Correction to: Branching-coalescing particle systems

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Abstract

In the article titled "Branching-Coalescing Particle Systems" published in *Probability Theory and Related Fields 131(3), pages 376–414, (2005)*, Theorem 7 as stated there is incorrect. Indeed, we show by counterexample that the equality that we claimed there to hold for all time, in general holds only for almost every time with respect to Lebesgue measure. We prove a weaker version of the theorem that is still sufficient for our applications in the mentioned paper.

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1 Introduction

In this note we repair an error in [AS05, Theorem 7]. We start by stating the corrected theorem.

If E be a metrizable space, we denote by $M(E), B(E)$ the spaces of real Borel measurable and bounded real Borel measurable functions on E, respectively. If A is a linear operator from a domain $\mathcal{D}(A) \subset M(E)$ into $M(E)$ and X is an E-valued process, then we say that X solves the martingale problem for A if X has cadlag sample paths and for each $f \in \mathcal{D}(A)$,

$$E\big[|f(X_t)|\big] < \infty \quad \text{and} \quad \int_0^t E\big[|Af(X_s)|\big]\mathrm{d}s < \infty \qquad (t \geq 0), \tag{1.1}$$

and the process $(M_t)_{t\geq 0}$ defined by

$$M_t := f(X_t) - \int_0^t Af(X_s)\mathrm{d}s \qquad (t \geq 0) \tag{1.2}$$

is a martingale with respect to the filtration generated by X.

We will prove the following theorem.

Theorem 1 (Duality with error term) *Assume that E_1, E_2 are metrizable spaces and that for $i = 1, 2$, A_i is a linear operator from a domain $\mathcal{D}(A_i) \subset B(E_i)$ into $M(E_i)$. Assume that $\Psi \in B(E_1 \times E_2)$ satisfies $\Psi(\cdot, x_2) \in \mathcal{D}(A_1)$ and $\Psi(x_1, \cdot) \in \mathcal{D}(A_2)$ for each $x_1 \in E_1$ and $x_2 \in E_2$, and that*

$$\Phi_1(x_1, x_2) := A_1\Psi(\,\cdot\,, x_2)(x_1) \quad \textit{and} \quad \Phi_2(x_1, x_2) := A_2\Psi(x_1, \,\cdot\,)(x_2) \qquad (x_1 \in E_1,\ x_2 \in E_2) \tag{1.3}$$

are jointly measurable in x_1 and x_2. Assume that X^1 and X^2 are independent solutions to the martingale problems for A_1 and A_2, respectively, and that

$$\int_0^T \mathrm{d}s \int_0^T \mathrm{d}t\, E\big[|\Phi_i(X^1_s, X^2_t)|\big] < \infty \qquad (T \geq 0,\ i = 1, 2). \tag{1.4}$$

Then

$$E[\Psi(X_T^1, X_0^2)] - E[\Psi(X_0^1, X_T^2)] = \int_0^T \mathrm{d}t \; E[R(X_t^1, X_{T-t}^2)] \tag{1.5}$$

holds for a.e. T with respect to Lebesgue measure, where $R(x_1, x_2) := \Phi_1(x_1, x_2) - \Phi_2(x_1, x_2) \quad (x_1 \in E_1,\ x_2 \in E_2)$. Moreover, the left-hand side of (1.5) is continuous in T.

Proof Although a bit of care is needed to see that all integrals are well-defined, the proof of [AS05, Theorem 7] is correct up to the last displayed formula ([AS05, formula (2.10)]), which says that

$$\int_0^S \mathrm{d}T \big(E[\Psi(X_T^1, X_0^2)] - E[\Psi(X_0^1, X_T^2)]\big) = \int_0^S \mathrm{d}T \int_0^T \mathrm{d}t \; E[R(X_t^1, X_{T-t}^2)] \qquad (S > 0). \tag{1.6}$$

Note that by our assumption (1.4),

$$\int_0^S \mathrm{d}T \int_0^T \mathrm{d}t \; E\big[|R(X_t^1, X_{T-t}^2)|\big] \le \sum_{i=1}^2 \int_0^S \mathrm{d}s \int_0^S \mathrm{d}t \; E\big[|\Phi_i(X_s^1, X_t^2)|\big] < \infty \tag{1.7}$$

$(S > 0)$, which shows that the right-hand side of (1.6) is well-defined for all $S > 0$. (The left-hand side of (1.6) is obviously well-defined by our assumption that $\Psi \in B(E_1 \times E_2)$.) Formula (1.7) also shows that

$$\int_0^T \mathrm{d}t \; E\big[|R(X_t^1, X_{T-t}^2)|\big] < \infty \qquad \text{for a.e. } T, \tag{1.8}$$

hence setting

$$f(T) := \int_0^T \mathrm{d}t \; E[R(X_t^1, X_{T-t}^2)] \tag{1.9}$$

yields an a.e. (w.r.t. Lebesgue measure) well-defined function f satisfying $\int_0^S |f(T)| \mathrm{d}T < \infty$ for each $S > 0$. Denoting the left-hand side of (1.5) by $g(T)$, formula (1.6) tells us that

$$\int_0^S g(T)\,\mathrm{d}T = \int_0^S f(T)\,\mathrm{d}T \qquad (S > 0), \tag{1.10}$$

which implies that $g(T) = f(T)$ for a.e. T.

To finish the proof, we need to show that $T \mapsto g(T)$ is continuous. By our assumption that X^1 solves the martingale problem for A_1,

$$E[\Psi(X_T^1, x_2)] = \int_0^T \mathrm{d}t\, E[\Phi_1(X_t^1, x_2)] \qquad (T \ge 0,\ x_2 \in E_2). \tag{1.11}$$

Being an integral (which is well-defined and finite by (1.1)), the right-hand side of this equation is continuous in T for each $x_2 \in E_2$, hence the same is true for the left-hand side. Now if $0 \le T_n \to T$, then by bounded pointwise convergence (using the fact that Ψ is bounded),

$$\begin{aligned} E[\Psi(X_{T_n}^1, X_0^2)] &= \int P[X_0^2 \in \mathrm{d}x_2]\, E[\Psi(X_{T_n}^1, x_2)] \\ &\underset{n\to\infty}{\longrightarrow} \int P[X_0^2 \in \mathrm{d}x_2]\, E[\Psi(X_T^1, x_2)] = E[\Psi(X_T^1, X_0^2)]. \end{aligned} \tag{1.12}$$

In the same way we see that $T \mapsto E[\Psi(X_0^1, X_T^2)]$ is continuous. ∎

Theorem 7 in [AS05] is applied at two places in that article: in proof of Theorem 1, pages 401–403, and in proof of Proposition 23, pages 404–405. Luckily, in both instances, all that is actually needed is the following corollary, which still holds.

Corollary 2 (Everywhere equality) *Under the assumptions of Theorem 1, if*

$$A_1\Psi(\,\cdot\,, x_2)(x_1) \ge A_2\Psi(x_1, \,\cdot\,)(x_2) \qquad (x_1 \in E_1,\ x_2 \in E_2), \tag{1.13}$$

then

$$E[\Psi(X_T^1, X_0^2)] \ge E[\Psi(X_0^1, X_T^2)] \qquad (T \in [0, \infty)). \tag{1.14}$$

The same statement holds with both inequality signs reversed.

Proof Set $g(T) := E[\Psi(X^1_T, X^2_0)] - E[\Psi(X^1_0, X^2_T)]$. Then Theorem 1 shows that g is a continuous function satisfying $g \geq 0$ a.e., hence $g(T) \geq 0$ for every $T \geq 0$. ∎

For completeness, we show by example that in general, the a.e. equality in (1.5) may fail to be an everywhere equality.

Counterexample 3 *There exists metric spaces E_i, linear operators A_i and processes X_i $(i = 1, 2)$ together with a function $\Psi : E_1 \times E_2 \to \mathbb{R}$ satisfying the assumptions of Theorem 1 such that (1.5) does not hold for $T = 1$.*

Proof We take $E_1 = E_2 = (0, \infty)$. For $r > 0$, we let $f_r : (0, \infty) \to \mathbb{R}$ be the function defined by $f_r(x) := e^{-rx}$. We define linear operators A_1, A_2 with domains $\mathcal{D}(A_1) = \mathcal{D}(A_2) := \mathrm{span}\{f_r : r > 0\}$ by

$$\begin{aligned} A_1 f_r(x) &:= 1_{\{rx \neq e\}} x \tfrac{\partial}{\partial x} f_r(x) = -1_{\{rx \neq e\}} rxe^{-rx}, \\ A_2 f_r(x) &:= x \tfrac{\partial}{\partial x} f_r(x) = -rxe^{-rx}. \end{aligned} \tag{1.15}$$

For X^1, X^2 we choose the deterministic processes

$$X^1_t = X^2_t := e^t \qquad (t \geq 0) \tag{1.16}$$

and we define $\Psi : E_1 \times E_2 \to \mathbb{R}$ by

$$\Psi(x_1, x_2) := e^{-x_1 x_2} \qquad (x_1, x_2, \geq 0). \tag{1.17}$$

It is straightforward to check that X^i solves the martingale problem for A_i $(i = 1, 2)$. Note that the factor $1_{\{rx \neq e\}}$ in the definition of A_1 is at this point irrelevant since for each $r > 0$ there is only one time t such that $rX^1_t = e$, hence this factor has no influence on the time integral in (1.2).

It is easy to check that (1.4) holds and, in the notation of Theorem 1,

$$R(x_1, x_2) = 1_{\{x_1 x_2 = e\}} x_1 x_2 e^{-x_1 x_2} \qquad (x_1, x_2 \geq 0). \tag{1.18}$$

Therefore, since $X^1_t X^2_{T-t} = e^t e^{T-t} = e^T$,

$$\int_0^T \mathrm{d}t \; E[R(X^1_t, X^2_{T-t})] = \int_0^T \mathrm{d}t \; 1_{\{T=1\}} e^T e^{-e^T} = 1_{\{T=1\}} e^{1-e} \qquad (T \geq 0), \tag{1.19}$$

while the left-hand side of (1.5) is in our example identically zero. ∎

References

[AS05] S.R. Athreya and J.M. Swart. Branching-coalescing particle systems. *Prob. Theory Relat. Fields.* 131(3), 376–414, 2005.

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