

# Zkouška z kvantové pravděpodobnosti

22 června, 2007

**Příklad 1. (Časový vývoj)** Bud'  $\mathcal{H}$  konečně rozměrný unitární prostor (t.j. prostor se skalárním součinem) nad  $\mathbb{C}$ . Necht' je  $K \in \mathcal{L}(\mathcal{H})$  hermitovské a  $\rho$  stav na  $\mathcal{L}(\mathcal{H})$ . Definujme

$$\rho_t(A) := \rho(e^{-itK} A e^{itK}) \quad (A \in \mathcal{L}(\mathcal{H})).$$

- (a) Ukažte, že  $\rho_t$  je stav na  $\mathcal{L}(\mathcal{H})$  pro každé  $t \in \mathbb{R}$ .  
(b) Ukažte, že  $\rho_t$  je pro každé  $t \in \mathbb{R}$  čistý stav, je-li  $\rho$  čistý stav.

**Příklad 2. (Excercise 4.3.3 ve skriptech)** Bud'  $\mathcal{A}$  algebra.

- (a) Dokažte, že  $\mathcal{A}$ , opatřená akcí  $(A, B) \mapsto AB$ , je věrnou representací sebe sama.  
(b) Předpokládejme, že  $\mathcal{A}$  je  $*$ -algebra a  $\tau$  je věrná pseudostopa na  $\mathcal{A}$ . Dokažte, že  $\mathcal{A}$  opatřená skalárním součinem  $\langle \cdot | \cdot \rangle_\tau$  je representací sebe sama ve smyslu  $*$ -algeber.

**Příklad 3. (Propletenost)**

(a) Bud' te  $\mathcal{H}_1$  a  $\mathcal{H}_2$  konečně rozměrné unitární prostory nad  $\mathbb{C}$ . Necht' je  $\rho$  stav na  $\mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_2) \cong \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ . Definujme marginálny  $\rho_1$  a  $\rho_2$  stavu  $\rho$  předpisem

$$\begin{aligned} \rho_1(A) &:= \rho(A \otimes 1) & (A \in \mathcal{L}(\mathcal{H}_1)), \\ \rho_2(A) &:= \rho(1 \otimes A) & (A \in \mathcal{L}(\mathcal{H}_2)). \end{aligned}$$

Ukažte, že  $\rho = \rho_1 \otimes \rho_2$ , pokud je  $\rho_1$  čistý stav. Návod: Stačí ukázat, že pro libovolné  $A \in \mathcal{L}(\mathcal{H}_1)$  a pro každý projektor  $P \in \mathcal{L}(\mathcal{H})$  platí  $\rho(A \otimes P) = \rho_1(A)\rho_2(P)$ . K tomu použijte formuli pro 'podmíněnou pravděpodobnost':

$$\rho(A \otimes 1) = \frac{\rho(A \otimes P)}{\rho(1 \otimes P)} \rho(1 \otimes P) + \frac{\rho(A \otimes (1 - P))}{\rho(1 \otimes (1 - P))} \rho(1 \otimes (1 - P)).$$

Co víte o 'podmíněném stavu'

$$A \mapsto \frac{\rho(A \otimes P)}{\rho(1 \otimes P)} \quad (A \in \mathcal{L}(\mathcal{H}_1)) ?$$

(b) Necht' jsou  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  a  $\mathcal{H}_3$  dvourozměrné unitární prostory nad  $\mathbb{C}$ , s orthonormálními bázemi  $\{e(1), e(2)\}$ ,  $\{f(1), f(2)\}$ , respektive  $\{g(1), g(2)\}$ . Necht' je  $\rho$  stav na  $\mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_2) \otimes \mathcal{L}(\mathcal{H}_3)$ . Uvažujte marginálny

$$\begin{aligned} \rho_{12}(A_1 \otimes A_2) &:= \rho(A_1 \otimes A_2 \otimes 1) & (A_1 \in \mathcal{L}(\mathcal{H}_1), A_2 \in \mathcal{L}(\mathcal{H}_2)), \\ \rho_{23}(A_2 \otimes A_3) &:= \rho(1 \otimes A_2 \otimes A_3) & (A_2 \in \mathcal{L}(\mathcal{H}_2), A_3 \in \mathcal{L}(\mathcal{H}_3)). \end{aligned}$$

Předpokládejme, že  $\rho_{12}$  je propletený stav odpovídající stavovému vektoru

$$\psi := \frac{1}{\sqrt{2}}(e(1) \otimes f(1) + e(2) \otimes f(2)).$$

Ukažte, že  $\rho_{23}$  není propletený. Návod: proved'te ideální měření na  $\mathcal{L}(\mathcal{H}_1)$ , abyste rozložili  $\rho_2$  na čisté stavy.

# Exam Quantum Probability

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**Exercise 1 (Time evolution)** Let  $\mathcal{H}$  be a finite-dimensional inner product space over  $\mathbb{C}$ . Let  $K \in \mathcal{L}(\mathcal{H})$  be hermitian and let  $\rho$  be a state on  $\mathcal{L}(\mathcal{H})$ . Define

$$\rho_t(A) := \rho(e^{-itK} A e^{itK}) \quad (A \in \mathcal{L}(\mathcal{H})).$$

(a) Show that  $\rho_t$  is a state on  $\mathcal{L}(\mathcal{H})$  for all  $t \in \mathbb{R}$ .

(b) If  $\rho$  is a pure state, then show that  $\rho_t$  is also pure, for all  $t \in \mathbb{R}$ .

**Exercise 2 (Excercise 4.3.3 from the lecture notes)** Let  $\mathcal{A}$  be an algebra.

(a) Show that  $\mathcal{A}$ , equipped with the action  $(A, B) \mapsto AB$ , becomes a faithful representation of itself.

(b) If  $\mathcal{A}$  is a  $*$ -algebra and  $\tau$  is a faithful pseudotrace on  $\mathcal{A}$ , then show that  $\mathcal{A}$  equipped with the inner product  $\langle \cdot | \cdot \rangle_\tau$  is a representation of itself as a  $*$ -algebra.

**Exercise 3 (Entanglement)**

(a) Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be finite-dimensional inner product spaces over  $\mathbb{C}$ . Let  $\rho$  be a state on  $\mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_2) \cong \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ . The *marginals*  $\rho_1$  and  $\rho_2$  of  $\rho$  are defined by

$$\begin{aligned} \rho_1(A) &:= \rho(A \otimes 1) \quad (A \in \mathcal{L}(\mathcal{H}_1)), \\ \rho_2(A) &:= \rho(1 \otimes A) \quad (A \in \mathcal{L}(\mathcal{H}_2)). \end{aligned}$$

If  $\rho_1$  is a pure state, then show that  $\rho = \rho_1 \otimes \rho_2$ . Hint: It suffices to show that  $\rho(A \otimes P) = \rho_1(A)\rho_2(P)$  for any  $A \in \mathcal{L}(\mathcal{H}_1)$  and for any projection  $P \in \mathcal{L}(\mathcal{H})$ . Now use the ‘conditional probability’ formula:

$$\rho(A \otimes 1) = \frac{\rho(A \otimes P)}{\rho(1 \otimes P)} \rho(1 \otimes P) + \frac{\rho(A \otimes (1 - P))}{\rho(1 \otimes (1 - P))} \rho(1 \otimes (1 - P)).$$

What do you know about the ‘conditioned state’

$$A \mapsto \frac{\rho(A \otimes P)}{\rho(1 \otimes P)} \quad (A \in \mathcal{L}(\mathcal{H}_1)) ?$$

(b) Let  $\mathcal{H}_1$ ,  $\mathcal{H}_2$ , and  $\mathcal{H}_3$  be two-dimensional inner product spaces over  $\mathbb{C}$ , with orthonormal bases  $\{e(1), e(2)\}$ ,  $\{f(1), f(2)\}$ , and  $\{g(1), g(2)\}$ , respectively. Let  $\rho$  be a state on  $\mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_2) \otimes \mathcal{L}(\mathcal{H}_3)$ . Consider the marginals

$$\begin{aligned} \rho_{12}(A_1 \otimes A_2) &:= \rho(A_1 \otimes A_2 \otimes 1) \quad (A_1 \in \mathcal{L}(\mathcal{H}_1), A_2 \in \mathcal{L}(\mathcal{H}_2)), \\ \rho_{23}(A_2 \otimes A_3) &:= \rho(1 \otimes A_2 \otimes A_3) \quad (A_2 \in \mathcal{L}(\mathcal{H}_2), A_3 \in \mathcal{L}(\mathcal{H}_3)). \end{aligned}$$

Assume that  $\rho_{12}$  is the entangled state corresponding to the state vector

$$\psi := \frac{1}{\sqrt{2}}(e(1) \otimes f(1) + e(2) \otimes f(2)).$$

Show that  $\rho_{23}$  is not entangled. Hint: perform an ideal measurement on  $\mathcal{L}(\mathcal{H}_1)$  to decompose  $\rho_2$  into pure states.

# Solutions

## Excercise 1

(a) Since  $K$  is hermitian we can find an orthonormal basis such that

$$K = \sum_i \lambda_i |e(i)\rangle\langle e(i)|,$$

where the eigenvalues  $\lambda_i$  are real. Now  $e^{itK}$ , defined with the functional calculus for normal operators, is given by

$$e^{itK} = \sum_i e^{it\lambda_i} |e(i)\rangle\langle e(i)|.$$

Since the eigenvalues of  $e^{itK}$  have norm one,  $e^{itK}$  is a unitary operator. Moreover,

$$e^{itK} e^{-itK} = \left( \sum_i e^{it\lambda_i} |e(i)\rangle\langle e(i)| \right) \left( \sum_i e^{-it\lambda_i} |e(i)\rangle\langle e(i)| \right) = \sum_i |e(i)\rangle\langle e(i)| = 1,$$

so  $e^{-itK}$  is the inverse of  $e^{itK}$ . For simplicity, we write  $U_t := e^{itK}$  and  $U_t^* = U_t^{-1} := e^{-itK}$ . We must check that  $A \mapsto \rho_t(A) := \rho(U_t^* A U_t)$  is linear, real, and positive, and that  $\rho_t(1) = 1$ . Linearity is trivial. Realness follows by writing

$$\rho_t(A^*) = \rho(U_t^* A^* U_t) = \rho((U_t^* A U_t)^*) = \rho(U_t^* A U_t)^* = \rho_t(A)^*.$$

Positivity follows by writing

$$\rho_t(A^* A) = \rho(U_t^* A^* A U_t) = \rho((A U_t)^*(A U_t)) \geq 0.$$

Finally,  $\rho_t(1) = \rho(U_t^* U_t) = \rho(1) = 1$ .

(b) If  $\rho$  is a pure state then there exists a vector  $\psi$  of norm one such that  $\rho(A) = \langle \psi | A | \psi \rangle$  ( $A \in \mathcal{L}(\mathcal{H})$ ). Now

$$\rho_t(A) = \rho(U_t^* A U_t) = \langle \psi | U_t^* A U_t | \psi \rangle = \langle U_t \psi | A | U_t \psi \rangle$$

so  $\rho_t$  is the pure state corresponding to the unit vector  $U_t \psi$ . (Note that since  $U_t$  is unitary, it preserves the norm.)

## Excercise 2

(a) Obviously

- |     |                          |   |
|-----|--------------------------|---|
| (a) | $A(bB + cC) = bAB + cAC$ | $(b, c \in \mathbb{K}, A, B, C \in \mathcal{A}),$ |
| (b) | $(aA + bB)C = aAC + bBC$ | $(a, b \in \mathbb{K}, A, B, C \in \mathcal{A}),$ |
| (c) | $(AB)C = A(BC)$          | $(A, B, C \in \mathcal{A}),$                      |
| (d) | $1A = \phi$              | $(A \in \mathcal{A}),$                            |

so we satisfy the conditions (a)–(d) from page 49 of the lecture notes, hence  $\mathcal{A}$ , equipped with the action  $(A, B) \mapsto AB$ , is a representation of itself. Let  $l : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A})$  be the

corresponding algebra homomorphism. To see that it is faithful it suffices to show that  $\text{Ker}(l) = \{0\}$ , i.e., we must show that  $A \neq 0$  implies that  $AB \neq 0$  for some  $B \in \mathcal{A}$ . This is obviously satisfied if we take  $B = 1$ .

**(b)** We must check condition (e) from page 50 of the lecture notes. We write

$$\langle A|BC\rangle_\tau = \tau(A^*(BC)) = \tau((A^*B)C) = \tau((B^*A)^*C) = \langle B^*A|C\rangle_\tau,$$

which is true for all  $A, B, C \in \mathcal{A}$ .

### Excercise 3

**(a)** The ‘conditional probability’ formula writes  $\rho_1$  as a convex combination of two states. Since  $\rho_1$  is pure, these states must both be equal to  $\rho_1$ . In particular,

$$\frac{\rho(A \otimes P)}{\rho(1 \otimes P)} = \rho(A \otimes 1),$$

which implies  $\rho(A \otimes P) = \rho_1(A)\rho_2(P)$ . Recall that the projections span an algebra so by linearity this implies  $\rho(A \otimes B) = \rho_1(A)\rho_2(B)$  for all  $A \in \mathcal{L}(\mathcal{H}_1)$  and  $B \in \mathcal{L}(\mathcal{H}_1)$ .

**(b)** Let  $P$  be the projection on  $e(1)$ . If we perform the ideal measurment  $\{P, 1 - P\}$  on  $\mathcal{L}(\mathcal{H}_1)$ , then this will tell us in which pure state the subsystem  $\mathcal{L}(\mathcal{H}_2)$  is. More precisely, we can write

$$\rho(1 \otimes A_2 \otimes A_3) = \frac{\rho(P \otimes A_2 \otimes A_3)}{\rho(P \otimes 1 \otimes 1)}\rho(P \otimes 1 \otimes 1) + \frac{\rho((1 - P) \otimes A_2 \otimes A_3)}{\rho((1 - P) \otimes 1 \otimes 1)}\rho((1 - P) \otimes 1 \otimes 1).$$

Hence

$$\rho_{23} = \rho(P \otimes 1 \otimes 1)\rho'_{23} + \rho((1 - P) \otimes 1 \otimes 1)\rho''_{23},$$

where

$$\begin{aligned}\rho'_{23}(A_2 \otimes A_3) &:= \frac{\rho(P \otimes A_2 \otimes A_3)}{\rho(P \otimes 1 \otimes 1)}, \\ \rho''_{23}(A_2 \otimes A_3) &:= \frac{\rho((1 - P) \otimes A_2 \otimes A_3)}{\rho((1 - P) \otimes 1 \otimes 1)}.\end{aligned}$$

Let

$$\begin{aligned}\rho'_2(A_2) &:= \rho'_{23}(A_2 \otimes 1) & (A_2 \in \mathcal{L}(\mathcal{H}_2)), \\ \rho''_2(A_2) &:= \rho''_{23}(A_2 \otimes 1) & (A_2 \in \mathcal{L}(\mathcal{H}_2)).\end{aligned}$$

We claim that  $\rho'_2$  is the pure state corresponding to the vector  $f(1)$  and  $\rho''_2$  is the pure state corresponding to the vector  $f(2)$ . Hence, by part (a),  $\rho'_{23}$  and  $\rho''_{23}$  are product states and  $\rho_{23}$  is not entangled. To see this, just write

$$\rho'_2(A_2) = \frac{\rho(P \otimes A_2 \otimes 1)}{\rho(P \otimes 1 \otimes 1)} = \frac{\rho_{12}(P \otimes A_2)}{\rho_{12}(P \otimes 1)},$$

where

$$\begin{aligned}\rho_{12}(P \otimes A_2) &= \frac{1}{2}\langle e(1) \otimes f(1) + e(2) \otimes f(2) | P \otimes A_2 | e(1) \otimes f(1) + e(2) \otimes f(2) \rangle \\ &= \frac{1}{2}\langle e(1) \otimes f(1) + e(2) \otimes f(2) | Pe(1) \otimes A_2 f(1) + Pe(2) \otimes A_2 f(2) \rangle \\ &= \frac{1}{2}\langle e(1) \otimes f(1) + e(2) \otimes f(2) | e(1) \otimes A_2 f(1) \rangle = \langle f(1) | A_2 | f(1) \rangle.\end{aligned}$$

We can calculate  $\rho''_2$  similarly.