



NORTH-HOLLAND

Conditional Independence and Natural Conditional Functions

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ABSTRACT

The concept of conditional independence (CI) within the framework of natural conditional functions (NCFs) is studied. An NCF is a function ascribing natural numbers to possible states of the world; it is the central concept of Spohn's theory of deterministic epistemology. Basic properties of CI within this framework are recalled, and further results analogous to the results concerning probabilistic CI are proved. Firstly, the intersection of two CI-models is shown to be a CI-model. Using this, it is proved that CI-models for NCFs have no finite complete axiomatic characterization (by means of a simple deductive system describing relationships among CI-statements). The last part is devoted to the marginal problem for NCFs. It is shown that (pairwise) consonancy is equivalent to consistency iff the running intersection property holds.

KEYWORDS: *natural conditional function, conditional independence, axiomatic characterization, marginal problem, running intersection property*

1. INTRODUCTION

Several recent works in AI have dealt with the concept of irrelevance, in particular conditional irrelevance among attributes. Pearl and Paz introduced the concept of a *dependency model* to describe such conditional irrelevance structures within various frameworks (undirected graphs, directed acyclic graphs, probability theory). In the probabilistic framework (we have probabilistic reasoning in expert systems in mind) the conditional irrelevance was interpreted as *conditional independence* (CI) among ran-

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dom variables (describing attributes). Although the concept of CI has been studied in probability theory and statistics for more than fifteen years [2, 21, 13, 17], its importance for probabilistic expert systems was highlighted relatively recently [14]. Pearl and Paz [15] proposed describing CI structures in an axiomatic way, i.e. by means of a simple deductive mechanism handling information about the CI structure. They conjectured that the CI structures for strictly positive measures coincide with a special type of dependency models, namely *graphoids* (which were introduced as dependency models closed under five concrete inference rules). This hypothesis was supported by several partial results, in that some substructures of CI structure were characterized in this way. Independently Matúš [12] and Geiger, Paz, and Pearl [3] characterized ordinary (unconditional) probabilistic independence; Geiger and Pearl [4] and Malvestuto [10] independently found an axiomatization for the class of so-called “fixed-context” CI-statements. Nevertheless, the original conjecture was refuted firstly by finding a further property of probabilistic CI [24] and finally by showing that the CI structures within the probabilistic framework cannot be characterized as dependency models closed under a finite number of inference rules [26]. For comprehensive survey see the recent paper of Geiger and Pearl [5].

Another framework in which the concept of CI was introduced in Spohn’s theory of *ordinal conditional functions* [22]. This theory, motivated from a philosophical point of view, provides a tool for the mathematical description of the dynamic handling of deterministic epistemology, and in this sense it is a counterpart of the probabilistic description of an epistemic state.¹ As soon as the concept of CI for ordinal conditional functions was introduced, researchers began to study its properties, especially for a special class of *natural conditional functions* (NCF) called “disbelief functions” in [19] or “ranking functions” in [6]. Hunter in [7] showed that any model of CI structure given by an NCF is a graphoid. After publishing the paper [25] with a further property of CI for strictly positive measures, the group of researchers around J. Pearl found that the new property also holds for NCFs. All these facts, together with the alleged homomorphism of NCFs to nonstandard probability measures, made Pearl formulate the hypothesis that the formal properties of CI for strictly positive measures and for NCFs coincide. Nevertheless, as recently shown by Spohn [23], the inference rule from [24] does not hold for NCFs (see also [27]).

The concept of CI can also be studied in other frameworks for dealing with uncertainty in AI, namely in the Dempster-Shafer theory of belief functions and possibility theory—for details see [20, 27].

¹Nevertheless, there exists a homomorphism between the class of ordinal conditional functions and the class of nonstandard probability measures—for explanation see [22].

In this article we try to extend some results from probabilistic CI into the framework of NCFs. Firstly, we recall basic concepts and results and give some equivalent definitions of CI within this framework. By examples we will show that in the case of three attributes all graphoids are representable in the framework of NCFs. In the third section we give a construction of an NCF allowing us to prove that the class of CI-models within the NCF framework is closed under intersection. This is used to prove the main result saying that CI-models within the NCF framework have no finite complete axiomatic characterization—i.e., the result analogous to the result from [26] for the probabilistic framework. We even show this by means of the same collection of inference rules.

In the fourth section we deal with the marginal problem for NCFs. We give a simple method for solving the problem of the existence of a simultaneous (multivariate) NCF with a prescribed set of marginal (less-dimensional) NCFs. This question has a far simpler solution than its counterpart in the probabilistic framework. Finally, we show that the *running intersection property* is a necessary and sufficient condition for the equivalence of the existence of a simultaneous NCF with the consonancy of marginal NCFs (this result is completely analogous to the probabilistic case).

2. BASIC CONCEPTS AND FACTS

We start with slightly modified definitions from [22].

DEFINITION 1 (Natural conditional function) *Let \mathbf{X} be a nonempty set, and $\exp \mathbf{X}$ denotes the class of all its subsets. Then a natural conditional function (NCF) on \mathbf{X} is a nonnegative integer set function $\kappa : (\exp \mathbf{X}) \setminus \{\emptyset\} \rightarrow \{0, 1, 2, \dots\}$ such that*

- (a) $\kappa(\mathbf{X}) = 0$,
- (b) $\kappa(\bigcup_{\gamma \in \Gamma} A_\gamma) = \min_{\gamma \in \Gamma} \kappa(A_\gamma)$ whenever $\emptyset \neq A_\gamma \subset \mathbf{X}$, $\gamma \in \Gamma$
(Γ is an arbitrary nonempty index set).

Having $A, B \subset \mathbf{X}$ with $A \cap B \neq \emptyset$ and an NCF κ on \mathbf{X} , the symbol $\kappa(A|B)$ will be used to denote the difference $\kappa(A \cap B) - \kappa(B)$.

Of course, an NCF is uniquely determined by its values on singletons. We can even define an NCF equivalently as a set function $\kappa : (\exp \mathbf{X}) \setminus \{\emptyset\} \rightarrow \{0, 1, 2, \dots\}$ extending some point function $\kappa : \mathbf{X} \rightarrow \{0, 1, \dots\}$ with $\min\{\kappa(x); x \in \mathbf{X}\} = 0$ by the formula

$$\kappa(A) = \min\{\kappa(a); a \in A\} \quad \text{for } \emptyset \neq A \subset \mathbf{X}.$$

The most general definition of CI (with respect to NCFs) introduces this

concept for complete algebras. In this paper we restrict our attention to perpendicular collections of algebras²:

DEFINITION 2 (Complete algebras, perpendicularity, independence) *A class \mathcal{S} of subsets of a nonempty set \mathbf{X} is a complete algebra on \mathbf{X} iff it contains \mathbf{X} and is closed under complement ($\mathbf{S} \in \mathcal{S} \Rightarrow \mathbf{X} \setminus \mathbf{S} \in \mathcal{S}$) and arbitrary union ($\mathbf{S}_\gamma \in \mathcal{S}, \gamma \in \Gamma \Rightarrow \bigcup_{\gamma \in \Gamma} \mathbf{S}_\gamma \in \mathcal{S}$).*

A nonempty set $\mathbf{A} \in \mathcal{S}$ is an atom of a complete algebra \mathcal{S} iff its only proper subset belonging to \mathcal{S} is the empty set ($\mathbf{B} \in \mathcal{S}, \mathbf{A} \neq \mathbf{B} \subset \mathbf{A} \Rightarrow \mathbf{B} = \emptyset$). The collection of atoms of a complete algebra \mathcal{S} will be denoted by $\text{at}(\mathcal{S})$.³

A collection of complete algebras $\{\mathcal{A}_\gamma; \gamma \in \Gamma\}$ on \mathbf{X} is perpendicular iff $\bigcap \{\mathbf{A}_\gamma; \gamma \in \Gamma'\} \neq \emptyset$ whenever $\mathbf{A}_\gamma \in \text{at}(\mathcal{A}_\gamma), \gamma \in \Gamma'$, and $\Gamma' \subset \Gamma$ is finite.

Having an NCF κ on \mathbf{X} and three perpendicular complete algebras $\mathcal{A}, \mathcal{B}, \mathcal{C}$ on \mathbf{X} (i.e. forming a perpendicular collection), we shall say that \mathcal{A} is conditionally independent of \mathcal{B} given \mathcal{C} with respect to κ and write $\mathcal{A} \perp \mathcal{B} | \mathcal{C}(\kappa)$ iff $\forall \mathbf{A} \in \text{at}(\mathcal{A}), \mathbf{B} \in \text{at}(\mathcal{B}), \mathbf{C} \in \text{at}(\mathcal{C}),$

$$\kappa(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}) + \kappa(\mathbf{C}) = \kappa(\mathbf{A} \cap \mathbf{C}) + \kappa(\mathbf{B} \cap \mathbf{C}).^4$$

REMARK The definition of CI can be formulated equivalently in apparently stronger form: $\forall \mathbf{A} \in \mathcal{A} \setminus \{\emptyset\}, \mathbf{B} \in \mathcal{B} \setminus \{\emptyset\}, \mathbf{C} \in \text{at}(\mathcal{C}),$

$$\kappa(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}) + \kappa(\mathbf{C}) = \kappa(\mathbf{A} \cap \mathbf{C}) + \kappa(\mathbf{B} \cap \mathbf{C}).$$

Indeed, owing to perpendicularity, we can write

$$\begin{aligned} \kappa(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}) &= \min\{\kappa(\mathbf{A}' \cap \mathbf{B}' \cap \mathbf{C}); \mathbf{A}' \in \text{at}(\mathcal{A}), \\ &\quad \mathbf{A}' \subset \mathbf{A}, \mathbf{B}' \in \text{at}(\mathcal{B}), \mathbf{B}' \subset \mathbf{B}\} \end{aligned}$$

and estimate each term from below (using the definition of CI):

$$\begin{aligned} \kappa(\mathbf{A}' \cap \mathbf{B}' \cap \mathbf{C}) &= \kappa(\mathbf{A}' \cap \mathbf{C}) + \kappa(\mathbf{B}' \cap \mathbf{C}) - \kappa(\mathbf{C}) \\ &\geq \kappa(\mathbf{A} \cap \mathbf{C}) + \kappa(\mathbf{B} \cap \mathbf{C}) - \kappa(\mathbf{C}). \end{aligned}$$

Thus $\kappa(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}) \geq \kappa(\mathbf{A} \cap \mathbf{C}) + \kappa(\mathbf{B} \cap \mathbf{C}) - \kappa(\mathbf{C})$, and the inverse inequality can be shown similarly by choosing $\mathbf{A}' \in \text{at}(\mathcal{A}), \mathbf{A}' \subset \mathbf{A}$ with

²Our reasons are explained in Remark 1 concluding this section.

³Note that every complete algebra \mathcal{S} is *atomic* in the sense that (different) atoms are mutually disjoint and every set from \mathcal{S} is decomposed into them: $\forall \mathbf{S} \in \mathcal{S} \mathbf{S} = \bigcup \{\mathbf{A}; \mathbf{A} \in \text{at}(\mathcal{S}), \mathbf{A} \subset \mathbf{S}\}$

⁴We can also write $\kappa(\mathbf{A} \cap \mathbf{B} | \mathbf{C}) = \kappa(\mathbf{A} | \mathbf{C}) + \kappa(\mathbf{B} | \mathbf{C})$ or $\kappa(\mathbf{A} | \mathbf{B} \cap \mathbf{C}) = \kappa(\mathbf{A} | \mathbf{C})$.

$\kappa(A \cap C) = \kappa(A' \cap C)$ and $B' \in \text{at}(\mathcal{B})$, $B' \subset B$ with $\kappa(B \cap C) = \kappa(B' \cap C)$.

However, we warn the reader that the condition

$$\begin{aligned} \forall A \in \mathcal{A} \setminus \{\emptyset\}, B \in \mathcal{B} \setminus \{\emptyset\}, C \in \mathcal{C} \setminus \{\emptyset\}, \\ \kappa(A \cap B \cap C) + \kappa(C) = \kappa(A \cap C) + \kappa(B \cap C) \end{aligned}$$

is strictly stronger than $\mathcal{A} \perp \mathcal{B} | \mathcal{C}(\kappa)$. It implies CI-statements $\mathcal{A} \perp \mathcal{B} | \mathcal{C}'(\kappa)$ for all complete subalgebras \mathcal{C}' of \mathcal{C} . [In general, $\mathcal{A} \perp \mathcal{B} | \mathcal{C}'(\kappa)$ is not implied by $\mathcal{A} \perp \mathcal{B} | \mathcal{C}(\kappa)$ with $\mathcal{C}' \subset \mathcal{C}$.]

Nevertheless, when the NCF-theory is applied in the area of AI a special framework is accepted: certain elementary variables or attributes are distinguished and the concept of (conditional) irrelevance among them is studied. Thus, in the following we will often consider this special situation:

(S) $\left\{ \begin{array}{l} \text{A nonempty finite set } N \text{ of attributes is given. A nonempty finite set} \\ \text{X}_i \text{ of possible states corresponds to each attribute } i \in N \text{ (to avoid} \\ \text{trivialities we suppose } \text{card X}_i \geq 2). \text{ Whenever } \emptyset \neq S \subset N, \text{ the} \\ \text{symbol X}_S \text{ will be used to denote the cartesian product } \prod_{i \in S} \text{X}_i, \text{ i.e.} \\ \text{the set of states for } S. \text{ Moreover, we introduce the coordinate algebra} \\ \mathcal{A}_S \text{ for every set of attributes } S \subset N: \\ \mathcal{A}_\emptyset = \{\emptyset, \text{X}_N\}, \quad \mathcal{A}_N = \text{exp X}_N, \\ \mathcal{A}_S = \{\text{T} \times \text{X}_{N \setminus S}; \text{T} \subset \text{X}_S\} \quad \text{for the remaining } S. \end{array} \right.$

Note that whenever sets of attributes S_1, \dots, S_k are pairwise disjoint, the collection of corresponding coordinate algebras $\{\mathcal{A}_{S_1}, \dots, \mathcal{A}_{S_k}\}$ is perpendicular. We can apply the general definitions above to the situation (S) and introduce (conditional) independence for attributes:

DEFINITION 3 (Dependency model, induced CI-model, graphoid) *Supposing (S), the symbol $T(N)$ will be used to denote the set of triplets $\langle A, B | C \rangle$ of pairwise disjoint subsets of N where the first two sets A and B are nonempty. Every subset of $T(N)$ will be called a dependency model over N .*

By an NCF over N we will understand an NCF on X_N . Having an NCF κ over N , then its marginal NCF κ^S (where $\emptyset \neq S \subset N$) is an NCF over S defined as follows ($\kappa^N \equiv \kappa$):

$$\begin{aligned} \kappa^S(\text{T}) = \min\{\kappa(x); x \in \text{T} \times \text{X}_{N \setminus S}\} = \kappa(\text{T} \times \text{X}_{N \setminus S}), \\ \text{where } \emptyset \neq \text{T} \subset \text{X}_S. \end{aligned}$$

Whenever $\langle A, B | C \rangle \in T(N)$ and κ is an NCF over N , we will write $A \perp B | C(\kappa)$ instead of $\mathcal{A}_A \perp \mathcal{A}_B | \mathcal{A}_C(\kappa)$. The dependency model $\{\langle A, B | C \rangle \in T(N); A \perp B | C(\kappa)\}$ is then called the CI-model induced by κ .

By an inference rule with r antecedents ($r \geq 1$) we understand an $(r + 1)$ -ary relation on $T(N)$ (specified concretely for every set of attributes N). We say that a dependency model $I \subset T(N)$ is closed under an inference rule \mathcal{R} iff for each instance of \mathcal{R} (i.e. every collection $[t_1, \dots, t_{r+1}]$ of elements of $T(N)$ belonging to \mathcal{R}) the following statement holds: whenever the antecedents (i.e. t_1, \dots, t_r) belong to I , then so does the consequent (i.e. t_{r+1}).

Usually, an inference rule is expressed by an informal schema, firstly antecedents and after an arrow the consequent. Thus, the schemata

$$\begin{aligned} \langle A, B|C \rangle &\rightarrow \langle B, A|C \rangle && \text{(symmetry),} \\ \langle A, B \cup C|D \rangle &\rightarrow \langle A, C|D \rangle && \text{(decomposition),} \\ \langle A, B \cup C|D \rangle &\rightarrow \langle A, B|C \cup D \rangle && \text{(weak union),} \\ [\langle A, B|C \cup D \rangle \&\langle A, C|D \rangle] &\rightarrow \langle A, B \cup C|D \rangle && \text{(contraction),} \\ [\langle A, B|C \cup D \rangle \&\langle A, C|B \cup D \rangle] &\rightarrow \langle A, B \cup C|D \rangle && \text{(intersection)} \end{aligned}$$

describe five inference rules. According to [15], we will call every dependency model closed under these inference rules a graphoid.

As suggested below Definition 1, every marginal NCF (over $\emptyset \neq S \subset N$) can be identified with a point function $\kappa^S : \mathbf{X}_S \rightarrow \{0, 1, 2, \dots\}$. We can formulate several equivalent definitions of CI (with respect to) in terms of these point functions.

LEMMA 1. *Supposing (S), let κ be an NCF over N and $\langle A, B|C \rangle \in T(N)$. Then the following three conditions are equivalent to $A \perp B|C(\kappa)$:*

(a) $\forall a \in \mathbf{X}_A, b \in \mathbf{X}_B, c \in \mathbf{X}_C,$

$$\kappa^{A \cup B \cup C}(abc) + \kappa^C(c) = \kappa^{A \cup C}(ac) + \kappa^{B \cup C}(bc).$$

(b) $\forall a, a' \in \mathbf{X}_A, b, b' \in \mathbf{X}_B, c \in \mathbf{X}_C,$

$$\kappa^{A \cup B \cup C}(abc) + \kappa^{A \cup B \cup C}(a'b'c) = \kappa^{A \cup B \cup C}(ab'c) + \kappa^{A \cup B \cup C}(a'bc).$$

(c) $\exists f : \mathbf{X}_{A \cup C} \rightarrow \{0, 1, \dots\}, g : \mathbf{X}_{B \cup C} \rightarrow \{0, 1, \dots\},$

$$\forall a \in \mathbf{X}_A, b \in \mathbf{X}_B, c \in \mathbf{X}_C, \quad \kappa^{A \cup B \cup C}(abc) = f(ac) + g(bc).$$

The reader has probably noticed that the conditions in the preceding lemma are analogous to well-known equivalent definitions of probabilistic CI: condition (b) can be interpreted as “cross interchangeability” and condition (c) as “factorization”.

Proof Condition (a) is a simple transcription of the definition $A \perp B|C(\kappa)$ in terms of marginal NCFs. To see (a) \Rightarrow (b), express the

$\kappa^{A \cup B \cup C}(\cdot)$'s using (a) and substitute them into (b). For (b) \Rightarrow (a), fix a, b, c and write, using (b),

$$\begin{aligned} & \kappa^{A \cup B \cup C}(abc) + \kappa^C(c) \\ &= \min\{\kappa^{A \cup B \cup C}(abc) + \kappa^{A \cup B \cup C}(a'b'c); a' \in \mathbf{X}_A, b' \in \mathbf{X}_B\} \\ &= \min\{\kappa^{A \cup B \cup C}(ab'c) + \kappa^{A \cup B \cup C}(a'bc); a' \in \mathbf{X}_A, b' \in \mathbf{X}_B\} \\ &= \min\{\kappa^{A \cup B \cup C}(ab'c); b' \in \mathbf{X}_B\} + \min\{\kappa^{A \cup B \cup C}(a'bc); a' \in \mathbf{X}_A\} \\ &= \kappa^{A \cup C}(ac) + \kappa^{B \cup C}(bc). \end{aligned}$$

For (a) \Rightarrow (c), put $f(ac) = \kappa^{A \cup C}(ac) - \kappa^C(c)$, $g(bc) = \kappa^{B \cup C}(bc)$. To see (c) \Rightarrow (a), fix a, b, c and, using (c), write

$$\begin{aligned} \kappa^{A \cup C}(ac) &= \min\{f(ac) + g(b'c); b' \in \mathbf{X}_B\} \\ &= f(ac) + \min\{g(b'c); b' \in \mathbf{X}_B\}, \\ \kappa^{B \cup C}(bc) &= \min\{f(a'c) + g(bc); a' \in \mathbf{X}_A\} \\ &= \min\{f(a'c); a' \in \mathbf{X}_A\} + g(bc), \\ \kappa^C(c) &= \min\{f(a'c) + g(b'c); a' \in \mathbf{X}_A, b' \in \mathbf{X}_B\} \\ &= \min\{f(a'c); a' \in \mathbf{X}_A\} + \min\{g(b'c); b' \in \mathbf{X}_B\}, \end{aligned}$$

and substitute these expressions together with (c) into (a). \blacksquare

Formal properties of CI arising in the NCF-theory are in many respects similar to the properties of probabilistic CI, namely, some basic properties are valid in both frameworks.

LEMMA 2 *Let κ be an NCF on a set $\mathbf{X} \neq \emptyset$, and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be a perpendicular collection of complete algebras on \mathbf{X} . Let $\mathcal{A} + \mathcal{B}$ denote the complete algebra generated by $\mathcal{A} \cup \mathcal{B}$.⁵ Then*

- (a) $\{\emptyset, \mathbf{X}\} \perp \mathcal{B} | \mathcal{C}(\kappa)$,
- (b) $\mathcal{A} \perp \mathcal{B} | \mathcal{C}(\kappa) \Leftrightarrow \mathcal{B} \perp \mathcal{A} | \mathcal{C}(\kappa)$,
- (c) $\mathcal{A} \perp (\mathcal{B} + \mathcal{C}) | \mathcal{D}(\kappa) \Leftrightarrow \mathcal{A} \perp \mathcal{C} | \mathcal{D}(\kappa) \ \& \ \mathcal{A} \perp \mathcal{B} | (\mathcal{C} + \mathcal{D})(\kappa)$,
- (d) $\mathcal{A} \perp \mathcal{B} | (\mathcal{C} + \mathcal{D})(\kappa) \ \& \ \mathcal{A} \perp \mathcal{C} | (\mathcal{B} + \mathcal{D})(\kappa)$
 $\Rightarrow \mathcal{A} \perp (\mathcal{B} + \mathcal{C}) | \mathcal{D}(\kappa)$.

The proof is left to the reader, who can also consult [22] or [7] [in the case of the special situation (S)]. Hence, we can easily deduce as a consequence the fact already mentioned in the Introduction.

⁵That is, the least complete algebra on \mathbf{X} containing $\mathcal{A} \cup \mathcal{B}$.

COROLLARY 1 *Supposing (S), let κ be an NCF over N . Then the CI-model induced by κ is a graphoid.*

One may ask which graphoids are CI-models in the NCF-theory. It is of interest to us that in the case of three attributes every graphoid is a CI-model (the same holds for probabilistic CI). The following example proves this claim.

EXAMPLE 1 (The case of three attributes) Firstly note that every graphoid is uniquely determined by its intersection with the set of elementary triplets:

$$E(N) = \{\langle\{a\}, \{b\}|C\rangle; a, b \in N, a \neq b, C \subset N, \{a, b\} \cap C = \emptyset\}$$

(for details see [11]). Thus, we leave it to the reader to verify that in the case $N = \{1, 2, 3\}$ there exist exactly 18 graphoids, which can be divided into eight groups (if we group together graphoids mutually transformable by means of a permutation of attributes).

In the following list we choose one representative of each group and give an example of an NCF inducing it as CI-model. Note that $\mathbf{X}_1 = \{a, a'\}$, $\mathbf{X}_2 = \{b, b'\}$, $\mathbf{X}_3 = \{c, c'\}$ in all eight items, and NCFs are given as point functions on $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$.

1. The empty graphoid is the CI-model induced by the following NCF:

$$\begin{aligned} \kappa(abc) &= \kappa(abc') = \kappa(ab'c) = \kappa(a'bc) = 0, \\ \kappa(a'b'c) &= \kappa(ab'c') = \kappa(a'bc') = \kappa(a'b'c') = 1, \end{aligned}$$

2. The graphoid $\{\langle\{1\}, \{2\}|\emptyset\rangle, \langle\{2\}, \{1\}|\emptyset\rangle\}$ is induced by

$$\begin{aligned} \kappa(abc) &= \kappa(a'bc) = \kappa(ab'c) = \kappa(a'b'c') = 0, \\ \kappa(abc') &= \kappa(a'b'c) = \kappa(a'bc') = \kappa(ab'c') = 1. \end{aligned}$$

3. The graphoid $\{\langle\{1\}, \{2\}|\{3\}\rangle, \langle\{2\}, \{1\}|\{3\}\rangle\}$ is the CI-model induced by κ :

$$\begin{aligned} \kappa(abc) &= \kappa(a'bc) = \kappa(abc') = \kappa(ab'c') = 0, \\ \kappa(a'b'c') &= \kappa(a'b'c) = \kappa(a'bc') = \kappa(ab'c) = 1. \end{aligned}$$

4. The graphoid $\{\langle\{1\}, \{2\}|\emptyset\rangle, \langle\{1\}, \{3\}|\emptyset\rangle + \text{sym. triplets}\}$ is the CI-model induced by the following NCF:

$$\begin{aligned} \kappa(abc) &= \kappa(abc') = \kappa(ab'c') = 0, \\ \kappa(a'b'c') &= \kappa(a'b'c) = \kappa(a'bc') = \kappa(a'bc) = \kappa(ab'c) = 1. \end{aligned}$$

5. The graphoid $\{\langle\{1\},\{2\}|\emptyset\rangle, \langle\{1\},\{2\}|\{3\}\rangle + \text{sym. triplets}\}$ is the CI-model induced by κ :

$$\kappa(abc) = \kappa(abc') = \kappa(ab'c) = \kappa(a'bc) = \kappa(a'b'c) = 0,$$

$$\kappa(ab'c') = \kappa(a'bc') = 1,$$

$$\kappa(a'b'c') = 2.$$

6. The graphoid $\{\langle\{1\},\{2\}|\emptyset\rangle, \langle\{1\},\{3\}|\emptyset\rangle, \langle\{2\},\{3\}|\emptyset\rangle + \text{sym. triplets}\}$ is the CI-model induced by the following NCF:

$$\kappa(abc) = \kappa(ab'c') = \kappa(a'bc') = \kappa(a'b'c) = \kappa(a'b'c') = 0,$$

$$\kappa(abc') = \kappa(ab'c) = \kappa(a'bc) = 1.$$

7. The graphoid generated by the triplet $\langle\{1\},\{2,3\}|\emptyset\rangle$ is the CI-model induced by the following NCF:

$$\kappa(abc) = \kappa(abc') = \kappa(ab'c) = \kappa(a'bc) = \kappa(a'b'c) = \kappa(a'bc') = 0,$$

$$\kappa(a'b'c') = \kappa(ab'c') = 1.$$

8. The full graphoid $T(N)$ is the CI-model induced by $\kappa \equiv 0$.

However, there are graphoids which are not CI-models in NCF-theory. Spohn in [23] claims that every CI-model induced by an NCF has to be closed under three further independent inference rules:

$$[\langle A, B|C \cup D \rangle \& \langle C, D|\emptyset \rangle \& \langle C, D|A \rangle \& \langle C, D|B \rangle] \rightarrow \langle C, D|A \cup B \rangle,$$

$$[\langle A, B|C \cup D \rangle \& \langle C, D|\emptyset \rangle \& \langle C, D|A \rangle \& \langle C, D|A \cup B \rangle] \rightarrow \langle C, D|B \rangle,$$

$$[\langle A, B|C \cup D \rangle \& \langle C, D|A \rangle \& \langle C, D|B \rangle \& \langle C, D|A \cup B \rangle] \rightarrow \langle C, D|\emptyset \rangle.$$

(The set of antecedents of such an inference rule can then give rise to an example of a graphoid which is not a CI-model in the NCF-theory). Nevertheless, the same result holds for probabilistic CI-models induced by strictly positive measures. This fact supported the hypothesis that CI-models arising in the NCF-theory coincide with probabilistic CI-models corresponding to strictly positive measures. But this hypothesis is incorrect, as the inference rule

$$[\langle A, B|C \cup D \rangle \& \langle C, D|A \rangle \& \langle C, D|B \rangle \& \langle A, B|\emptyset \rangle] \rightarrow \langle C, D|\emptyset \rangle,$$

which “holds” for each probabilistic CI-model (see [24]), fails in the case of CI-models induced by NCFs. A counterexample can be found in [23] or [27].

We conclude the section with a remark explaining why we restrict ourselves to perpendicular collections of algebras.

REMARK 1 Attempts to extend the definition of CI for nonperpendicular collections of algebras lead to unpleasant problems.

The first possibility is to accept the “weak definition” (Definition 10 in [22]): $\mathcal{A} \perp \mathcal{B} | \mathcal{C} (\kappa) \equiv$ for all $\mathbf{A} \in \text{at}(\mathcal{A})$, $\mathbf{B} \in \text{at}(\mathcal{B})$, $\mathbf{C} \in \text{at}(\mathcal{C})$ with $\mathbf{A} \cap \mathbf{B} \cap \mathbf{C} \neq \emptyset$ it holds that $\kappa(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}) + \kappa(\mathbf{C}) = \kappa(\mathbf{A} \cap \mathbf{C}) + \kappa(\mathbf{B} \cap \mathbf{C})$. But in the case of this definition the “intersection” property $\mathcal{A} \perp \mathcal{B} | \mathcal{C} \ \& \ \mathcal{A} \perp \mathcal{C} | \mathcal{B} \Rightarrow \mathcal{A} \perp (\mathcal{B} + \mathcal{C}) | (\mathcal{B} \cap \mathcal{C})$ does not hold. The counterexample is easy: take $\mathbf{X} = \{a, b, c, d\}$ with an NCF given by a point function $\kappa(a) = 1$, $\kappa(b) = \kappa(c) = \kappa(d) = 0$, and consider the following algebras given by decompositions: $\text{at}(\mathcal{A}) = \{\{a, b\}, \{c, d\}\}$, $\text{at}(\mathcal{B}) = \{\{a, c\}, \{b, d\}\}$, $\text{at}(\mathcal{C}) = \{\{b\}, \{a, c, d\}\}$. Note for explanation that Theorem 13 from [22], claiming that the “intersection” property holds, implicitly uses the assumption of perpendicularity.

The second possibility is the “strong definition” (this approach is used in Definition 8 of [22]—the definition of unconditional independence): $\mathcal{A} \perp \mathcal{B} | \mathcal{C} (\kappa) \equiv$ for all $\mathbf{A} \in \text{at}(\mathcal{A})$, $\mathcal{B} \in \text{at}(\mathcal{B})$, $\mathbf{C} \in \text{at}(\mathcal{C})$ it holds that $[\mathbf{A} \cap \mathbf{B} \cap \mathbf{C} \neq \emptyset \ \& \ \kappa(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}) + \kappa(\mathbf{C}) = \kappa(\mathbf{A} \cap \mathbf{C}) + \kappa(\mathbf{B} \cap \mathbf{C})]$. Nevertheless, in the case of this definition even the “weak union” property $\mathcal{A} \perp (\mathcal{B} + \mathcal{C}) | \mathcal{D} \Rightarrow \mathcal{A} \perp \mathcal{B} | (\mathcal{C} + \mathcal{D})$ fails. To see this take $\mathbf{X} = \{a, b, c, d\}$, $\kappa \equiv 0$ and consider the following algebras: $\text{at}(\mathcal{A}) = \{\mathbf{X}\}$, $\text{at}(\mathcal{B}) = \text{at}(\mathcal{C}) = \{\{a, b\}, \{c, d\}\}$, $\text{at}(\mathcal{D}) = \{\{a, c\}, \{b, d\}\}$. Note that the “trivial” property $\{\emptyset, \mathbf{X}\} \perp \mathcal{B} | \mathcal{C}$ also fails in this case.

3. NONAXIOMATIZABILITY OF CI-MODELS ARISING IN THE NCF-THEORY

In this section we show that CI-models induced by NCFs cannot be characterized as dependency models closed under a finite number of inference rules. This result has an analogy both in the probabilistic case [26] and in the case of EMVD-models in the theory of database relations⁶ [18].

Firstly we give a construction allowing us to generate CI-models in the NCF-theory very simply. The result also has an analogy in the probabilistic case—see [4, 26].

⁶The abbreviation EMVD means *embedded multivalued dependency*. It has an analogous meaning for relational databases as CI for probability measures.

PROPOSITION 1 *Having fixed a nonempty finite set of attributes N , the intersection of two CI-models induced by NCFs is a CI-model induced by an NCF.⁷*

Proof Suppose that two NCFs over N are given: $\kappa_1 : \mathbf{X}_N \rightarrow \{0, 1, \dots\}$ and $\kappa_2 : \mathbf{Y}_N \rightarrow \{0, 1, \dots\}$, where $\mathbf{X}_N = \prod_{i \in N} \mathbf{X}_i$, $\mathbf{Y}_N = \prod_{i \in N} \mathbf{Y}_i$ —as in the situation (S). Put $\mathbf{Z}_i = \mathbf{X}_i \times \mathbf{Y}_i$ for $i \in N$, and define on $\mathbf{Z}_N = \prod_{i \in N} \mathbf{Z}_i$ the function κ :

$$\kappa([x_i, y_i]_{i \in N}) = \kappa_1([x_i]_{i \in N}) + \kappa_2([y_i]_{i \in N}), \quad x_i \in \mathbf{X}_i, \quad y_i \in \mathbf{Y}_i.$$

Of course, κ defined as NCF (over N). Moreover, it is no problem to see that for each $\emptyset \neq S \subset N$ it holds that

$$\kappa^S([x_i, y_i]_{i \in S}) = \kappa_1^S([x_i]_{i \in S}) + \kappa_2^S([y_i]_{i \in S}), \quad x_i \in \mathbf{X}_i, \quad y_i \in \mathbf{Y}_i.$$

Now, to show $[A \perp B|C(\kappa_1) \ \& \ A \perp B|C(\kappa_2)] \Leftrightarrow A \perp B|C(\kappa)$ for an arbitrary $\langle A, B|C \rangle \in T(N)$, use the equivalent definition of CI (b) from Lemma 1. The implication \Rightarrow is then a simple summing of equations for κ_1 and κ_2 ; the implication $A \perp B|C(\kappa) \Rightarrow A \perp B|C(\kappa_1)$ is evident if we eliminate terms with κ_2 by a proper choice of states for κ_2 (for example $[y_i]_{i \in A} = [y'_i]_{i \in A} = [y'_i]_{i \in B} = [y'_i]_{i \in B}$). The same idea yields $A \perp B|C(\kappa) \Rightarrow A \perp B|C(\kappa_2)$. ■

Moreover, we shall need two special constructions of NCFs.

LEMMA 3 *Supposing that N is a finite set of attributes and $\emptyset \neq S \subset N$, there exists an NCF κ over N such that:*

(a) $\neg[A \perp B|C(\kappa)]$ whenever $\langle A, B|C \rangle \in T(N)$ with

$$S \subset A \cup B \cup C \quad \& \quad A \cap S \neq \emptyset \neq B \cap S;$$

(b) $A \perp B|C(\kappa)$ for the remaining $\langle A, B|C \rangle \in T(N)$.

Proof Put $\mathbf{X}_i = \{0, 1\}$ for each $i \in N$ and

$$\kappa([x_i]_{i \in N}) = \begin{cases} 1 & \text{if } [\forall i \in S, x_i = 1], \\ 0 & \text{otherwise.} \end{cases}$$

To verify (a) use Lemma 1(b): consider $a \in \mathbf{X}_A$, $b \in \mathbf{X}_B$, $c \in \mathbf{X}_C$ to be made of units (i.e., $a_i = 1$ for all $i \in A$) and $a' \in \mathbf{X}_A$, $b' \in \mathbf{X}_B$ to be made of zeros (i.e., $a'_i = 0$ for all $i \in A$). Then $\kappa^{A \cup B \cup C}(abc) = 1$ but $\kappa^{A \cup B \cup C}(a'b'c) = \kappa^{A \cup B \cup C}(a'bc) = \kappa^{A \cup B \cup C}(ab'c) = 0$. Whenever $\emptyset \neq T \subset N$ with $S \setminus T \neq \emptyset$, then $\kappa^T \equiv 0$. Thus, we have $A \perp B|C(\kappa)$ for each

⁷Here we have in mind intersection in ordinary set-theoretical sense—that is, simply intersection of subsets of $T(N)$.

$\langle A, B|C \rangle \in T(N)$ with $S \setminus (A \cup B \cup C) \neq \emptyset$. In the case $[S \subset A \cup B \cup C \ \& \ A \cap S = \emptyset]$ we have $S \subset B \cup C$ and, using Lemma 1(c), conclude $A \perp B|C(\kappa)$. Similarly in the dual case $B \cap S = \emptyset$. ■

LEMMA 4 *Supposing $N = \{0, 1, \dots, n\}$ where $n \geq 3$, there exists an NCF κ over N such that:*

- (a) $\{0\} \perp \{i\}|\{j\}(\kappa)$ for $j > i \geq 1$,
- (b) $\neg[\{0\} \perp \{j\}|\{i\}(\kappa)]$ for $j > i \geq 1$,
- (c) $\neg[\{i\} \perp \{j\}|\{0\}(\kappa)]$ for $i \neq j, i, j \geq 1$,
- (d) $\neg[K \perp L|\emptyset(\kappa)]$ for $K \neq \emptyset \neq L, K \cap L = \emptyset, K, L \subset N$.

Proof Put $X_i = \{0, 1\}$ for each $i \in N$, and define κ as follows:

$$\kappa([x_i]_{i \in N}) = \begin{cases} 0, & \text{if } x_0 = 1 \ \& \ [\forall i \geq 1, x_i = 0], \\ 1, & \text{if } x_0 = 1 \ \& \ [\exists i \geq 1, x_i = 1], \\ 1, & \text{if } x_0 = 0 \ \& \ [\exists 1 \leq j < n, x_j = 1 \ \& \ x_{j+1} = 0], \\ 0, & \text{if } x_0 = 0 \ \& \ [\forall j < n, x_j \leq x_{j+1}]. \end{cases}$$

It is easy to verify that the marginal on $S = \{0, i, j\}$ for $j > i \geq 1$ has the form

$$\begin{aligned} \kappa^S(000) &= \kappa^S(100) = \kappa^S(001) = \kappa^S(011) = 0, \\ \kappa^S(111) &= \kappa^S(110) = \kappa^S(101) = \kappa^S(010) = 1. \end{aligned}$$

Hence, by definition, we get $\{0\} \perp \{i\}|\{j\}(\kappa)$, but $\neg[\{0\} \perp \{j\}|\{i\}(\kappa)]$, $\neg[\{i\} \perp \{j\}|\{0\}(\kappa)]$, $\neg[\{0\} \perp \{i\}|\emptyset(\kappa)]$, $\neg[\{0\} \perp \{j\}|\emptyset(\kappa)]$, and $\neg[\{i\} \perp \{j\}|\emptyset(\kappa)]$. Thus conditions (a), (b), (c) are satisfied [we use symmetry in (c)]. Also condition (d) is valid; otherwise, by the decomposition property (see Corollary 1) we can derive $\{k\} \perp \{l\}|\emptyset(\kappa)$ for $k \neq l$, and this contradicts the conclusions above [either $\{k, l\} = \{0, i\}$ or $\{k, l\} = \{i, j\}$]. ■

In the rest of this section we will deal with the set of attributes $N = \{0, 1, \dots, n\}$ where $n \geq 3$. In this situation we use the following successor operation $\text{suc}: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$:

$$\text{suc}(i) = i + 1 \quad \text{whenever } i = 1, \dots, n - 1; \quad \text{suc}(n) = 1.$$

COROLLARY 2 *Supposing $N = \{0, 1, \dots, n\}$ ($n \geq 3$) and $s \in \{1, \dots, n\}$, there exists an NCF κ over N such that the dependency model*

$$\bigcup_{j \in \{1, \dots, n\} \setminus \{s\}} \{ \langle \{0\}, \{j\} | \{\text{suc}(j)\} \rangle, \langle \{j\}, \{0\} | \{\text{suc}(j)\} \rangle \}$$

is the CI-model induced by κ .

Proof As the successor operation permutes $\{1, \dots, n\}$, we can suppose without loss of generality $s = n$. Put $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where

$$\mathcal{D}_1 = \{D \subset N; \text{card } D \geq 4\},$$

$$\mathcal{D}_2 = \{D \subset N; \text{card } D = 3 \text{ \& } D \neq \{0, j, \text{suc}(j)\} \text{ for every } j = 1, \dots, n-1\}.$$

For each $D \in \mathcal{D}$ we find by Lemma 3 (where we put $S = D$) the corresponding NCF $\kappa[D]$ over N , and similarly apply Lemma 4 to find an NCF $\kappa[-]$ over N . Finally, using Proposition 1, construct an NCF κ over N whose induced CI-model is the intersection of CI-models induced by $\kappa[-]$ and $\kappa[D]$'s for $D \in \mathcal{D}$. Clearly, triplets $\langle \{0\}, \{j\} \{ \text{suc}(j) \} \rangle$ for $j = 1, \dots, n-1$ [and $\langle \{j\}, \{0\} \{ \text{suc}(j) \} \rangle$] belong to all these CI-models. It suffices to verify that no other triplet $\langle A, B|C \rangle \in T(N)$ does so:

if $A \cup B \cup C \in \mathcal{D}$, then $\langle A, B|C \rangle$ is not induced by $\kappa[A \cup B \cup C]$

if $C = \emptyset$ (especially $\text{card } A \cup B \cup C = 2$), then $\langle A, B|C \rangle$ is not induced by $\kappa[-]$.

Thus, only $\langle A, B|C \rangle \in T(N)$ with $A \cup B \cup C = \{0, j, \text{suc}(j)\}$ for some $j = 1, \dots, n-1$ and with $C \neq \emptyset$ remain. By Lemma 4(b),(c) we simply get that $\langle A, B|C \rangle$ is not induced by $\kappa[-]$. ■

The second important step is to prove that CI-models arising in the NCF-theory are closed under the collection of inference rules from [26] and [18]. This can be obtained as a consequence of the following proposition.

PROPOSITION 2 *Let $\{\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_n\}$ ($n \geq 3$) be a perpendicular collection of complete algebras on a nonempty set \mathbf{X} , and κ be an NCF on \mathbf{X} . Then it holds that*

$$\left[\forall i = 1, \dots, n, \mathcal{A} \perp \mathcal{B}_i | \mathcal{B}_{\text{suc}(i)}(\kappa) \right] \Rightarrow \left[\forall i = 1, \dots, n, \mathcal{A} \perp \mathcal{B}_{\text{suc}(i)} | \mathcal{B}_i(\kappa) \right]$$

The proof is in the Appendix (the fifth section). We were inspired in the proof by ideas from [23], where an uncomplete proof (in a special case) is given.

COROLLARY 3 *Supposing (S), let $N = \{0, \dots, n\}$ ($n \geq 3$), and let κ be an NCF over N . Then the following conditions are equivalent:*

- (a) $\forall j = 1, \dots, n, \{0\} \perp \{j\} \{ \text{suc}(j) \}(\kappa)$
- (b) $\forall j = 1, \dots, n, \{0\} \perp \{ \text{suc}(j) \} \{j\}(\kappa)$.

Now, the main result can be proved.

THEOREM 1 *Every system \mathcal{S} of inference rules characterizing the CI-models induced by NCFs as dependency models closed under \mathcal{S} has to be infinite.*

Proof Suppose by contradiction that \mathcal{S} is finite, and find $r \geq 3$ which exceeds the maximal number of antecedents in \mathcal{S} . Put

$$I = \bigcup_{j \in \{1, \dots, r\}} \{ \langle \{0\}, \{j\} \{ \text{suc}(j) \} \rangle, \langle \{j\}, \{0\} \{ \text{suc}(j) \} \rangle \}.$$

To show that I is closed under some inference rule \mathcal{R} from \mathcal{S} , consider a set of triplets $K \subset I$ which can be the set of antecedents of an instance of \mathcal{R} ; let $t \in T(N)$ be the corresponding consequent. As $\text{card } K < r$, by Corollary 2 find a CI-model J (induced by some NCF) such that $K \subset J \subset I$. Necessarily J is closed under \mathcal{R} (by the assumption that \mathcal{S} characterizes the CI-models induced by NCFs) and hence $t \in J \subset I$. Thus, I is closed under each inference rule \mathcal{R} from \mathcal{S} . Nevertheless, by Corollary 3, I is not a CI-model, and this contradicts the assumption about \mathcal{S} . ■

REMARK 2 Nevertheless, the CI-models induced by NCFs can be characterized by a countable system of inference rules with one consequent⁸ under the platonic assumption that all CI-models arising in NCF-theory are known. One can then construct these inference rules from so-called *minimal sound inference instances*⁹ exactly as in Proposition 2 in [26], where the proof is made for the probabilistic case. The property from Proposition 1 of this paper is the crucial fact enabling that construction.

4. MARGINAL PROBLEM

Dealing with the integration of knowledge in probabilistic expert systems [16, 8], we naturally meet with the problem of how to recognize whether for a system of prescribed less-dimensional probability measures there exists a “simultaneous” multidimensional probability measure having the prescribed measures as marginal measures, often called the *marginal problem*. The same process can be expected when we try to model epistemic states using the NCF-theory. Therefore this section is devoted to the analogous problem in the framework of NCFs.

DEFINITION 4 (Consistency, consonancy, solvability) *Suppose (S) and $\emptyset \neq \mathcal{Z} \subset (\text{exp } N) \setminus \{\emptyset\}$. A system of NCFs $\{\kappa_Z; Z \in \mathcal{Z}\}$, where κ_Z is an NCF on \mathbf{X}_Z , is called consistent iff there exists an NCF κ on \mathbf{X}_N having $\{\kappa_Z; Z \in \mathcal{Z}\}$ as marginals, i.e., $\forall Z \in \mathcal{Z}, \kappa^Z = \kappa_Z$.*

⁸Inference rules with one consequent are also called *Horn clauses* in the literature [5].

⁹It is a collection $[t_1, \dots, t_{r+1}] \in T(N)^{r+1}$ ($r \geq 1$), where $\{t_1, \dots, t_r\}$ “implies” t_{r+1} (i.e., each CI-model containing $\{t_1, \dots, t_r\}$ also contains t_{r+1}) but no proper subset of $\{t_1, \dots, t_r\}$ does so.

A system $\{\kappa_Z; Z \in \mathcal{Z}\}$ is called consonant iff its marginals coincide, i.e., for each couple $S, T \in \mathcal{Z}$ with $S \cap T \neq \emptyset$ it holds that $(\kappa_S)^{S \cap T} = (\kappa_T)^{S \cap T}$. Supposing that $N \neq \emptyset$ is a finite set of attributes, we shall say that a class $\emptyset \neq \mathcal{Z} \subset (\exp N) \setminus \{\emptyset\}$ is reduced iff each pair of its sets incomparable.¹⁰ Moreover, a class \mathcal{Z} will be called solvable iff for each assignment of sets of possible states $\mathbf{X}_i, i \in N$ (i.e. in any corresponding situation (S)) every consonant system of NCFs $\{\kappa_Z; Z \in \mathcal{Z}\}$, where κ_Z is an NCF on \mathbf{X}_Z for $Z \in \mathcal{Z}$, is also consistent.

By the marginal problem we shall understand the task of recognizing whether a given system of NCFs $\{\kappa_Z; Z \in \mathcal{Z}\}$ is consistent. Of course, the condition of consonancy, which can be easily verified, is a necessary condition. However, it is not sufficient, as the following example shows:

EXAMPLE 2 (Nonsolvable class) Put $N = \{1, 2, 3\}$ and $\mathcal{Z} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Then \mathcal{Z} is not solvable. To this end consider $\mathbf{X}_1 = \{a, a'\}$, $\mathbf{X}_2 = \{b, b'\}$, $\mathbf{X}_3 = \{c, c'\}$ and the following NCFs (given as point functions):

$$\kappa_{\{1,2\}}(ab) = \kappa_{\{1,2\}}(a'b') = 1, \quad \kappa_{\{1,2\}}(ab') = \kappa_{\{1,2\}}(a'b) = 0,$$

$$\kappa_{\{1,3\}}(ac) = \kappa_{\{1,3\}}(a'c') = 1, \quad \kappa_{\{1,3\}}(ac') = \kappa_{\{1,3\}}(a'c) = 0,$$

$$\kappa_{\{2,3\}}(bc) = \kappa_{\{2,3\}}(b'c') = 1, \quad \kappa_{\{2,3\}}(bc') = \kappa_{\{2,3\}}(b'c) = 0.$$

As their one-dimensional marginals are zero, $\{\kappa_Z; Z \in \mathcal{Z}\}$ is consonant. However, supposing that κ is an NCF on $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ having $\{\kappa_Z; Z \in \mathcal{Z}\}$ as marginals, we derive by the definition of the marginal NCF that $\kappa(abc) \geq \kappa^{\{1,2\}}(ab) = \kappa_{\{1,2\}}(a, b) = 1$, and similarly for the other points of $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$. Thus $\kappa \geq 1$, and this contradicts the primary condition on NCFs, $\min\{\kappa(x), x \in \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3\} = 0$.

Thus, as in the probabilistic case, we face the problem of how to recognize whether a consonant system is consistent. The solution of this problem in the probabilistic framework can be obtained asymptotically: the method defines by the so-called *iterative proportional fitting procedure* a sequence of multidimensional probability measures, and this sequence is shown [1] to converge iff there exists a "simultaneous" measure and even converges to one of the possible solutions.

¹⁰That is, neither $A \subset B$ nor $B \subset A$ for $A, B \in \mathcal{Z}$.

Nevertheless, the marginal problem in the NCF-theory has much simpler solution—we need not make iterations:

PROPOSITION 3 *Supposing (S), let $\{\kappa_Z; Z \in \mathcal{Z}\}$, where $\mathcal{Z} \subset (\exp N) \setminus \{\emptyset\}$ and $\bigcup \mathcal{Z} = N$, be a system of NCFs (κ_Z is an NCF on \mathbf{X}_Z). Then $\{\kappa_Z; Z \in \mathcal{Z}\}$ is consistent iff the formula*

$$\tilde{\kappa}([x_i]_{i \in N}) = \max\{\kappa_Z([x_i]_{i \in Z}); Z \in \mathcal{Z}\} \quad \text{for } [x_i]_{i \in N} \in \mathbf{X}_N$$

defines an NCF on \mathbf{X}_N having $\{\kappa_Z; Z \in \mathcal{Z}\}$ as marginals.

Proof Supposing that κ is an NCF on \mathbf{X}_N having $\{\kappa_Z; Z \in \mathcal{Z}\}$ as marginals, for each $[x_i]_{i \in N} \in \mathbf{X}_N$ and $Z \in \mathcal{Z}$ we have $\kappa([x_i]_{i \in N}) \geq \kappa^Z([x_i]_{i \in Z}) = \kappa_Z([x_i]_{i \in Z})$ and therefore $\kappa(x) \geq \tilde{\kappa}(x)$ for $x \in \mathbf{X}_N$. Hence $\min\{\kappa(x); x \in \mathbf{X}_N\} = 0$ implies $\min\{\tilde{\kappa}(x); x \in \mathbf{X}_N\} = 0$, i.e., κ is an NCF on \mathbf{X}_N , and also $\kappa^Z(y) \geq \tilde{\kappa}^Z(y)$ for $Z \in \mathcal{Z}$, $y \in \mathbf{X}_Z$. Nevertheless, it is not hard to see that $\tilde{\kappa}^Z(y) \geq \kappa_Z(y)$ for $Z \in \mathcal{Z}$, $y \in \mathbf{X}_Z$. Thus $\kappa^Z(y) \geq \tilde{\kappa}^Z(y) \geq \kappa_Z(y) = \kappa^Z(y)$ implies that $\tilde{\kappa}$ has $\{\kappa_Z; Z \in \mathcal{Z}\}$ as marginals. ■

REMARK Note that it may happen that $\tilde{\kappa}$ is an NCF but it does not have $\{\kappa_Z; Z \in \mathcal{Z}\}$ as marginals. We can obtain an example through a modification of Example 2, by changing $\kappa_{\{2,3\}}$ as follows: $\kappa_{\{2,3\}}(b'c') = 1$, $\kappa_{\{2,3\}}(bc) = \kappa_{\{2,3\}}(bc') = \kappa_{\{2,3\}}(b'c) = 0$.

It is a well-known old result of probability theory [9] that for a system $\mathcal{Z} \subset (\exp N) \setminus \{\emptyset\}$ the (probabilistic consonancy) is equivalent to (probabilistic) consistency iff the system \mathcal{Z} satisfies so-called *running intersection property* [8]:

$$(*) \left\{ \begin{array}{l} \text{there exists an ordering } Z_1, \dots, Z_n \text{ of elements of } \mathcal{Z} \text{ such that} \\ \forall j \geq 2, \exists i \ 1 \leq i < j \quad Z_j \cap \left(\bigcup_{k < j} Z_k \right) \subset Z_i. \end{array} \right.$$

The same characterization of solvable systems holds in the framework of NCFs. We will show it below,¹¹ where the following notation will be used.

NOTATION *Having a class $\emptyset \neq \mathcal{Z} \subset (\exp N) \setminus \{\emptyset\}$ and $i \in N$, we use the abbreviation $t_{\mathcal{Z}}(i)$ for $\text{card}\{Z \in \mathcal{Z}; i \in Z\}$. Moreover, the contraction of \mathcal{Z} to a nonempty set $S \subset N$, denoted by $\mathcal{Z} \wedge S$, is defined as the class of maximal sets of $\{Z \cap S; Z \in \mathcal{Z}, Z \cap S \neq \emptyset\}$.*

¹¹We were inspired by the method from [9]. We observed the main features of Kellerer's proof and simplified it a little. The essence of the method does not depend on a particular formalism—in fact we used it in [28] to prove an analogous result also for other calculi for dealing with uncertainty in AI.

LEMMA 5 *Whenever $\emptyset \neq \mathcal{Z} \subset (\exp N) \setminus \{\emptyset\}$ is a solvable class, then its contraction to a set $\emptyset \neq S \subset N$ is also solvable.*

Proof Having a consonant system of NCFs $\{\kappa_T; T \in \mathcal{Z} \wedge S\}$, consider the system $\{\kappa'_Z; Z \in \mathcal{Z}\}$, where we put $[x_i \in \mathbf{X}_i$ for $i \in Z]$:

$$\kappa'_Z([x_i]_{i \in Z}) = \begin{cases} (\kappa_R)^{Z \cap S}([x_i]_{i \in Z \cap S}) & \text{for } Z \cap S \subset R \in \mathcal{Z} \wedge S, \\ & \text{if } Z \cap S \neq \emptyset, \\ 0 & \text{if } Z \cap S = \emptyset. \end{cases}$$

The definition does not depend on the choice of R , as $\{\kappa_T; T \in \mathcal{Z} \wedge S\}$ is consonant. Clearly, $\{\kappa'_Z; Z \in \mathcal{Z}\}$ is consonant, and as \mathcal{Z} is solvable, there exists an NCF κ over N having $\{\kappa'_Z; Z \in \mathcal{Z}\}$ as marginals. Of course, it also has $\{\kappa_T; T \in \mathcal{Z} \wedge S\}$ as marginals. ■

LEMMA 6 *Suppose that $\emptyset \neq \mathcal{Z} \subset (\exp N) \setminus \{\emptyset\}$ contains a sequence Z_1, \dots, Z_n ($n \geq 3$) where for all $j = 1, \dots, n$ it holds $Z_j \cap Z_{j+1} \setminus \cup(\mathcal{Z} \setminus \{Z_j, Z_{j+1}\}) \neq \emptyset$ ($Z_{n+1} \equiv Z_1$). Then \mathcal{Z} is not solvable.*

Proof For every $J \in \{1, \dots, n\}$ choose $z_j \in Z_j \cap Z_{j+1} \setminus \cup(\mathcal{Z} \setminus \{Z_j, Z_{j+1}\})$ and put $S = \{z_j; j = 1, \dots, n\}$ ($z_{n+1} \equiv z_1$). By Lemma 5 it suffices to show that $\mathcal{Z} \wedge S = \{\{z_j, z_{j+1}\}; j \in \{1, \dots, n\}\}$ is not solvable. To this end consider $\mathbf{X}_i = \{0, 1\}$ for $i \in S$ and the following system of NCFs:

$$\begin{aligned} \kappa_T(00) = \kappa_T(11) = 0, \quad \kappa_T(01) = \kappa_T(10) = 1 \\ \text{for } T = \{z_j, z_{j+1}\}, \quad j \neq n, \\ \kappa_T \equiv 0 \quad \text{for } T = \{z_1, z_n\}. \end{aligned}$$

As the one-dimensional marginals are zero, $\{\kappa_T; T \in \mathcal{Z} \wedge S\}$ is consonant. Nevertheless, the NCF $\tilde{\kappa}$ over S (see Proposition 3) expressed by

$$\tilde{\kappa}([x_i]_{i \in S}) = \begin{cases} 0 & \text{if } [\forall i \in S, x_i = 0] \text{ or } [\forall i \in S, x_i = 1], \\ 1 & \text{otherwise} \end{cases}$$

has a nonzero marginal over $\{z_1, z_n\}$, and therefore $\{\kappa_T; T \in \mathcal{Z} \wedge S\}$ is not consistent. ■

LEMMA 7 *Let $\emptyset \neq \mathcal{Z} \subset (\exp N) \setminus \{\emptyset\}$ be a reduced class satisfying*

$$\forall i, j \in \cup \mathcal{Z} \quad \exists Z \in \mathcal{Z} \quad i, j \in Z.$$

Then \mathcal{Z} is solvable iff $\text{card } \mathcal{Z} = 1$.

Proof Suppose $\text{card } \mathcal{Z} \geq 2$, and put $S = N \setminus \cap \mathcal{Z}$. By Lemma 5 it suffices to show that $\mathcal{Z} \wedge S$ is not solvable, i.e., we can suppose $\cap \mathcal{Z} = \emptyset$.

Thus, consider $\mathbf{X}_i = \{0, 1\}$ for $i \in N$, and put for each $Z \in \mathcal{Z}$

$$\kappa_Z([\mathbf{x}_i]_{i \in Z}) = \begin{cases} 0 & \text{if } \sum_{i \in Z} x_i = 1, \\ 1 & \text{otherwise.} \end{cases}$$

It is no problem to see that for each $T \subset Z$, $\emptyset \neq T \neq Z$, it holds that

$$(\kappa_Z)^T([\mathbf{x}_i]_{i \in T}) = \begin{cases} 0 & \text{if } \sum_{i \in T} x_i \leq 1, \\ 1 & \text{otherwise,} \end{cases}$$

and therefore $\{\kappa_Z; Z \in \mathcal{Z}\}$ is consonant. Nevertheless, it is no problem to see that the function $\tilde{\kappa}$ from Proposition 3 is in this case identically equal to 1 and therefore $\{\kappa_Z; Z \in \mathcal{Z}\}$ is not consistent. ■

LEMMA 8 *Let $\mathcal{Z} \subset (\exp N) \setminus \{\emptyset\}$ be a reduced solvable class and $i \in \cup \mathcal{Z}$. If $k \in N$ satisfies $t_{\mathcal{Z} \wedge (N \setminus \{i\})}(k) = 1$, then at most one set $K \in \mathcal{Z}$ with $i, k \in K$ may exist, and the inequality $t_{\mathcal{Z}}(k) \leq 2$ holds.*

Proof Consider $\mathcal{B} = \mathcal{Z} \wedge (N \setminus \{i\})$; clearly there exists a unique $B \in \mathcal{B}$ with $k \in B$ and a unique $Z \in \mathcal{Z}$ with $B = Z \cap (N \setminus \{i\})$ [as \mathcal{Z} is reduced]. Now, suppose the existence of $I \in \mathcal{Z}$ with $i, k \in I$, and put $C = \cup \{I \in \mathcal{Z}; i, k \in I\}$. Of course $C \setminus \{i\} \subset B \subset Z$, and hence $\mathcal{Z} \wedge C$ satisfies the assumption of Lemma 7. Therefore there exists $K \in \mathcal{Z}$ with $\mathcal{Z} \wedge C = \{K \cap C\}$, i.e., $C \subset K$ —as \mathcal{Z} is reduced, and that implies the first conclusion. To see $t_{\mathcal{Z}}(k) \leq 2$ it suffices to realize that the only $I \in \mathcal{Z}$ with $k \in I$, $i \notin I$ would have to be Z . ■

LEMMA 9 *Let $\mathcal{Z} \subset (\exp N) \setminus \{\emptyset\}$ be a reduced solvable class with $\text{card } \mathcal{Z} \geq 2$. Then there exist two different sets $I, J \in \mathcal{Z}$ with*

$$\min_{i \in I \setminus J} t_{\mathcal{Z}}(i) = 1 = \min_{j \in J \setminus I} t_{\mathcal{Z}}(j).$$

Proof We will prove this lemma by induction on $n = \text{card } \cup \mathcal{Z}$. In the case $n \leq 2$ it is trivial; therefore suppose $n \geq 3$. The conclusions will be derived in three steps.

I. *Supposing $\min_{i \in \cup \mathcal{Z}} t_{\mathcal{Z}}(i) > 1$, there exists $z \in N$ with $t_{\mathcal{Z}}(z) = 2$, and for each such z the uniquely determined pair $Z_1, Z_2 \in \mathcal{Z}$ with $z \in Z_1 \cap Z_2$ satisfies $\min_{i \in Z_1 \setminus Z_2} t_{\mathcal{Z}}(i) = 2 = \min_{j \in Z_2 \setminus Z_1} t_{\mathcal{Z}}(j)$. Indeed, choose $z \in N$ with $t_{\mathcal{Z}}(z) = \min_{i \in \cup \mathcal{Z}} t_{\mathcal{Z}}(i)$, and put $\mathcal{B} = \mathcal{Z} \wedge (N \setminus \{z\})$. The hypothesis $\text{card } \mathcal{B} = 1$ leads to the conclusion $(\cup \mathcal{Z}) \setminus \{z\} \equiv I \in \mathcal{Z}$. As $t_{\mathcal{Z}}(i) \geq t_{\mathcal{Z}}(z) \geq 2$ for each $i \in I$, and \mathcal{Z} is reduced, we simply derive that the condition from Lemma 7 is fulfilled. Hence $\text{card } \mathcal{Z} = 1$, and this contradicts the assumption. Therefore \mathcal{B} is a reduced, solvable (Lemma 5)*

class with $\text{card } \mathcal{B} \geq 2$, and by the induction hypothesis there exist $K, L \in \mathcal{B}$ and $k \in K \setminus L, l \in L \setminus K$ with $t_{\mathcal{B}}(k) = 1 = t_{\mathcal{B}}(l)$. Clearly, by Lemma 8, $t_{\mathcal{Z}}(k) \leq 2$, and hence $t_{\mathcal{Z}}(z) = 2$. Now, again by Lemma 8, we see that there exists exactly one set $Z_1 \in \mathcal{Z}$ with $z, k \in Z_1$ [the existence follows from $t_{\mathcal{Z}}(k) \geq 2$ and $t_{\mathcal{B}}(k) = 1$]. Similarly, the only $Z_2 \in \mathcal{Z}$ contains both z and l . Moreover, $Z_1 \setminus \{z\} \subset K$ and $Z_2 \setminus \{z\} \subset L$ gives $k \notin Z_2$ and $l \notin Z_1$.

II. *There exists $i \in N$ with $t_{\mathcal{Z}}(i) = 1$.* Indeed, by contradiction we suppose $\min_{i \in \cup \mathcal{Z}} t_{\mathcal{Z}}(i) > 1$, and by repeated application of step I we find a sequence of sets $Z_1, \dots, Z_k \in \mathcal{Z}$ ($k \geq 3$) with $\min_{j \in Z_m \cap Z_{m+1}} t_{\mathcal{Z}}(j) = 2$ for $m = 1, \dots, k$ ($Z_{k+1} \equiv Z_1$). By Lemma 6 this contradicts the assumption that \mathcal{Z} is solvable.

III. *There exist $I, J \in \mathcal{Z}, I \neq J$, with*

$$\min_{i \in I \setminus J} t_{\mathcal{Z}}(i) = 1 = \min_{j \in J \setminus I} t_{\mathcal{Z}}(j).$$

Indeed, suppose $\text{card } \mathcal{Z} \geq 3$ (otherwise the result is trivial), and by step II find $i \in N$ with $t_{\mathcal{Z}}(i) = 1$ and put $\mathcal{B} = \mathcal{Z} \wedge N \setminus \{i\}$. As $\text{card } \mathcal{B} \geq 2$ (otherwise $\text{card } \mathcal{Z} \leq 2$), by the induction assumption there exist $K, J \in \mathcal{B}$ and $k \in K \setminus J, j \in J \setminus K$ with $t_{\mathcal{B}}(k) = 1 = t_{\mathcal{B}}(j)$. We can choose j in such a way that $j \notin I$, where I is the only set from \mathcal{Z} containing i . Then necessarily $J \in \mathcal{Z}$ and $j \notin I$ implies $t_{\mathcal{Z}}(j) = t_{\mathcal{B}}(j) = 1$. ■

THEOREM 2 *A nonempty class $\mathcal{Z} \subset (\exp N) \setminus \{\emptyset\}$ is solvable iff it satisfies the running intersection property:*

$$(*) \left\{ \begin{array}{l} \text{there exists an ordering } Z_1, \dots, Z_n \text{ of elements of } \mathcal{Z} \text{ such that} \\ \forall j \geq 2 \quad \exists i \ 1 \leq i < j \quad Z_j \cap \left(\bigcup_{k < j} Z_k \right) \subset Z_i. \end{array} \right.$$

Proof As a class is solvable iff the class of its maximal sets is, and the same principle holds for validity of (*), we can suppose that \mathcal{Z} is reduced. To show the necessity of (*), suppose $\text{card } \mathcal{Z} \geq 2$. The sequence in (*) can be constructed (backwards) if we show the existence of $I, J \in \mathcal{Z}, I \neq J$ with $I \cap (\cup(\mathcal{Z} \setminus \{I\})) \subset J$ (the class $\mathcal{Z} \setminus \{I\}$ is solvable by Lemma 5). To this end put $S = \{i \in N; t_{\mathcal{Z}}(i) \geq 2\}$. Suppose $S \neq \emptyset$ (otherwise the conclusion is clear), and put $\mathcal{B} = \mathcal{Z} \wedge S$. By Lemma 9 there exist $j \in N$ covered by a unique set $B \in \mathcal{B}$. Find some $J \in \mathcal{Z}$ with $B = J \cap S$. But $t_{\mathcal{Z}}(j) \geq 2$ implies the existence of $I \in \mathcal{Z}, I \neq J$, with $j \in I$. Nevertheless $I \cap \cup(\mathcal{Z} \setminus \{I\}) \subset I \cap S \subset B \subset J$ gives the desired conclusion.

To show the sufficiency of (*), first realize that a pair of sets $\{I, J\}$ is always solvable: whenever $\{\kappa_I, \kappa_J\}$ is a consonant system of NCFs, the formula $\kappa([x_i]_{i \in I \cup J}) = \max\{\kappa_I([x_i]_{i \in I}), \kappa_J([x_i]_{i \in J})\}$ defines an NCF over

$I \cup J$ having $\{\kappa_I, \kappa_J\}$ as marginals. Now, having an ordering $\{Z_1, \dots, Z_n\}$ from (*), consider a consonant system of NCFs $\{\kappa_{Z_i}; i = 1, \dots, n\}$. By induction on $k = 1, \dots, n$ we can construct an NCF κ^k over $\bigcup_{i \leq k} Z_i$ having $\{\kappa_{Z_i}; i \leq k\}$ as marginals: put $\kappa^1 \equiv \kappa_{Z_1}$, and for $k \geq 2$ construct κ^k from κ^{k-1} and κ_{Z_k} [κ^{k-1} and κ_{Z_k} are consonant owing to (*)], as $\{\bigcup_{i \leq k-1} Z_i, Z_k\}$ is a solvable class. ■

5. APPENDIX: PROOF OF PROPOSITION 2

LEMMA 10 *Let $\{\mathcal{A}, \mathcal{B}, \mathcal{E}\}$ be a perpendicular collection of complete algebras on $\mathbf{X} \neq \emptyset$, and κ be an NCF on \mathbf{X} . Then $\forall \mathbf{A} \in \text{at}(\mathcal{A}), \mathbf{B} \in \text{at}(\mathcal{B})$ one has*

$$\begin{aligned} & \min\{\kappa(\mathbf{A}|\mathbf{B} \cap \mathbf{C}); \mathbf{C} \in \text{at}(\mathcal{E})\} \\ & \leq \kappa(\mathbf{A}|\mathbf{B}) \leq \max\{\kappa(\mathbf{A}|\mathbf{B} \cap \mathbf{C}), \mathbf{C} \in \text{at}(\mathcal{E})\}. \end{aligned}$$

Proof Find $\mathbf{C}_0 \in \text{at}(\mathcal{E})$ with $\kappa(\mathbf{A} \cap \mathbf{B}) = \kappa(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}_0)$, and write

$$\begin{aligned} \kappa(\mathbf{A}|\mathbf{B}) &= \{\kappa(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}_0) - \kappa(\mathbf{B} \cap \mathbf{C}_0)\} + \{\kappa(\mathbf{B} \cap \mathbf{C}_0) - \kappa(\mathbf{B})\} \\ &\geq \kappa(\mathbf{A}|\mathbf{B} \cap \mathbf{C}_0) \geq \min\{\kappa(\mathbf{A}|\mathbf{B} \cap \mathbf{C}), \mathbf{C} \in \text{at}(\mathcal{E})\}. \end{aligned}$$

Analogously, find $\mathbf{C}_1 \in \text{at}(\mathcal{E})$ with $\kappa(\mathbf{B}) = \kappa(\mathbf{B} \cap \mathbf{C}_1)$, and write

$$\begin{aligned} \kappa(\mathbf{A}|\mathbf{B}) &= \{\kappa(\mathbf{A} \cap \mathbf{B}) - \kappa(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}_1)\} + \{\kappa(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}_1) - \kappa(\mathbf{B} \cap \mathbf{C}_1)\} \\ &\leq \kappa(\mathbf{A}|\mathbf{B} \cap \mathbf{C}_1) \leq \max\{\kappa(\mathbf{A}|\mathbf{B} \cap \mathbf{C}), \mathbf{C} \in \text{at}(\mathcal{E})\}. \quad \blacksquare \end{aligned}$$

LEMMA 11 *Let $\{\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_n\}$ ($n \geq 3$) be a perpendicular collection of algebras on $\mathbf{X} \neq \emptyset$, and κ be an NCF on \mathbf{X} . Suppose that for each $i = 1, \dots, n$ the class of atoms of \mathcal{B}_i is nontrivially decomposed into two parts:*

$$\text{at}(\mathcal{B}_i) = \mathcal{B}_i^- \cup \mathcal{B}_i^+ \quad \text{where } \mathcal{B}_i^- \neq \emptyset \neq \mathcal{B}_i^+ \text{ and } \mathcal{B}_i^- \cap \mathcal{B}_i^+ = \emptyset,$$

and consider the successor operation on $\{1, \dots, n\}$ (defined just before Corollary 2). Then $\forall \mathbf{A} \in \text{at}(\mathcal{A})$ one has

$$\begin{aligned} & \min_{i=1, \dots, n} \min\{\kappa(\mathbf{A} \cap \mathbf{B}^- \cap \mathbf{B}^+); \mathbf{B}^- \in \mathcal{B}_i^-, \mathbf{B}^+ \in \mathcal{B}_{\text{succ}(i)}^+\} \\ & = \min_{j=1, \dots, n} \min\{\kappa(\mathbf{A} \cap \mathbf{B}^+ \cap \mathbf{B}^-); \mathbf{B}^+ \in \mathcal{B}_j^+, \mathbf{B}^- \in \mathcal{B}_{\text{succ}(j)}^-\}. \end{aligned}$$

Moreover

$$\begin{aligned} \min_{i=1,\dots,n} \min\{\kappa(\mathbf{B}^- \cap \mathbf{B}^+); \mathbf{B}^- \in \mathcal{B}_i^-, \mathbf{B}^+ \in \mathcal{B}_{\text{suc}(i)}^+\} \\ = \min_{j=1,\dots,n} \min\{\kappa(\mathbf{B}^+ \cap \mathbf{B}^-); \mathbf{B}^+ \in \mathcal{B}_j^+, \mathbf{B}^- \in \mathcal{B}_{\text{suc}(j)}^-\}. \end{aligned}$$

Proof Consider the set \mathbf{Y} made of “mixed” atoms of $\mathcal{B}_1 + \dots + \mathcal{B}_n$:

$$\mathbf{Y} = \bigcup \left\{ \bigcap_{k=1}^n \mathbf{B}_k; [\forall k \mathbf{B}_k \in \text{at}(\mathcal{B}_k)] \& [\exists i \mathbf{B}_i \in \mathcal{B}_i^-] \& [\exists j \mathbf{B}_j \in \mathcal{B}_j^+] \right\}.$$

Owing to the perpendicularly assumption for all $\mathbf{A} \in \text{at}(\mathcal{A})$ it holds that

$$\begin{aligned} \mathbf{A} \cap \mathbf{Y} = \bigcup \left\{ \mathbf{A} \cap \bigcap_{k=1}^n \mathbf{B}_k; \right. \\ \left. [\forall k \mathbf{B}_k \in \text{at}(\mathcal{B}_k)] \& [\exists i \mathbf{B}_i \in \mathcal{B}_i^-] \& [\exists j \mathbf{B}_j \in \mathcal{B}_j^+] \right\}. \end{aligned}$$

Nevertheless, for each set $\mathbf{S} = \mathbf{A} \cap \bigcap_{k=1}^n \mathbf{B}_k$ included in this union there exists $i \in \{1, \dots, n\}$ with $\mathbf{B}_i \in \mathcal{B}_i^-$ and $\mathbf{B}_{\text{suc}(i)} \in \mathcal{B}_{\text{suc}(i)}^+$, i.e., $\mathbf{S} \subset \mathbf{A} \cap \mathbf{B}^- \cap \mathbf{B}^+$ for some $\mathbf{B}^- \in \mathcal{B}_i^-$, $\mathbf{B}^+ \in \mathcal{B}_{\text{suc}(i)}^+$, $i \in \{1, \dots, n\}$. Hence

$$\mathbf{A} \cap \mathbf{Y} = \bigcup_{i=1}^n \bigcup \{ \mathbf{A} \cap \mathbf{B}^- \cap \mathbf{B}^+; \mathbf{B}^- \in \mathcal{B}_i^-, \mathbf{B}^+ \in \mathcal{B}_{\text{suc}(i)}^+ \}. \quad (1)$$

Quite analogously we derive

$$\mathbf{A} \cap \mathbf{Y} = \bigcup_{j=1}^n \bigcup \{ \mathbf{A} \cap \mathbf{B}^+ \cap \mathbf{B}^-; \mathbf{B}^+ \in \mathcal{B}_j^+, \mathbf{B}^- \in \mathcal{B}_{\text{suc}(j)}^- \}. \quad (2)$$

Thus, if we express $\kappa(\mathbf{A} \cap \mathbf{Y})$ considering (1), then we get the left-hand side of the first desired equality, while if we express $\kappa(\mathbf{A} \cap \mathbf{Y})$ using (2) we get its right-hand side. The same consideration, this time with omitted \mathbf{A} , proves the second desired equality. \blacksquare

Proof of Proposition 2 This will be performed in three steps. Suppose that the collection $\{\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_n\}$ is perpendicular and $\forall i = 1, \dots, n$, $\mathcal{A} \perp \mathcal{B}_i | \mathcal{B}_{\text{suc}(i)}(\kappa)$.

I. Introduce for each $\mathbf{A} \in \text{at}(\mathcal{A})$ and $i \in \{1, \dots, n\}$

$$l_i(\mathbf{A}) = \min\{\kappa(\mathbf{A}|\mathbf{B}); \mathbf{B} \in \text{at}(\mathcal{B}_i)\}, \quad u_i(\mathbf{A}) = \max\{\kappa(\mathbf{A}|\mathbf{B}); \mathbf{B} \in \text{at}(\mathcal{B}_i)\}$$

Then

$$\forall A \in \text{at}(\mathcal{A}),$$

$$l_1(A) = \dots = l_n(A) \equiv l(A), \quad u_1(A) = \dots = u_n(A) \equiv u(A).$$

Indeed, having fixed A , i , and $B \in \text{at}(\mathcal{B}_i)$ write—by $\mathcal{A} \perp \mathcal{B}_i | \mathcal{B}_{\text{suc}(i)}(\kappa)$, Lemma 10, and again $\mathcal{A} \perp \mathcal{B}_i | \mathcal{B}_{\text{suc}(i)}(\kappa)$ —

$$\begin{aligned} l_{\text{suc}(i)}(A) &= \min\{\kappa(A|B'); B' \in \text{at}(\mathcal{B}_{\text{suc}(i)})\} \\ &= \min\{\kappa(A|B \cap B'); B' \in \text{at}(\mathcal{B}_{\text{suc}(i)})\} \leq \kappa(A|B) \\ &\leq \max\{\kappa(A|B \cap B'); B' \in \text{at}(\mathcal{B}_{\text{suc}(i)})\} \\ &= \max\{\kappa(A|B'); B' \in \text{at}(\mathcal{B}_{\text{suc}(i)})\} = u_{\text{suc}(i)}(A). \end{aligned}$$

Hence, by minimizing, respectively maximizing, through $B \in \text{at}(\mathcal{B}_i)$ we get

$$\forall A \in \text{at}(\mathcal{A}) \quad \forall i \in \{1, \dots, n\}, \quad l_{\text{suc}(i)}(A) \leq l_i(A) \leq u_i(A) \leq u_{\text{suc}(i)}(A).$$

As the successor operation is “cyclic”, this implies the equalities to be proved.

II. Introduce for each $A \in \text{at}(\mathcal{A})$ and $i \in \{1, \dots, n\}$

$$\mathcal{B}_i^-(A) = \{B \in \text{at}(\mathcal{B}_i); \kappa(A|B) = l_i(A)\},$$

$$\mathcal{B}_i^+(A) = \{B \in \text{at}(\mathcal{B}_i); \kappa(A|B) > l_i(A)\}.$$

Then for each $A \in \text{at}(\mathcal{A})$ there exists $i \in \{1, \dots, n\}$ with $\mathcal{B}_i^+(A) = \emptyset$. Indeed, fix $A \in \text{at}(\mathcal{A})$, and suppose by contradiction that $\mathcal{B}_i^+(A) \neq \emptyset$ for all $i \in \{1, \dots, n\}$. Thus, $\text{at}(\mathcal{B}_i) = \mathcal{B}_i^-(A) \cup \mathcal{B}_i^+(A)$ is a nontrivial decomposition of $\text{at}(\mathcal{B}_i)$, and we can apply Lemma 11 in the following. By step I there exists a shared value for $l_i(A)$, $i = 1, \dots, n$, denoted by $l(A)$. Thus, owing to $\mathcal{A} \perp \mathcal{B}_j | \mathcal{B}_{\text{suc}(j)}(\kappa)$ and the definition of $\mathcal{B}_{\text{suc}(j)}^-$, we get

$$\forall j \in \{1, \dots, n\} \quad \forall B^+ \in \mathcal{B}_j^+ \quad \forall B^- \in \mathcal{B}_{\text{suc}(j)}^-$$

$$\kappa(A|B^+ \cap B^-) = \kappa(A|B^-) = l(A),$$

i.e., $\kappa(A \cap B^+ \cap B^-) = \kappa(B^+ \cap B^-) + l(A)$ for $j \in \{1, \dots, n\}$, $B^+ \in \mathcal{B}_j^+$, $B^- \in \mathcal{B}_{\text{suc}(j)}^-$. Now, denoting

$$x = \min_{j=1, \dots, n} \min\{\kappa(A \cap B^+ \cap B^-); B^+ \in \mathcal{B}_j^+, B^- \in \mathcal{B}_{\text{suc}(j)}^-\},$$

$$y = \min_{j=1, \dots, n} \min\{\kappa(B^+ \cap B^-); B^+ \in \mathcal{B}_j^+, B^- \in \mathcal{B}_{\text{suc}(j)}^-\},$$

we easily get by minimization $x = y + l(\mathbf{A})$. Nevertheless, owing to $\mathcal{A} \perp \mathcal{B}_i | \mathcal{B}_{\text{suc}(i)}(\kappa)$ and the definition of $\mathcal{B}_{\text{suc}(i)}^+$, we can write

$$\begin{aligned} \forall i \in \{1, \dots, n\} \quad \forall \mathbf{B}^- \in \mathcal{B}_i^- \quad \forall \mathbf{B}^+ \in \mathcal{B}_{\text{suc}(i)}^+, \\ \kappa(\mathbf{A} | \mathbf{B}^- \cap \mathbf{B}^+) = \kappa(\mathbf{A} | \mathbf{B}^+) \geq l(\mathbf{A}) + 1, \end{aligned}$$

i.e.,

$$\begin{aligned} \kappa(\mathbf{A} \cap \mathbf{B}^- \cap \mathbf{B}^+) \geq \kappa(\mathbf{B}^- \cap \mathbf{B}^+) + l(\mathbf{A}) + 1 \\ \text{for } i \in \{1, \dots, n\}, \quad \mathbf{B}^- \in \mathcal{B}_i^-, \quad \mathbf{B}^+ \in \mathcal{B}_{\text{suc}(i)}^+. \end{aligned}$$

Hence, by minimization (now we actually use the equalities in Lemma 11) we derive $x \geq y + l(\mathbf{A}) + 1$, and this contradicts the previously derived equality. Thus, necessarily $\exists i \in \{1, \dots, n\} \mathcal{B}_i^+(\mathbf{A}) = \emptyset$.

III. $\forall i \in \{1, \dots, n\}, \mathcal{A} \perp \mathcal{B}_{\text{suc}(i)} | \mathcal{B}_i(\kappa)$. Indeed, having fixed $\mathbf{A} \in \text{at}(\mathcal{A})$ by step II, there exists $i \in \{1, \dots, n\}$ with $l_i(\mathbf{A}) = u_i(\mathbf{A})$ (see the notation in step I). But this, by step I, means that $\forall i \in \{1, \dots, n\}, l_i(\mathbf{A}) = u_i(\mathbf{A})$, i.e., there exists a number $l(\mathbf{A})$ such that

$$\forall i = 1, \dots, n \quad \forall \mathbf{B} \in \text{at}(\mathcal{B}_i), \quad \kappa(\mathbf{A} | \mathbf{B}) = l(\mathbf{A})$$

i.e.,

$$\kappa(\mathbf{A} \cap \mathbf{B}) = l(\mathbf{A}) + \lambda(\mathbf{B}).$$

The condition above means just $\mathcal{A}_i \perp \mathcal{B}_i | \{\emptyset, \mathbf{X}\}(\kappa)$ for all $i = 1, \dots, n$ [clearly $\kappa(\mathbf{A}) = l(\mathbf{A})$]. Thus $\mathcal{A} \perp \mathcal{B}_{\text{suc}(i)} | \{\emptyset, \mathbf{X}\}(\kappa)$ with $\mathcal{A} \perp \mathcal{B}_i | \mathcal{B}_{\text{suc}(i)}(\kappa)$ gives, by Lemma 2(c), $\mathcal{A} \perp (\mathcal{B}_i + \mathcal{B}_{\text{suc}(i)}) | \{\emptyset, \mathbf{X}\}(\kappa)$, and hence by Lemma 2(c) we finally get $\mathcal{A} \perp \mathcal{B}_{\text{suc}(i)} | \mathcal{B}_i(\kappa)$. ■

6. CONCLUSION

The significance of the main results proved here is as follows. Theorem 1 has above all a theoretical value. It says that, although in the NCF-theory different CI-models from those in probabilistic reasoning can arise (an example is in [27] or [23]), they cannot be characterized by means of a simple finite axiomatic system (similarly to the probabilistic case). Thus, the description of all CI-models in the NCF-theory seems to be a rather complicated problem.

On the other hand, one can restrict attention to special classes of CI-models. For example, Hunter [7] is interested in CI-models described by influence diagrams (i.e. directed acyclic graphs). Another possible approach to the description of CI-models (practiced in probabilistic reasoning) is to use undirected graphs, especially so-called triangulated or

chordal graphs, which give rise to the class of decomposable models (for details see [14]). These models correspond uniquely to (or can be equivalently described by) classes satisfying the running intersection property. In fact, this identification of triangulated graphs (or classes satisfying the running intersection property) with probabilistic CI-models is also possible owing to the result from [9] which is analogous to our Theorem 2. Thus, our second main result suggests that the very useful tool of decomposable models can be also transferred to the framework of NCFs.

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