# Uniqueness for isotropic diffusions with a linear drift

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**Abstract**

This paper proves that $D$-valued solutions to the SDE $\mathrm{d}X = c(\theta - X)\mathrm{d}t + \sqrt{2g(X)}\mathrm{d}B$ are unique in distribution, when $D \subset \mathbb{R}^d$ is convex with nonempty interior $D^\circ$, $\theta \in D^\circ$, $c > 0$, $g : D \to [0, \infty)$ is positive and locally Lipschitz on $D^\circ$ and $\{x \in D : g(x) \geq r\}$ is convex for $r$ sufficiently small. The proof is based on the transformation $X_t \mapsto e^{ct}X_t$, which removes the drift, and a random time change. Although the set-up is rather specialized the result gives uniqueness for some SDE's that cannot be treated by any of the conventional techniques.

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## 1 Main result

Assume that $D \subset \mathbb{R}^d$ $(d \geq 1)$ is convex, with nonempty interior $D^\circ$, and let $\partial D := D \backslash D^\circ$ denote its boundary. Let $g : D \to [0, \infty)$ be continuous and assume that $g = 0$ on $\partial D$ and $g > 0$ on $D^\circ$. Fix $\theta \in D$ and $c \geq 0$. Consider the operator

$$A_\theta^{c,g} f(x) := c \sum_{i=1}^{d} (\theta_i - x_i) \tfrac{\partial}{\partial x_i} f(x) + g(x) \sum_{i=1}^{d} \tfrac{\partial^2}{\partial {x_i}^2} f(x) \qquad (x \in D,\ f \in \mathcal{C}_c^2(D)), \tag{1}$$

where $\mathcal{C}_c^2(D)$ denotes the class of real continuous compactly supported functions on $D$, whose first and second partial derivatives exist on $D^\circ$ and can be extended to continuous functions on $D$. Our main result is the following.

**Theorem 1 (Uniqueness for isotropic diffusions with a linear drift)** *Assume that $\theta \in D^\circ$ and $c > 0$. Assume that $g$ is positive and locally Lipschitz on $D^\circ$ and that there exists an $\varepsilon > 0$ such that for all $r \in (0, \varepsilon)$, the level sets $\{x \in D : g(x) \geq r\}$ are convex. Then solutions to the martingale problem for $A_\theta^{c,g}$ are unique.*

Here, a $D$-valued process $X = (X_t)_{t \geq 0}$ with continuous sample paths is said to solve the martingale problem for $A_\theta^{c,g}$ if for every $f \in \mathcal{C}_c^2(D)$, the process $(M_t^f)_{t \geq 0}$ given by

$$M_t^f := f(X_t) - \int_0^t (A_\theta^{c,g} f)(X_s)\mathrm{d}s \qquad (t \geq 0) \tag{2}$$

is a martingale with respect to the filtration generated by $X$. One says that solutions to the martingale problem for $A^{c,g}_\theta$ are unique, if any two solutions with identical initial laws are equal in distribution. As is well-known [EK86, Theorem 5.3.3], every solution to the martingale problem for $A^{c,g}_\theta$ is equal in distribution to a solution of the stochastic differential equation (SDE)

$$\mathrm{d}X_t = c(\theta - X_t)\mathrm{d}t + \sqrt{2g(X_t)}\mathrm{d}B_t \qquad (t \geq 0), \tag{3}$$

where $B$ is $d$-dimensional Brownian motion. Thus, uniqueness for the martingale problem for $A^{c,g}_\theta$ is equivalent to distribution uniqueness for the SDE (3). Note that our result does not follow from the standard results on pathwise uniqueness, since the function $\sqrt{g}$ may not be Lipschitz at $\partial D$. Solutions to (3) are *isotropic* diffusions since the random fluctuations are equally strong in all directions, i.e., the diffusion coefficient $\sqrt{2g}$ is a function and not a matrix.

For the information of the reader, we provide here a quick discussion of *existence* of solutions to the martingale problem for $A^{c,g}_\theta$. If $D$ is compact, then there exists a solution for each initial law on $D$ (see [EK86, Theorem 4.5.4 and Problem 4.19]). If $D$ is not compact, then the same is true provided that one can show that solutions to the SDE (3) are nonexplosive. For this, it suffices to find functions $f_n \in \mathcal{C}^2_c(D)$ and $f, h \in \mathcal{C}(D)$ such that $f_n \to f$ and $Af_n \to h$ uniformly on compacta, $f \geq 0$, $f(x) \to \infty$ as $x \to \infty$, and $h$ is bounded from above. For example, if $D = [0,\infty)^d$, then one may take $f(x) = \sum_{i=1}^d x_i$. In dimension one, one may take $f(x) := |x|$ and in dimension two $f(x) := \log(|x|)$, for $x$ outside some neighborhood of 0. For arbitrary domains and dimensions, solutions to the SDE (3) are nonexplosive if $g$ satisfies a quadratic growth condition [EK86, Proposition 5.3.5].

## 2 Applications and discussion

Our interest in SDE's of the form (3) is motivated by models for catalytic branching and resampling. For mutually catalytic models, uniqueness is usually hard to prove. This motivated, for example, the recent work in [ABBP02, BP02]. However, their perturbation technique does not cover our result. The following proposition gives an application of our Theorem 1 to a model for mutually catalytic branching.

**Proposition 2 (Mutually catalytic Feller's branching diffusions)** *Let $d \geq 1$, $c > 0$, and $\theta \in (0,\infty)^d$. Then, for each initial law on $[0,\infty)^d$, there exists a unique (in distribution) $[0,\infty)^d$-valued (weak) solution to the SDE*

$$\mathrm{d}X_i(t) = c(\theta_i - X_i(t))\mathrm{d}t + \sqrt{\textstyle\prod_{k=1}^d X_k(t)}\,\mathrm{d}B_i(t) \qquad (i = 1, \ldots, d). \tag{4}$$

**Proof** Existence of a (weak) solution follows from the remarks at the end of the last section. Since $x \mapsto \log \prod_{k=1}^d x_k = \sum_{k=1}^d \log x_k$ is a concave function, the level sets of $x \mapsto \prod_{k=1}^d x_k$ are convex. Thus, distribution uniqueness of solutions to (4) follows from Theorem 1. ∎

Infinite systems of mutually catalytic Feller's branching diffusions with $d = 2$ and a linear interaction between the components have been the subject of extensive study [DP98, DKMPX01]. Uniqueness for these systems follows from Mytnik's self-duality [Myt98], but this argument works only for $d = 2$.

We also give an application of our Theorem 1 to a model with multitype resampling.

**Proposition 3 (A $p$-type resampling model)** *For $p \geq 2$, let $K_p := \{x \in \mathbb{R}^d : \sum_{i=1}^p x_i = 1\}$ denote the $(p-1)$-dimensional simplex and write $K_p^\circ := \{x \in K_p : x_i > 0 \ \forall i = 1, \ldots, p\}$. Let $c > 0$ and $\theta \in K_p^\circ$. Then, for each initial law on $K_p$, there exists a unique (in distribution) $K_p$-valued (weak) solution to the SDE*

$$\mathrm{d}X_i(t) = c(\theta_i - X_i(t))\mathrm{d}t + \sqrt{\prod_{k=1}^p X_k(t)}\,\Big(\mathrm{d}B_i(t) - \frac{1}{p}\sum_{j=1}^p \mathrm{d}B_j(t)\Big) \qquad (i = 1, \ldots, p). \tag{5}$$

**Proof** A simple calculation shows that solutions to (5) solve the martingale problem for the operator

$$Af(x) := c(\theta - x) \cdot \nabla f(x) + \Big(\prod_{k=1}^p x_k\Big)\Delta f(x) \qquad (x \in K_p,\ f \in \mathcal{C}^2(K_d)), \tag{6}$$

where $\nabla$ is the gradient and $\Delta$ is the Laplacian in the plane $\{x : \sum_{i=1}^p x_i = 1\}$. In the proof of Proposition 2 we have already seen that $x \mapsto \prod_{k=1}^p x_k$ has convex level sets, and therefore Theorem 1 is applicable. ∎

For $p = 2$, the diffusion in (5) is the well-known Wright-Fisher diffusion with a linear drift. For general $p \geq 2$, the diffusion in (5) arises from a $p$-tuple resampling procedure [Swa99, Section 1.1]. It seems that there are at present no other techniques available that would give uniqueness for this diffusion when $p \geq 3$.

# 3 Proof and further results

We will derive Theorem 1 from a result that is stronger, but the assumptions of which are more complicated to verify. Let

$$S_d := \{x \in \mathbb{R}^d : |x| = 1\} \tag{7}$$

denote the surface of the unit ball in $\mathbb{R}^d$. Assume that $R : S_d \to (0, \infty)$ is continuous, and put

$$D := \{rx : x \in S_d,\ 0 \leq r \leq R_x\}. \tag{8}$$

Then $D \subset \mathbb{R}^d$ is compact and the closure of its interior $D^\circ$ (but $D$ need not be convex). Let $\partial D := D \backslash D^\circ$ denote its boundary and write $\mathcal{C}^2(D)$ for the class of real functions on $D$ that can be extended to a twice continuously differentiable function on $\mathbb{R}^d$. Assume that $g : D \to [0, \infty)$ is continuous and satisfies $g(x) = 0 \Leftrightarrow x \in \partial D$. For any $c > 0$, let $A^{c,g}$ be defined by

$$A^{c,g} f(x) := -c\sum_i x_i \tfrac{\partial}{\partial x_i} f(x) + g(x)\sum_{i=1}^d \tfrac{\partial^2}{\partial x_i{}^2} f(x) \qquad (x \in D,\ f \in \mathcal{C}^2(D)). \tag{9}$$

**Theorem 4 (Basic uniqueness result)** *Assume that $c > 0$. Set*

$$h_x(r) := \frac{r}{g(rx)} \qquad (x \in S_d,\ r \in (0, R_x)). \tag{10}$$

*Assume that there exists a constant $M < \infty$ such that*

$$h_x(r_1) - h_x(r_2) \leq M(r_2 - r_1) \qquad (x \in S_d,\ 0 < r_1 < r_2 < R_x). \tag{11}$$

*Then uniqueness holds for the martingale problem for $A^{c,g}$.*

Note that $\lim_{r\to 0} h_x(r) = 0$ and $\lim_{r\to R_x} h_x(r) = \infty$. Assumption (11) says that $h$ *decreases* at most with speed $M$. There is no bound on the speed with which $h$ increases. The proof of Theorem 4 is based on the following three lemmas.

**Lemma 5 (Process leaves the boundary immediately)** *Assume that $c > 0$. Let $X$ be a solution to the martingale problem for $A^{c,g}$ with initial condition $X_0 = x \in \partial D$, and put $\tau := \inf\{t \geq 0 : X_t \in D^\circ\}$. Then $\tau = 0$ a.s.*

The next lemma shows that the process $(e^{ct}X_t)_{t\geq 0}$ is a random time-changed Brownian motion. The fact that the function $\psi$ below is strictly increasing will follow from Lemma 5.

**Lemma 6 (Random time change)** *Assume that $c > 0$ and that $X$ solves the martingale problem for $A^{c,g}$. Then $X$ may be coupled to a Brownian motion $B$ in such a way that*

$$e^{ct}X_t = B_{\psi(t)} \qquad (t \geq 0), \tag{12}$$

*where $\psi : [0,\infty) \to [0,\infty)$ is a (random) continuous, strictly increasing function satisfying $\psi(0) = 0$ and moreover, if $\psi(\infty) := \lim_{t\to\infty} \psi(t)$ and $\psi^{-1} : [0,\psi(\infty)) \to [0,\infty)$ denotes the inverse of $\psi$, then $\psi^{-1}$ solves the equation*

$$\psi^{-1}(\tau) = \int_0^\tau \Big(e^{c\psi^{-1}(\sigma)} g\big(e^{-c\psi^{-1}(\sigma)}B_\sigma\big)\Big)^{-1} \mathrm{d}\sigma \qquad (\tau \in [0,\psi(\infty))). \tag{13}$$

Our final and crucial lemma shows that solutions to (13) are unique.

**Lemma 7 (Uniqueness of the random time change)** *Assume that $c > 0$ and that $g$ satisfies (11). Let $b : [0,\infty) \to \mathbb{R}^d$ be continuous and let $b_0 \in D$. Then there exists at most one constant $T \leq \infty$ and continuous strictly increasing function $\phi : [0,T) \to [0,\infty)$ with $\lim_{\tau\to T} \phi(\tau) = \infty$, such that $e^{-c\phi(\tau)}b_\tau \in D$ for all $\tau \in [0,T)$ and*

$$\phi(\tau) = \int_0^\tau \Big(e^{c\phi(\sigma)} g\big(e^{-c\phi(\sigma)}b_\sigma\big)\Big)^{-1} \mathrm{d}\sigma \qquad (\tau \in [0,T)). \tag{14}$$

We will now proceed as follows. First, we show that Lemmas 5–7 imply Theorem 4. Then we prove Lemmas 5–7. Finally, we show that Theorem 4 implies Theorem 1.

**Proof of Theorem 4** Let $X$ be a solution to the martingale problem for $A^{c,g}$ and let $\psi$ and $B$ be as in Lemma 6. Since there is only one Brownian motion with initial condition $\mathcal{L}(B_0) = \mathcal{L}(X_0)$, $B$ is determined uniquely in distribution. It follows from (12) that

$$X_t = e^{-ct}B_{\psi(t)} \qquad (t \geq 0). \tag{15}$$

Since solutions to (14) are unique, for every path of the Browninan motion $B$ there exists a unique $\psi$ such that (13) holds, and therefore the distribution of $X$ is determined uniquely by (15). ∎

**Proof of Lemma 5** Represent $X$ as a solution to the stochastic differential equation (written in integral form)

$$X_t = x - c\int_0^t X_s \,\mathrm{d}s + \int_0^t \sqrt{2g(X_s)}\,\mathrm{d}B_s \qquad (t \geq 0). \tag{16}$$

Since $X_t \in \partial D$ for $t \leq \tau$ and $g = 0$ on $\partial D$, the second term on the right-hand side of (16) is zero for $t \leq \tau$. Differentiating and solving for $(X_t)_{t \leq \tau}$ we find that

$$X_\tau = e^{-c\tau} x \quad \text{a.s.} \tag{17}$$

Since $e^{-ct}x \in D^\circ$ for all $t > 0$ and $X_\tau \in \partial D$, it follows that $\tau = 0$ a.s. ∎

**Proof of Lemma 6** Represent $X$ as a solution to the stochastic differential equation (16). Set

$$Y_t := e^{ct} X_t \qquad (t \geq 0). \tag{18}$$

Then, by Itô's formula

$$\begin{aligned} \mathrm{d}Y_t &= c e^{ct} X_t \mathrm{d}t + e^{ct} \mathrm{d}X_t \\ &= c e^{ct} X_t \mathrm{d}t - c e^{ct} X_t \mathrm{d}t + e^{ct} \sqrt{2g(X_t)} \mathrm{d}B_t, \end{aligned} \tag{19}$$

and therefore

$$\mathrm{d}Y_t = e^{ct} \sqrt{2g(e^{-ct} Y_t)} \mathrm{d}B_t. \tag{20}$$

It follows that for all $f \in \mathcal{C}_c^2(\mathbb{R}^d)$, the process $(M_t^f)_{t \geq 0}$ given by

$$M_t^f := f(Y_t) - \int_0^t e^{cs} g(e^{-cs} Y_s) \Delta f(Y_s) \mathrm{d}s \tag{21}$$

is a martingale with respect to the filtration generated by $X$, where $\Delta f(x) := \sum_{i=1}^d \frac{\partial^2}{\partial x_i{}^2} f(x)$ denotes the Laplacian. Define $\psi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) := \int_0^t e^{cs} g(e^{-cs} Y_s) \, \mathrm{d}s \qquad (t \geq 0). \tag{22}$$

Then $\psi$ is continuously differentiable, and by Lemma 5, $\psi$ is strictly increasing. Put $\psi(\infty) := \lim_{t \to \infty} \psi(t)$.[1] Let $\psi^{-1}$ denote the inverse of $\psi$, i.e.,

$$\psi^{-1}(\tau) := \inf \big\{ t \geq 0 : \int_0^t e^{cs} g(e^{-cs} Y_s) \mathrm{d}s = \tau \big\} \qquad (\tau \in [0, \psi(\infty))). \tag{23}$$

Note that $\psi^{-1}$ is continuous and strictly increasing since $\psi$ is. Define a process $(B_\tau)_{\tau \in [0, \psi(\infty))}$ by

$$B_\tau := Y_{\psi^{-1}(\tau)} \qquad (\tau \in [0, \psi(\infty))). \tag{24}$$

Using the substitution of variables

$$\begin{aligned} \sigma &= \psi(s) & \mathrm{d}\sigma &= e^{cs} g(e^{-cs} Y_s) \, \mathrm{d}s, \\ \psi^{-1}(\sigma) &= s & \big(e^{c\psi^{-1}(\sigma)} g(e^{-c\psi^{-1}(\sigma)} Y_{\psi^{-1}(\sigma)})\big)^{-1} \, \mathrm{d}\sigma &= \mathrm{d}s, \end{aligned} \tag{25}$$

we see that

$$\psi^{-1}(\tau) = \int_0^{\psi^{-1}(\tau)} \mathrm{d}s = \int_0^\tau \big(e^{c\psi^{-1}(\sigma)} g(e^{-c\psi^{-1}(\sigma)} B_\sigma)\big)^{-1} \mathrm{d}\sigma \qquad (\tau \in [0, \psi(\infty))). \tag{26}$$

[1] It is probably true that $\psi(\infty) = \infty$ a.s. but this requires some work to prove. Since this is a problem concerning the long-time behavior of $X$ that has little to do with the uniqueness problems that we are studying at present, we will not touch this subject here.

Again using (25) we see that for every $f \in \mathcal{C}_c^2(\mathbb{R}^d)$,

$$
\begin{aligned}
\tilde{M}^f_\tau &:= f(B_\tau) - \int_0^\tau \Delta f(B_\sigma)\,\mathrm{d}\sigma \\
&= f(Y_{\psi^{-1}(\tau)}) - \int_0^\tau \Delta f(Y_{\psi^{-1}(\sigma)})\,\mathrm{d}\sigma \\
&= f(Y_{\psi^{-1}(\tau)}) - \int_0^{\psi^{-1}(\tau)} e^{cs} g(e^{-cs}Y_s)\Delta f(Y_s)\,\mathrm{d}s = M^f_{\psi^{-1}(\tau)} \qquad (\tau \in [0, \psi(\infty))).
\end{aligned} \tag{27}
$$

It follows that for any $T < \infty$,

$$
\tilde{M}^f_{\psi(T)\wedge\tau} = M^f_{T\wedge\psi^{-1}(\tau)} \qquad (\tau \geq 0). \tag{28}
$$

Let $(\mathcal{F}_t)_{t\geq 0}$ denote the filtration generated by $X$. By optional stopping and the fact that (21) is a martingale,

$$
E[M^f_{T\wedge\psi^{-1}(\tau)} | \mathcal{F}_{T\wedge\psi^{-1}(\sigma)}] = M^f_{T\wedge\psi^{-1}(\sigma)} \qquad (0 \leq \sigma \leq \tau). \tag{29}
$$

Thus, we see that $(\tilde{M}^f_{\psi(T)\wedge\tau})_{\tau\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_{T\wedge\psi^{-1}(\tau)})_{\tau\geq 0}$. Now, by [EK86, Lemma 4.5.16], there exists (possibly on an extended probability space) a solution $B'$ to the martingale problem for $\Delta$, that is, a Brownian motion, such that

$$
B_{\psi(T)\wedge\tau} = B'_{\psi(T)\wedge\tau} \qquad (\tau \geq 0). \tag{30}
$$

Letting $T \uparrow \infty$ we see that $B$ can be extended to a Brownian motion. ∎

**Proof of Lemma 7** Note that (11) holds more generally for all $0 < r_1 < r_2 \leq R_x$ if we put $h_x(R_x) := \infty$. We start by showing that (11) implies that

$$
\frac{e^{-\lambda}}{g(e^{-\lambda}x)} - \frac{1}{g(x)} \leq M\lambda \qquad (x \in D,\ \lambda \geq 0). \tag{31}
$$

Indeed, (31) is trivial if $x = 0$ so assume that $x \neq 0$, in which case $x = r\tilde{x}$ for some $\tilde{x} \in S_d$ and $r \in (0, R_{\tilde{x}}]$, and

$$
\begin{aligned}
\frac{e^{-\lambda}}{g(e^{-\lambda}x)} - \frac{1}{g(x)} &= \frac{e^{-\lambda}}{g(e^{-\lambda}r\tilde{x})} - \frac{1}{g(r\tilde{x})} \\
&= r^{-1}h_{\tilde{x}}(e^{-\lambda}r) - r^{-1}h_{\tilde{x}}(r) \\
&\leq r^{-1}M(r - e^{-\lambda}r) = M(1 - e^{-\lambda}) \leq M\lambda.
\end{aligned} \tag{32}
$$

Define $F : [0,\infty) \times [0,\infty) \to [0,\infty]$ by

$$
F_t(\lambda) := \Big(e^{c\lambda} g\big(e^{-c\lambda}b_t\big)\Big)^{-1} \qquad (t, \lambda \geq 0). \tag{33}
$$

Then $F$ is jointly continuous in $t$ and $\lambda$. Note that $F_t(\lambda) < \infty$ if $\lambda > 0$. Moreover, for every $0 \leq \lambda_1 < \lambda_2$ and $t \geq 0$ one has, by (31),

$$
\begin{aligned}
F_t(\lambda_2) - F_t(\lambda_1) &= \Big(e^{c\lambda_2} g\big(e^{-c\lambda_2}b_t\big)\Big)^{-1} - \Big(e^{c\lambda_1} g\big(e^{-c\lambda_1}b_t\big)\Big)^{-1} \\
&= \left(\frac{e^{-c(\lambda_2-\lambda_1)}}{g\big(e^{-c(\lambda_2-\lambda_1)}e^{-c\lambda_1}b_t\big)} - \frac{1}{g\big(e^{-c\lambda_1}b_t\big)}\right) e^{-c\lambda_1} \leq Mc(\lambda_2 - \lambda_1).
\end{aligned} \tag{34}
$$

Now let $\phi, \tilde{\phi}$ be solutions to (14) defined up to explosion times $T, \tilde{T}$, respectively. Assume that $\phi(t) > \tilde{\phi}(t)$ for some $t < T \wedge \tilde{T}$ and set $u := \sup\{s \leq t : \phi(s) = \tilde{\phi}(s)\}$. Then

$$\begin{aligned} \phi(t) - \tilde{\phi}(t) &= \int_u^t F_s(\phi(s))\mathrm{d}s - \int_u^t F_s(\tilde{\phi}(s))\mathrm{d}s \\ &\leq Mc \int_u^t (\phi(s) - \tilde{\phi}(s))\mathrm{d}s. \end{aligned} \tag{35}$$

It follows from Gronwall's lemma that $\phi(t) = \tilde{\phi}(t)$ and thus we arrive at a contradiction. In the same way we see that $\phi(t) \not< \tilde{\phi}(t)$ for all $t < T \wedge \tilde{T}$ and we conclude that $\phi = \tilde{\phi}$ and $T = \tilde{T}$. ∎

**Proof of Theorem 1** By a simple translation of our space we may choose $\theta = 0$. Fix $K > 0$ and for $x \in S_d$, set $R_x := \sup\{r \geq 0 : rx \in D\}$ if $Kx \notin D^\circ$ and $R_x := K + g(Kx)$ if $Kx \in D^\circ$. It is not hard to see that $R : S_d \to (0, \infty)$ is continuous. Put $\tilde{D} := \{rx : x \in S_d,\ 0 \leq r \leq R_x\}$ and for $x \in S_d$, put $\tilde{g}(rx) := g(rx)$ if $0 \leq r \leq R_x \wedge K$ and $\tilde{g}(rx) := g(Kx) - (r - K)$ if $K < r \leq R_x$. We wil show that $\tilde{D}, \tilde{g}$ satisfy the assumptions of Theorem 4. Since $g$ and $\tilde{g}$ coincide on $D \cap \{x : |x| \leq K\} = \tilde{D} \cap \{x : |x| \leq K\}$, this implies that solutions $X$ to the martingale problem for $A_0^{c,\theta}$ are unique up to the stopping time $\tau_K := \inf\{t \geq 0 : |X_t| \geq K\}$. Letting $K \uparrow \infty$ we arrive at Theorem 1.

To see that $\tilde{D}, \tilde{g}$ satisfy the assumptions of Theorem 4, choose $\varepsilon \in (0, g(0))$ such that the level sets $\{x \in D : g(x) \geq r\}$ are convex for all $r \in (0, \varepsilon]$. For $x \in S_d$, let $\tilde{g}_x : [0, R_x] \to [0, \infty)$ denote the function

$$\tilde{g}_x(r) := \tilde{g}(rx) \qquad (x \in S_d,\ r \in [0, R_x]). \tag{36}$$

Then, for each $x \in S_d$, there exists a unique $\delta_x > 0$ such that $\tilde{g}_x(r) \geq \varepsilon \Leftrightarrow r \leq R_x - \delta_x$. Note that the map $x \mapsto \delta_x > 0$ is continuous and set $\delta := \inf_{x \in S_d} \delta_x$. Since $g$ is locally Lipschitz on $D^\circ$ there exists an $L < \infty$ such that

$$\begin{array}{rl} \text{(i)} & \tilde{g}_x(r_2) - \tilde{g}_x(r_1) \leq L(r_2 - r_1) \qquad \forall 0 \leq r_1 \leq r_2 \leq R_x - \delta, \\ \text{(ii)} & \tilde{g}_x \text{ is nonincreasing on } [R_x - \delta, R_x]. \end{array} \tag{37}$$

Set

$$\begin{aligned} G &:= \inf\{\tilde{g}(rx) : x \in S_d,\ r \leq R_x - \delta\}, \\ R &:= \sup\{R_x : x \in S_d\}. \end{aligned} \tag{38}$$

Then, for all $0 \leq r_1 < r_2 < R_x - \delta$, with $h_x(r) := r/\tilde{g}_x(r)$,

$$h_x(r_1) - h_x(r_2) = \frac{r_1}{\tilde{g}_x(r_1)} - \frac{r_2}{\tilde{g}_x(r_2)} = \frac{r_1(\tilde{g}_x(r_2) - \tilde{g}_x(r_1)) - (r_2 - r_1)\tilde{g}_x(r_1)}{\tilde{g}_x(r_1)\tilde{g}_x(r_2)} \leq \frac{RL}{G^2}(r_2 - r_1). \tag{39}$$

Moreover, $h_x$ is nondecreasing on $[R_x - \delta, R_x]$, since:

$$h_x(r_1) = \frac{r_1}{\tilde{g}_x(r_1)} \leq \frac{r_2}{\tilde{g}_x(r_1)} \leq \frac{r_2}{\tilde{g}_x(r_2)} = h_x(r_2) \qquad (R_x - \delta < r_1 < r_2 < R_x). \tag{40}$$

Combining this with (39) we see that (11) holds with $M = RL/G^2$, and therefore Theorem 4 is applicable. ∎

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