)$ can be majorized out, cf. Lemma 2, one can work equivalently with $\mathfrak{c}(\bar{\mathcal{L}}_N)$ for $N = P_2 - \{(\blacksquare,\blacksquare),(\cdot,\cdot)\}$. This semigraphoid is, however, an isomorphic copy of the semigraphoid $\mathcal{K}_{\boxplus}$ examined in detail in Sections 3, 6 and 9.

A combination of the expansion technique with the notion of pure minor gives the following main result which reduces study of the semigraphoid closure in general to study of the semigraphoid closure of $\bar{\mathcal{L}}_N$ for $P_r \subseteq N \subseteq O^r$.

Theorem 1. *Every semigraphoid $\mathcal{K}$ over M with r generators is an expansion of a pure minor of an r-canonical semigraphoid.*

Proof. Let $\mathcal{K} = \mathfrak{c}(\mathcal{M})$ where $\mathcal{M} = (I_1, J_1|K_1)_\star \cup \ldots \cup (I_r, J_r|K_r)_\star$. If $r = 0$ then $\mathcal{K} = \mathcal{M} = \emptyset$ is obviously an expansion of the 0-canonical semigraphoid. Let $r \geqslant 1$. The nonempty intersections $A_1 \cap \ldots \cap A_r$ where A_t is one of the sets I_t, J_t, K_t, $M - I_tJ_tK_t$ and t is between 1 and r, give rise to a factorset of M. This factorset can be identified with the subset L of O^r of those r-tuples $i = (i^1, \ldots, i^r)$ that render $A_1 \cap \ldots \cap A_r$ nonempty where A_t equals I_t, J_t, K_t and $M - I_tJ_tK_t$, according to i^t equals $+$, $-$, $\blacksquare$ and $\cdot$, respectively. As for the factormap $i^\rho = A_1 \cap \ldots \cap A_r$ for $i \in L$ and $I^\rho = \bigcup_{i \in I} i^\rho$ for $I \subseteq L$. Each I_t, J_t and K_t is union of blocks of the factorset and it is not difficult to see that $I_t = (L \cap O^r_{t+})^\rho$, $J_t = (L \cap O^r_{t-})^\rho$ and $K_t = (L \cap O^r_{t\blacksquare})^\rho$.

Hence, $\mathcal{M} \subseteq (\bar{\mathcal{L}}_L)^\rho$ and $\mathcal{M}_\rho \subseteq ((\bar{\mathcal{L}}_L)^\rho)_\rho = \bar{\mathcal{L}}_L$ using Lemma 7. Actually, the factor of $\mathcal{M}$ by ρ is equal to $\bar{\mathcal{L}}_L$ because $(ij|K) \in (L \cap O^r_{t+}, L \cap O^r_{t-}|L \cap O^r_{t\blacksquare})_\star$ implies that $(i^\rho, j^\rho|K^\rho)_\star$ is contained in the subset $(I_t, J_t|K_t)_\star$ of $\mathcal{M}$. In turn, $(ij|K) \in \mathcal{M}_\rho$, and thus $\bar{\mathcal{L}}_L \subseteq \mathcal{M}_\rho$.

Since $\mathcal{M}$ is a subset of $(\bar{\mathcal{L}}_L)^\rho = (\mathcal{M}_\rho)^\rho$ the semigraphoid $\mathcal{K} = \mathfrak{c}(\mathcal{M})$ is contained in $\mathfrak{c}((\mathcal{M}_\rho)^\rho) = (\mathfrak{c}(\mathcal{M}_\rho))^\rho$ using Corollary 1. Obviously, $\mathcal{M}_\rho \subseteq \mathfrak{c}(\mathcal{M})_\rho = \mathcal{K}_\rho$ and hence $\mathfrak{c}(\mathcal{M}_\rho)$ is contained in $\mathcal{K}_\rho$ as $\mathcal{K}_\rho$ is a semigraphoid, cf. Remark 4. Thus $\mathcal{K} \subseteq (\mathfrak{c}(\mathcal{M}_\rho))^\rho \subseteq (\mathcal{K}_\rho)^\rho$. By Lemma 6, $(\mathcal{K}_\rho)^\rho \subseteq \mathcal{K}$, and thus $\mathcal{K}$ is the expansion of the semigraphoid $\mathfrak{c}(\bar{\mathcal{L}}_L) = \mathfrak{c}(\mathcal{M}_\rho)$ by ρ.

Let $N = L \cup P_r$, $I = L$ and $J = L \cup \{+,-,\blacksquare\}^r$. The final aim is to show that the canonical semigraphoid $\mathfrak{c}(\bar{\mathcal{L}}_N)$ has its minor $\mathsf{co}_I\, \mathsf{re}_J\, \mathfrak{c}(\bar{\mathcal{L}}_N)$ equal to $\mathfrak{c}(\bar{\mathcal{L}}_L)$ and that this minor is pure. In fact, restricting by some elements of $\{+,-,\cdot\}^r$

$$\mathsf{re}_J\,(N \cap O^r_{t+}, N \cap O^r_{t-}|N \cap O^r_{t\blacksquare})_\star = (J \cap O^r_{t+}, J \cap O^r_{t-}|J \cap O^r_{t\blacksquare})_\star\ ,$$

and thus $\mathsf{re}_J\, \bar{\mathcal{L}}_N = \bar{\mathcal{L}}_J$. Contracting by some elements of $\{+,-,\blacksquare\}^r$

$$\mathsf{co}_I\,(J \cap O^r_{t+}, J \cap O^r_{t-}|J \cap O^r_{t\blacksquare})_\star = (I \cap O^r_{t+}, I \cap O^r_{t-}|I \cap O^r_{t\blacksquare})_\star\ ,$$

and thus $\mathsf{co}_I\, \bar{\mathcal{L}}_J = \bar{\mathcal{L}}_L$. Therefore, $\bar{\mathcal{L}}_L$ is the minor $\mathsf{co}_I\, \mathsf{re}_J\, \mathfrak{c}(\bar{\mathcal{L}}_N)$ of $\bar{\mathcal{L}}_N$. To see that it is pure, let us suppose that $ij \subseteq I$ and $(ij|K)$ belongs to $\bar{\mathcal{L}}_N$, say to $(N \cap O^r_{t+}, N \cap O^r_{t-}|N \cap O^r_{t\blacksquare})_\star$ for some t. Then

$$(ij|K \cap J) \in (J \cap O^r_{t+}, J \cap O^r_{t-}|J \cap O^r_{t\blacksquare})_\star$$

and, in turn, $(ij|K \cap I)$ is in $(L \cap O^r_{t+}, L \cap O^r_{t-}|L \cap O^r_{t\blacksquare})_\star$. This implies that $\bar{\mathcal{L}}_L$ contains $(ij|K \cap I)$, and thus is a pure minor of $\bar{\mathcal{L}}_N$. By Lemma 2, $\mathfrak{c}(\bar{\mathcal{L}}_L)$ is a pure minor of $\mathfrak{c}(\bar{\mathcal{L}}_N)$. To summarize, $\mathcal{K}$ equals $(\mathsf{co}_I\, \mathsf{re}_J\, \mathfrak{c}(\bar{\mathcal{L}}_N))^\rho$. $\square$

The strategy to construct any semigraphoid from a canonical one applies also to the notion of dominance.

Corollary 4. *If $\mathcal{K}$ is an expansion of a pure minor $\mathsf{co}_I\,\mathsf{re}_J\,\mathfrak{c}(\bar{\mathcal{L}}_N)$ of an r-canonical semigraphoid $\mathfrak{c}(\bar{\mathcal{L}}_N)$, and $\{(I_t, J_t|K_t)_\star\,;\ 1 \leqslant t \leqslant s\}$ is a dominant family in $\mathfrak{c}(\bar{\mathcal{L}}_N)$ then $\{((I_t \cap I)^\rho, (J_t \cap I)^\rho|(K_t \cap I)^\rho)_\star\,;\ 1 \leqslant t \leqslant s\}$ is a dominant family in $\mathcal{K}$.*

Proof. Lemma 8 brings a dominant family in $\mathfrak{c}(\bar{\mathcal{L}}_N)$ down to a dominant family in its pure minor $\mathsf{co}_I\,\mathsf{re}_J\,\mathfrak{c}(\bar{\mathcal{L}}_N)$ and Proposition 6 lifts it to a dominant family in $\mathcal{K}$. □

Let us say that two elementary semigraphoids $(I_1, J_1|K_1)_\star$ and $(I_2, J_2|K_2)_\star$ are *sticky* if $K_1 \subseteq I_2J_2K_2$ and $K_2 \subseteq I_1J_1K_1$.

Remark 13. Let $\mathcal{K} = \mathfrak{c}((I_1, J_1|K_1)_\star \cup (I_2, J_2|K_2)_\star)$ be a semigraphoid with two sticky generators. By Theorem 1, $\mathcal{K}$ is an expansion of a pure minor of the 2-canonical semigraphoid $\mathfrak{c}(\bar{\mathcal{L}}_N)$ with $N = O^2 - \{(\blacksquare, \cdot), (\cdot, \blacksquare)\}$. Using Corollary 4, a dominant family in $\mathcal{K}$ can be found from a dominant family in $\mathfrak{c}(\bar{\mathcal{L}}_N)$. Since this canonical semigraphoid differs from $\mathcal{K}_{\boxdot}$ only by a majorization its dominators obtain easily from the dominators of $\mathcal{K}_{\boxdot}$ described in Lemma 11. Therefore

$$\Big\{(I_1, J_1|K_1)_\star, (I_2, J_2|K_2)_\star, \big((I_1 \cap I_2)(J_1 \cap J_2), (I_1 \cap J_2)(J_1 \cap I_2)|K_1K_2\big)_\star,$$
$$\big((I_1 \cap I_2), J_1(I_1 \cap J_2)|K_1(I_1 \cap K_2)\big)_\star, \ldots\,,$$
$$\big((I_1 \cap I_2), (J_2 - K_2)(J_1 \cap I_2K_2)|K_1(I_1 \cap K_2)\big)_\star, \ldots\Big\}$$

is a dominant family in $\mathcal{K}$; the missing terms are obtained by switching $I_1 \leftrightarrow J_1$, or $I_2 \leftrightarrow J_2$, or $I_1 \leftrightarrow I_2$ and $J_1 \leftrightarrow J_2$ and $K_1 \leftrightarrow K_2$ simultaneously. This result was stated in Corollary 15 to Lemma 10 in [23]. That Lemma 10 (its 'tree-like' proof occupies over eleven pages) describes, in our terminology, the inclusion minimal elementary semigraphoids not contained in $\mathcal{K}$.

For further progress on the topic is seems crucial to understand isomorphisms between pure minors of canonical semigraphoids. This is probably difficult in general and remains open.

11. Linear and probabilistic representations

A relation $\mathcal{L} \subseteq \mathcal{R}(N)$ is *linearly representable*, or simply *linear*, if there exists a collection $D = (D_i)_{i \in N}$ of finite-dimensional linear subspaces of a linear vector space E over a field F such that $\mathcal{L} = [D]$ where

$$[D] = \{(ij|K) \in \mathcal{R}(N)\,;\ D_{iK} \cap D_{jK} = D_K\}$$

and D_K is the inner sum of subspaces D_k for $k \in K$. Obviously, $[D]$ is a semigraphoid. The class of linear semigraphoids was introduced in [14] where its closure operator, forbidden minors and linear semigraphoids over a four-element set N were studied. In this section, the attention is focused on linearity of semigraphoids with special generators.

If all semigraphoids of a class $\boldsymbol{C}$ are linear and each construction of a *list* conserves the linearity then all semigraphoids of the class $\langle\!\langle \boldsymbol{C} \mid \textit{list} \rangle\!\rangle$ are linear as well. Since our building blocks have been the uniform semigraphoids first concern is about their representability. As an illustration, a linear representation of $\mathsf{U}_{2,5}$ over a two-element field can be found in Example 6 of [14].

Lemma 12. *The uniform semigraphoids are linear over any field.*

Proof. Each uniform matroid is obviously linear over any infinite field and also, see Proposition 6.8.2 of [19], over any finite field of sufficiently large cardinality. Therefore, given $0 < r \leqslant n$, for any finite field F of large cardinality there exists a configuration of n vectors in F^r such that any r of them are linearly independent. Let $D_1, \dots, D_n$ be the one-dimensional subspaces of F^r spanned by the vectors. The collection D of these spaces obviously represents $\mathsf{U}_{r,n}$ over $N = \{1, \dots, n\}$. If a field F has cardinality not large enough then the same argumentation goes through with its algebraic extension F_a of sufficiently large finite degree s over F. Then D_i's are s-dimensional linear subspaces of F_a^r when this linear space is considered over F and, in addition, $\dim_F D_K = \min\{|K|s, rs\}$ for $K \subseteq N$. Immediately, $[D] = \mathsf{U}_{r,n}$. □

Let us recall that linear semigraphoids are closed to intersections and minors, cf. [14] Lemma 7.

Lemma 13. *Direct sums of linear semigraphoids over a fixed field are linear.*

Proof. If $(D_i)_{i \in N_1}$ in E_1 represents $\mathcal{L}_1 \subseteq \mathcal{R}(N_1)$ and $(D_i)_{i \in N_2}$ in E_2 represents $\mathcal{L}_2 \subseteq \mathcal{R}(N_2)$, N_1, N_2 disjoint, then $(D_i)_{i \in N_1 \cup N_2}$ in $E_1 \oplus E_2$ represents $\mathcal{L}_1 \oplus \mathcal{L}_2$. □

As a consequence, the semigraphoids from the class $\boldsymbol{A}$ described in Proposition 1 are linear.

Lemma 14. *The dual of a linear semigraphoid is linear.*

Proof. Let $D = (D_i)_{i \in N}$ be a linear representation of $\mathcal{L}$. Further, let C_i for $i \in N$ be disjoint sets of cardinality $\dim D_i$, and $C_I = \bigcup_{i \in I} C_i$ for $I \subseteq N$. For each $i \in N$ let $(e_c;\, c \in C_i)$ be a base of D_i. The linear matroid (C_N, r), where $r(A)$ equals $\dim(e_c;\, c \in A)$ for $A \subseteq C_N$, has the dual $(C_N, r^\rceil)$, where $r^\rceil(A)$ equals $|A| + r(C_N - A) - r(C_N)$, which is linear by Theorem 2 of [25, p. 141]. Let $(e'_c;\, c \in C_N)$ be a linear representation of the dual, i.e. $r^\rceil(A)$ equals $\dim(e'_c;\, c \in A)$ for $A \subseteq C_N$. Then $D' = (D'_i)_{i \in N}$ where D'_i is the span of $(e'_c; c \in C_i)$ represents $\mathcal{L}^\rceil$. In fact, $(ij|K) \in \mathcal{L} = [D]$ is equivalent to the equality between $\dim D_{iK} + \dim D_{jK}$ and $\dim D_{ijK} + \dim D_K$. Owing to $r^\rceil(C_K) = |C_K| + \dim D_{N-K} - r(C_N)$ this amounts the equality between $r^\rceil(C_{jL}) + r^\rceil(C_{iL})$ and $r^\rceil(C_L) + r^\rceil(C_{ijL})$ where $L = N - ijK$. This means $D'_{iL} \cap D'_{jL} = D'_L$. Therefore $(ij|L) \in \mathcal{L}^\rceil$ if and only if $(ij|L) \in [D']$. □

Combining Proposition 2 with the previous three lemmas one obtains the following assertion.

Proposition 7. *Every semigraphoid with solitary generators is linearly representable over any field.*

To show linearity of semigraphoids from other classes of Sections 5 and 6, additional lemmas on linear representability are worked out.

Lemma 15. *Linear semigraphoids are closed to majors.*

Proof. For $\mathcal{L} \subseteq \mathcal{R}(N)$ one can restrict only to two situations $\mathcal{K} = \mathsf{re}_{N-i}\, \mathcal{L}$ and $\mathcal{K} = \mathsf{co}_{N-i}\, \mathcal{L}$ where $i \in N$ and the cardinalities of $\mathcal{K}$ and $\mathcal{L}$ coincide. Here, from the linearity of $\mathcal{K}$ over $N - i$ the linearity of $\mathcal{L}$ over N should be derived. By duality, Lemma 14, it suffices to deal only with the restriction. The semigraphoid $\mathcal{R}(N - i)$ is linear because it belongs to the class $\boldsymbol{A}$. The major $\mathcal{L} = (\mathcal{K} \oplus \mathcal{R}(i)) \cap \mathcal{R}(N - i)$ of $\mathcal{K}$ is then linear by Lemma 13. □

Lemma 16. *Parallel extensions of linear semigraphoids are linear.*

Proof. Let $D = (D_i)_{i\in N}$ be a linear representation of $\mathcal{L}$, $k \in N$, $l \notin N$, and $D^{k\|l}$ contain in comparison to D one more subspace D_l which is identical to D_k. The ascending semigraphoid $\{(ij|K) \in \mathcal{R}(lN);\ ij \neq kl\} = \mathcal{M}_{kl}$ belongs to the class $\boldsymbol{A}$ and is therefore linear. It is easy to see that the semigraphoid $[D^{k\|l}] \cap \mathcal{M}_{kl}$ is linear and equals $\mathcal{L}_{k\|l}$. □

Lemma 17. *Factors and expansions of linear semigraphoids are linear.*

Proof. If $(D_{i_\diamond})_{i_\diamond\in M}$ represents $\mathcal{K}$ then $\big(\bigoplus_{i_\diamond\in i^\rho} D_{i_\diamond}\big)_{i\in N}$ represents the factor $\mathcal{K}_\rho$ of $\mathcal{K}$.

Let a semigraphoid $\mathcal{L}$ over $N = \{1, 2, \ldots, n\}$, $n \geqslant 1$, be linear and let $\mathcal{L} = [D]$ with $D = (D_i)_{i\in N}$ living in a linear space E. By Corollary 3, it suffices to prove that the expansion $\mathcal{K} = \mathcal{L}^\rho$ over $M = \{1', 1'', 2, \ldots, n\}$ where $1^\rho = 1'1''$ and $r^\rho = r$ for $2 \leqslant r \leqslant n$ is linear. Where E' and E'' are two subspaces of E, let $D^{E',E''} = (D_{i_\diamond})_{i_\diamond\in M}$ have $D_{1'} = E'$, $D_{1''} = E''$ and $D_{i_\diamond} = D_i$, $2 \leqslant i_\diamond = i \leqslant n$. It is easy to see that $[D^{E',E''}]$ contains $\mathcal{L}^\rho$ once $E' \oplus E'' = D_1$. Hence

$$\mathcal{L}^\rho \ \subseteq\ \big[\emptyset_{1'1''} \oplus \mathcal{R}(M - 1'1'')\big] \ \cap\ \bigcap\nolimits_{E'\oplus E''=D_1} [D^{E',E''}]$$

where $\emptyset_{1'1''}$ is the empty semigraphoid over $1'1''$.

To prove the opposite inclusion, let $(i_\diamond j_\diamond|L_\diamond) \notin \mathcal{L}^\rho$. If $i_\diamond j_\diamond = 1'1''$ then $(i_\diamond j_\diamond|L_\diamond)$ is not in $\emptyset_{1'1''} \oplus \mathcal{R}(M - 1'1'')$. If $i_\diamond j_\diamond = i1'$, $i \geqslant 2$, then $(i1|L_\diamond - 1'')$ is not in $\mathcal{L}$, and thus $(i_\diamond j_\diamond|L_\diamond)$ not in $[D^{E',E''}]$ where $E' = D_1$ and $E'' = \{0\}$ (contains only the zero vector). By symmetry, if $i_\diamond j_\diamond = i1''$, $i \geqslant 2$, then $(i_\diamond j_\diamond|L_\diamond)$ does not belong to $[D^{\{0\},D_1}]$. Hence, let $i_\diamond j_\diamond$ do not intersect $1'1''$. If $L_\diamond \subseteq N - 1$ or $L_\diamond \supseteq 1'1''$ then $(i_\diamond j_\diamond|L_\diamond)$ is not in $[D^{E',E''}]$ where E', E'' summing to D_1 can be arbitrary.

In the last case, by symmetry, $(i_\diamond j_\diamond|L_\diamond) = (ij|1'K) \notin \mathcal{L}^\rho$ where $ijK \subseteq N - 1$. If $(ij|K)$ is not in $\mathcal{L}$ then $(i_\diamond j_\diamond|L_\diamond)$ is not in $[D^{\{0\},D_1}]$ and if $(ij|1K)$ is not in $\mathcal{L}$ then $(i_\diamond j_\diamond|L_\diamond)$ is not in $[D^{D_1,\{0\}}]$. Hence one can assume $(ij|K) \in \mathcal{L}$ and $(ij|1K) \in \mathcal{L}$. Then $(i1|K)$ is not in $\mathcal{L}$ and $(j1|K)$ not in $\mathcal{L}$, otherwise $(i_\diamond j_\diamond|L_\diamond) \in \mathcal{L}^\rho$ can be easily obtained. As $D_{iK} \cap D_{1K}$ and $D_{jK} \cap D_{1K}$ are different from D_K it is possible to choose two vectors d_i, d_j out of D_K such that d_i belongs to $D_{iK} \cap D_1$ and d_j to $D_{jK} \cap D_1$. Let E' be the one-dimensional space spanning the sum $d_i + d_j$; note that $E' \subseteq D_1$. The intersection of $(E' \oplus D_{iK})$ and $(E' \oplus D_{jK})$ contains obviously d_i, d_j and D_K. Its dimension is at least $2 + \dim D_K$ because $d_i \notin D_{jK}$ (otherwise $d_i \in D_{iK} \cap D_{jK} = D_K$) where D_{jK} contains d_j and $d_j \notin D_K$. Therefore the intersection cannot equal $E' \oplus D_K$, the latter having its dimension at most $1 + \dim D_K$. Thus $(ij|1'K) \notin [D^{E',D_1}]$.

To conclude, the extension $\mathcal{L}^\rho$ was expressed as intersection of linear semigraphoids and is therefore linear. □

Theorem 2. *Every semigraphoid with at most two generators is linearly representable over any field.*

Proof. The case of solitary generators follows from Proposition 7. As the uniform semigraphoids are linear by Lemma 12 and their factors are linear by Lemma 17, using Lemmas 13, 14, and Proposition 3 the case of two inferenceless generators is settled.

If Lemmas 12, 13, 14, 16 are combined with Proposition 4 linearity of the semigraphoid $\mathcal{K}_{⧉}$ is obtained. Then Lemma 15 implies that the canonical semigraphoid $\mathfrak{c}(\bar{\mathcal{L}}_N)$ for $N = O^2 - \{(\blacksquare, \cdot), (\cdot, \blacksquare)\}$ is linear. Finally, Lemma 17 and Theorem 1 provide also the remaining case of two sticky generators. □

Remark 14. A relation $\mathcal{L} \subseteq \mathcal{R}(N)$ is *probabilistically representable*, or simply *probabilistic*, if there exists a system of random variables $\xi = (\xi_i)_{i\in N}$ taking only a finite number of values such that $(ij|K) \in \mathcal{L}$ is equivalent to 'ξ_i is conditionally independent of ξ_j given $\xi_K = (\xi_k)_{k \in K}$'. For an overview and recent results on probabilistic semigraphoids see [12, 14, 17, 13, 15]. Since every linear semigraphoid is probabilistic by Lemma 10 of [14] Theorem 2 implies that the semigraphoids with two generators are also probabilistic. This assertion was the main result, Theorem 16, of [23]. Its proof was based on the technical Lemma 10, cf. Remark 13. In our approach this monstrous lemma and the ten sophisticated constructions of random vectors from Section 5 of [23] are bypassed.

Remark 15. The analogue of Lemma 14 for probabilistic semigraphoids is not true, for a counterexample see [13]. Lemmas 13 and 16 hold trivially also for p-representable semigraphoids. The intuition is that Lemmas 15 and 17 are valid also within the probabilistic framework.

Remark 16. A natural class of semigraphoids originates from polymatroids. A real function h defined on the subsets of N is said to be (the rank function of) a *polymatroid* if $h(\emptyset) = 0$, $h(iK) \geqslant h(K)$ for $i \in N$, $K \subseteq N$, and $h(iK) + h(jK) \geqslant h(ijK) + h(K)$ for $(ij|K) \in \mathcal{R}(N)$. If h is integer-valued and $h(i) \leqslant 1$, $i \in N$, then h is a *matroid*, see [19] or [25]. Given a polymatroid h, the set of all $(ij|K) \in \mathcal{R}(N)$ rendering the equality between $h(iK) + h(jK)$ and $h(ijK) + h(K)$ is obviously a semigraphoid. These semigraphoids are examples of *semimatroids*, introduced in [12] in a slightly more general framework; in [22] these semigraphoids are called structural. Let us note in passing that the uniform matroids give rise in this way to the uniform semigraphoids.

Each linear semigraphoid $\mathcal{L} = [D]$ is a semimatroid as it can be obtained from the (multilinear) polymatroid which maps K to $\dim D_K$. Even each probabilistic semigraphoid, constructed from some ξ, is a semimatroid as it originates from the polymatroid, called the entropy function of ξ, which maps K to the Shannon entropy of ξ_K. From this point of view this section and Section 5 are, actually, dealing with semimatroids.

Remark 17. Expansions of polymatroids were studied e.g. in [18] and related to semigraphoids in [12]. Let us remark that if h' is an expansion of a polymatroid h then the semimatroid constructed from h' as above need not be an expansion, according to Definition 4, of the semimatroid constructed from h. We have found no way to parallel the notion of polymatroid expansion in semigraphoids.

References

[1] S.A. Andersson and M.D. Perlman (1993) Lattice models for conditional independence in a multivariate normal distribution. *The Annals of Statistics* **21** 1318–1358.

[2] Z. Ann, D.A. Bell and J.G. Hughes (1992) On the axiomization of conditonal independence. *Int. J. of Systems and Cybernetics (Kybernetes)* **21** 48–58.

[3] D.R. Cox and N. Wermuth (1996) *Multivariate dependencies.* Chapman and Hall, London.

[4] A.P. Dawid (1979) Conditional independence in statistical theory (with discussion). *J. Roy. Statist. Soc. Ser. B* **41** 1–31.

[5] A.P. Dawid (2002) Separoids: A mathematical framework for conditional independence and irrelevance. *Annals of Math. and AI* **32** 335–372.

[6] D. Geiger (1990) Graphoids: a qualitative framework for probabilistic inference. Ph.D. Dissertation, Computer Science Department, University of California, Los Angeles.

[7] D. Geiger and J. Pearl (1993) Logical and algorithmic properties of conditional independence and graphical models. *The Annals of Statistics* **21** 2001–2021.

[8] J.T.A. Koster (1996) Markov properties of nonrecursive causal models. *Ann. Statist.* **24** 2148–2177.

[9] S.L. Lauritzen (1996) *Graphical Models.* Oxford University Press, Oxford.

[10] F. Matúš (1992) Ascending and descending conditional independence relations. *Transactions of the 11-th Prague Conference on Information Theory, Statistical Decision Functions and Random Processes*, Vol. B, 189–200, Academia, Prague (also Kluwer, Dordrecht).

[11] F. Matúš (1992) On equivalence of Markov properties over undirected graphs. *J. Appl. Prob.* **29** 745–749.

[12] F. Matúš (1994) Probabilistic conditional independence structures and matroid theory: background. *Int. J. of General Systems* **22** 185–196.

[13] F. Matúš (1995) Conditional independences among four random variables II. *Combinatorics, Probability and Computing* **4** 407–417.

[14] F. Matúš (1997) Conditional independence structures examined via minors. *Annals of Math. and AI* **21** 99–128.

[15] F. Matúš (1999) Conditional independences among four random variables III: final conclusion. *Combinatorics, Probability and Computing* **8** 269–276.

[16] F. Matúš (2002) Lengths of semigraphoid inferences. *Annals of Math. and AI* **35** 287–294.

[17] F. Matúš and M. Studený (1995) Conditional independences among four random variables I. *Combinatorics, Probability and Computing* **4** 269–278.

[18] H.Q. Nguyen (1986) Semimodular functions. in: *Theory of Matroids* (N. White, ed.), Cambridge University Press, Cambridge,London.

[19] J.G. Oxley (1992) *Matroid Theory.* Oxford University Press, Oxford, New York, Tokyo.

[20] J. Pearl (1988) *Probabilistic Reasoning in Intelligent Systems.* Morgan Kaufman: San Meteo, California.

[21] W. Spohn (1980) Stochastic independence, causal independence and shieldability. *J. of Philosophical Logic* **9** 73–99.

[22] M. Studený (1994) Structural semigraphoids. *Int. J. of General Systems* **22** 207–217.

[23] M. Studený (1997) Semigraphoids and structures of probabilistic conditional independence. *Annals of Math. and AI* **21** 71–98.

[24] M. Studený and R.R. Bouckaert (1998) On chain graph models for description of conditional independence structure. *Annals of Statistics* **26** 1434–1495.

[25] D.J.A. Welsh (1976) *Matroid Theory.* Academic Press, London.