The contact process seen from a typical infected site

Jan M. Swart

Institute of Information Theory and Automation of the ASCR (ÚTIA)
Pod vodárenskou věží 4, 18208 Praha 8, Czech Republic
swart@utia.cas.cz

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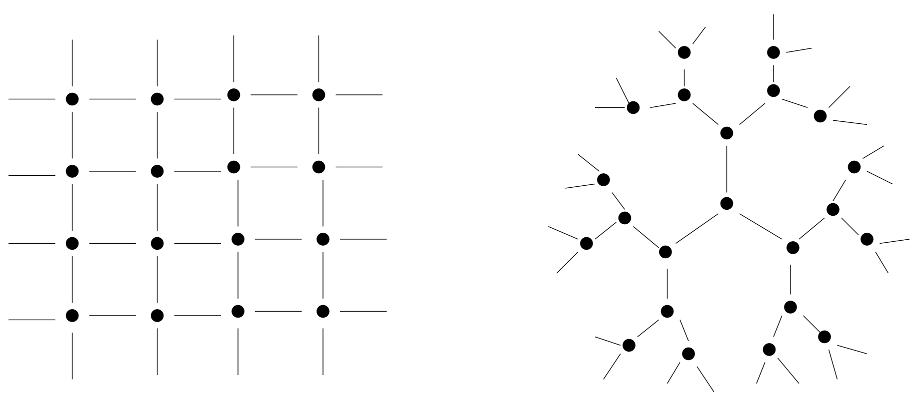
Abstract

We consider contact processes on general Cayley graphs. It is shown that any such contact process has a well-defined exponential growth rate, which can be related to the configuration seen from a 'typical' infected site at a 'typical' late time. Using this quantity, it is proved that on any nonamenable Cayley graph, the critical contact process dies out.

Cayley graphs

Let Λ be a countable, finitely generated group, with group action denoted by $(i, j) \mapsto ij$, inverse operation $i \mapsto i^{-1}$, and unit element (origin) 0. Let $\Delta \subset \Lambda$ be a finite generating set for Λ that is symmetric in the sense that $i \in \Delta$ implies $i^{-1} \in \Delta$. Then the (left) *Cayley graph* $\mathcal{G}(\Lambda, \Delta)$ associated with Λ and Δ is the graph with vertex set Λ, where there is an edge connecting $i, j \in \Lambda$ if and only if there is a $k \in \Delta$ such that $j = ki$. Examples of Cayley graphs are $\mathbb{Z}^d$, equipped with the usual nearest-neighbor structure, or the regular tree $\mathbb{T}_d$ in which each site has $d + 1$ neighbors.



$\mathbb{Z}^2$ $\qquad$ $\mathbb{T}_2$

Amenability and exponential growth

A Cayley graph $\mathcal{G}(\Lambda, \Delta)$ is called *amenable* if for every $\varepsilon > 0$ there exists a finite nonzero $A \subset \Lambda$ such that

$$\frac{|\partial A|}{|A|} \leq \varepsilon \quad \text{where} \quad \partial A := \big\{ i \notin A : \exists \text{ edge } (i,j) \text{ s.t. } j \in A \big\}.$$

This says that there exist large 'blocks' A whose surface can be made arbitrarily small compared to their volume. For example, $\mathbb{Z}^d$ is amenable but $\mathbb{T}_d$ ($d \geq 2$) is not.

A subadditivity argument shows that for each Cayley graph $\mathcal{G}(\Lambda, \Delta)$, the limit

$$\lim_{n \to \infty} \frac{1}{n} \log \big| \{ i \in \Lambda : d(0,i) \leq n \} \big|$$

exists, where $d(0,i)$ denotes the usual graph distance of a site i to the origin. The Cayley graph $\mathcal{G}(\Lambda, \Delta)$ is said to have *exponential growth* (resp. *subexponential growth*) if this limit is positive (resp. zero).

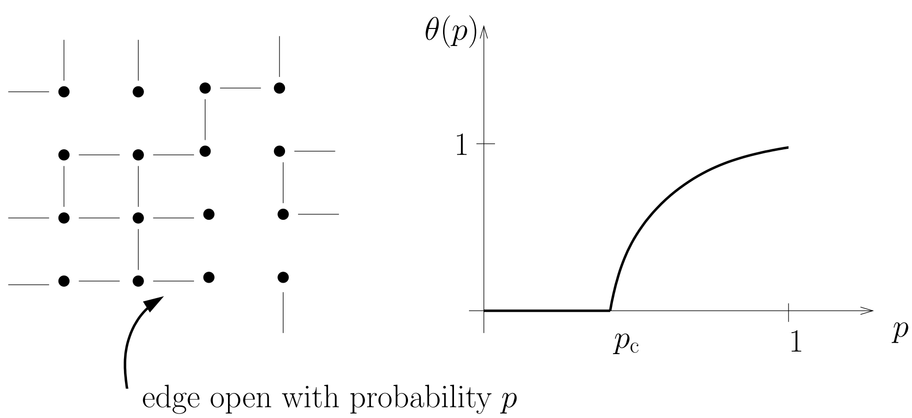
It can be shown that subexponential growth implies amenability, but the converse is not true. A counterexample is the lamplighter group.

Percolation

In (nearest-neighbor, Bernoulli) percolation on a Cayley graph $\mathcal{G}(\Lambda, \Delta)$, we independently make edges open with probability p and closed with the remaining probability. We let

$$\theta(p) := \mathbb{P}[0 \leftrightarrow \infty]$$

denote the probability that the origin is part of an infinite open cluster.

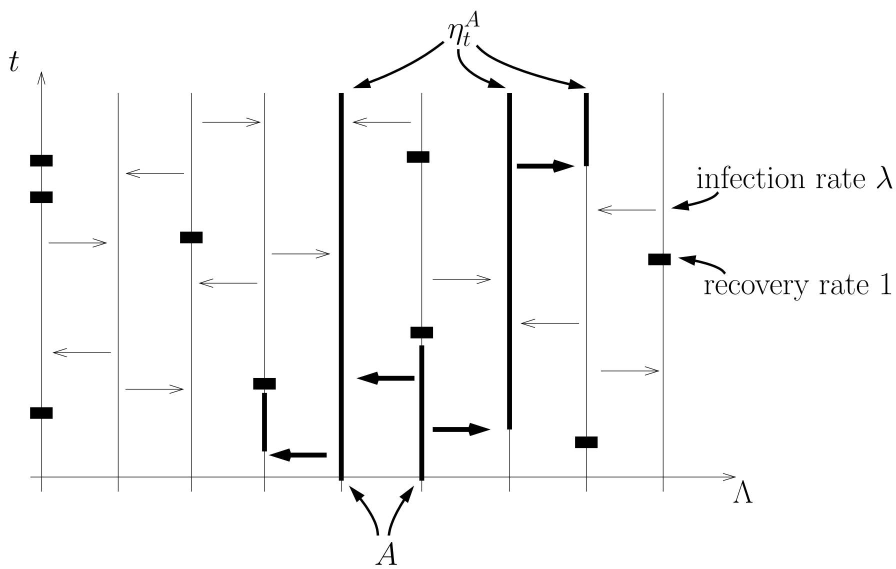


The graph of $\theta(p)$ is believed to be roughly as drawn above. In particular, there exists a critical percolation parameter $0 < p_c < 1$ such that $\theta(p) = 0$ for $p < p_c$ and $\theta(p) > 0$ for $p > p_c$. On $\mathbb{Z}^d$, it is known that $\theta(p_c) = 0$ in dimension $d = 2$ and dimensions $d \geq 19$. Proving this for $3 \leq d \leq 18$ is an open problem. For nonamenable graphs, the problem turns out to be easier than on $\mathbb{Z}^d$. In 1999, Benjamini, Lyons, Peres, and Schramm [BLPS99] proved that $\theta(p_c) = 0$ on any nonamenable Cayley graph.

Contact processes

The contact process with infection rate λ on a Cayley graph $\mathcal{G}(\Lambda, \Delta)$ is a Markov process $(\eta_t)_{t\geq 0}$ taking values in the subsets of Λ. If $i \in \eta_t$ then we say that the site i is infected at time $t \geq 0$; otherwise we say that the site is healthy. Infected sites infect healthy neighboring sites with rate λ, and infected sites become healthy with recovery rate 1.

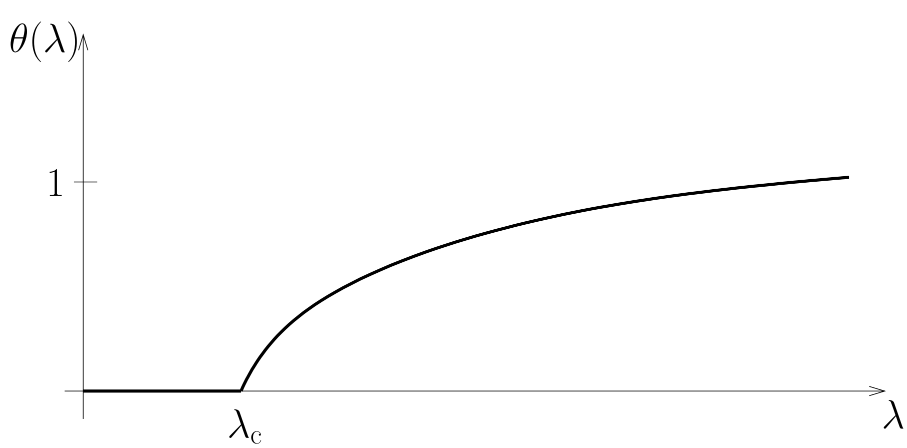
The contact process $(\eta_t^A)_{t\geq 0}$ started from the initial state A can be constructed with the help of a graphical representation. Here, a site i is infected at time t if there is a site $j \in A$ and an upward path from $(j, 0)$ to (i, t) that may follow infection arrows but must avoid recovery symbols.



We let

$$\begin{aligned}\theta(\lambda) &:= \mathbb{P}\big[\eta_t^{\{0\}} \neq \emptyset \ \forall t \geq 0\big] \\ &= \mathbb{P}\big[(0,0) \to \infty\big]\end{aligned}$$

denote the probability that the process started with one infected site survives. The graph of $\theta(\lambda)$ is believed to be roughly as follows:



In particular, there exists a critical infection rate $0 < \lambda_c < \infty$ such that $\theta(\lambda) = 0$ for $\lambda < \lambda_c$ and $\theta(\lambda) > 0$ for $\lambda > \lambda_c$. In the celebrated paper of Bezuidenhout and Grimmett [BG90], it is proved that $\theta(\lambda_c) = 0$ for the process on $\mathbb{Z}^d$ in all dimensions $d \geq 1$. The analogue result for trees has been proved by Morrow, Schinazi, and Zhang in [MSZ94]. The next result generalizes this, in the spirit of [BLPS99], to any nonamenable Cayley graph:

Theorem [Swa08] Assume that Λ is nonamenable. Then $\theta(\lambda_c) = 0$.

The exponential growth rate

A simple argument using subadditivity shows that each contact process on a Cayley graph has a well-defined exponential growth rate. More precisely, there exists a real constant $r = r(\lambda)$ such that the process started in any finite nonzero initial state satisfies:

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[|\eta_t|\big] = r.$$

If the Cayley graph has subexponential growth, then it is not hard to show that $r \leq 0$. In particular, on $\mathbb{Z}^d$, it is known that $r(\lambda) < 0$ for $\lambda < \lambda_c$ and $r(\lambda) = 0$ for $\lambda \geq \lambda_c$ [BG91]. On the other hand, on graphs with exponential growth, it is possible that $r > 0$. In many ways, the function $r(\lambda)$ is easier to study than the function $\theta(\lambda)$. The Theorem above is a result of the following proposition.

Proposition 1 For any Cayley graph $\mathcal{G}(\Lambda, \Delta)$:

(a) The function $\lambda \mapsto r(\lambda)$ is Lipschitz continuous.

(b) $r(\lambda) > 0$ implies $\theta(\lambda) > 0$.

(c) If Λ is nonamenable and $\theta(\lambda) > 0$, then $r(\lambda) > 0$.

The process seen from a typical site

Proposition 1 (c) is proved by relating the exponential growth rate r to the configuration seen from a typical infected site at a typical late time.

Definitions

The space $\mathcal{P}(\Lambda) := \{A : A \subset \Lambda\}$ of all subsets of Λ can in a natural way be identified with $\{0,1\}^\Lambda$, which is a compact space under the product topology. In this topology, $\mathcal{P}_+(\Lambda) := \{A \in \mathcal{P}(\Lambda) : A \neq \emptyset\}$ is a locally compact space. We define locally finite measures μ_t on $\mathcal{P}_+(\Lambda)$ by

$$\mu_t := \sum_{i\in\Lambda} \mathbb{P}[\eta_t^{\{i\}} \in \cdot\,]\big|_{\mathcal{P}_+(\Lambda)} \qquad (t \geq 0),$$

where $|_{\mathcal{P}_+(\Lambda)}$ denotes restriction of a measure to $\mathcal{P}_+(\Lambda)$. Think of μ_t as the law at time t of the process started with one infected site, distributed according to the uniform distribution on Λ. Conditioning μ_t on the origin being infected yields a probability measure, describing the configuration seen from a typical infected site at time t.

We set

$$\hat{\mu}_\alpha := \frac{1}{Z_\alpha}\int_0^\infty \mu_t\, e^{-\alpha t}\mathrm{d}t \qquad (\alpha > r),$$

where Z_α is a normalization constant such that $\hat{\mu}_\alpha\{A : 0 \in A\} = 1$.

Proposition 2 The measures $\{\hat{\mu}_\alpha : \alpha > r\}$ are tight in the topology of vague convergence, and each vague limit as $\alpha \downarrow r$ yields an 'eigenmeasure' with eigenvalue r (as defined below).

Eigenmeasures

By definition, we say that a locally finite measure μ on $\mathcal{P}_+(\Lambda)$ is an *eigenmeasure* with eigenvalue α of a contact process if

$$\int \mu(\mathrm{d}A)\,\mathbb{P}\big[\eta_t^A \in \cdot\,\big]\big|_{\mathcal{P}_+(\Lambda)} = e^{\alpha t}\mu \qquad (t \geq 0).$$

Note that this says that if we start the process in the (possibly infinite) measure μ, then, up to an exponential factor, we get back the same law at any later time. Consider the 'spectrum'

$$\mathcal{E}(\lambda) := \big\{\alpha \in \mathbb{R} : \begin{array}{l}\text{there exists a spatially homogeneous eigenmeasure}\\ \text{with eigenvalue } \alpha \text{ for the contact process with}\\ \text{infection rate } \lambda\big\}.\end{array}$$

Proposition 2 shows that $r \in \mathcal{E}(\alpha)$. More generally, one has:

Proposition 3 $\mathcal{E}(\lambda)$ is a compact subset of $\mathbb{R}$ and $r(\lambda) = \max \mathcal{E}(\lambda)$.

The upper invariant law

Extending the graphical representation to negative times and setting

$$\overline{\eta}_t := \{i \in \Lambda : -\infty \to (i,t)\} \qquad (t \in \mathbb{R})$$

defines a stationary contact process $(\overline{\eta}_t)_{t\geq 0}$, whose stationary law $\overline{\nu} := \mathbb{P}[\eta_t \in \cdot\,]$ is called the *upper invariant law*. By reversing the direction of all arrows and turning the graphical representation upside down, it is not hard to see that $\overline{\nu}$ is nontrivial (i.e., concentrated on $\mathcal{P}_+(\Lambda)$) if and only if the contact process survives.

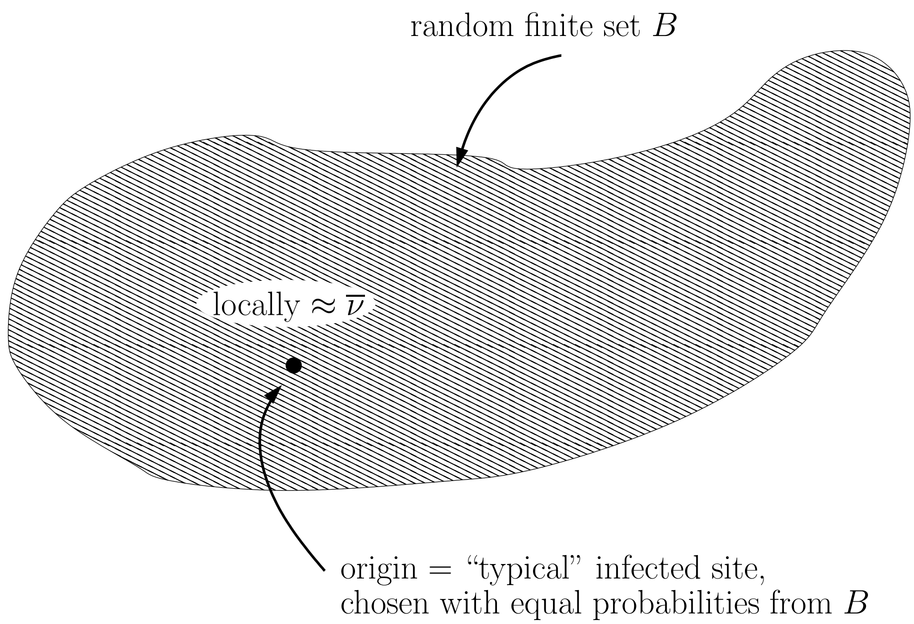
There exists a not too difficult proof, that works on any Cayley graph, that if a contact process survives, then $\overline{\nu}$ is its unique nontrivial spatially homogeneous invariant law. In a similar fashion, one can prove the following, stronger fact:

Proposition 4 If a contact process on a Cayley graph survives, then, up to a multiplicative constant, the upper invariant law $\overline{\nu}$ is the only spatially homogeneous eigenmeasure with eigenvalue zero.

Proof of Proposition 1 (c) (sketch)

Assume that a contact process on a Cayley graph survives, and its exponential growth rate $r(\lambda)$ is zero. Then, by Propositions 2 and 4, the vague limit $\lim_{\alpha\downarrow 0}\hat\mu_\alpha$ exists and is up to a multiplicative constant equal to $\overline{\nu}$.

Consider the law $\hat\mu_\alpha$, conditioned on the event $\{A : 0 \in A\}$. By our previous remarks, for α close to zero, this law describes a random finite set B, containing the origin, that looks something like this:



Since seen from the origin, we see something that looks like the spatially homogeneous law $\overline{\nu}$, we conclude that 0 lies with high probability far from the outer boundary of B. Since 0 is a 'typical' site, this contradicts nonamenability, which says that in any finite set B, a positive fraction of the sites must lie near the boundary.

References

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