Functionals of empirical process defined implicitly and their study by means of Skorokhod topology

Abstract

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We shall illustrate the behavior of statistical functionals on the simplest case, what is the M-estimator of the real parameter

$$\theta_0 = T(F) = \arg\min \int \rho(x,t)dF_t(x) : t \in \mathbf{\Theta},\ \mathbf{\Theta} \text{ open set in } \mathbf{R}_1.$$

There ρ is a function, absolutely continuous in t, with derivative $\psi(x,t)$, and $F_t(x)$ is a generally unknown probability distribution function of a random variable X, indexed by parameter t. If we observe n independent realizations $X_1,...,X_n$ of X, then we estimate $T(F)$ by the functional $T_n = T(F_n)$ of the empirical distribution function $F_n(x) = \frac{1}{n}\sum_{i=1}^n I[X_n \leq x]$. This has the form $T(F_n) = \arg\min \left\{\sum_{i=1}^n \rho(X_i,t) : t \in \mathbf{\Theta}\right\}$. Under smoothness assumptions on ψ in t, if θ_0 above is the unique minimum, it is possible to derive an expansion of T_n through its Hadamard or Fréchet derivatives, and to prove that

- $n^{1/2}(T_n - \theta_0) = \mathcal{O}_p(1)$ as $n \to \infty$
- $T_n = \theta_0 - \frac{1}{n}(\gamma(\theta_0))^{-1}\sum_{i=1}^n \psi(X_i,\theta_0) + R_n,\ R_n = \mathcal{O}_p(n^{-1})$
- The probability distribution function of nR_n converges to a nondenerate distribution function, corresponding to product of two specific normal random variables.

To prove that, we use the weak convergence of the process

$$Y_n(t) = \left\{1/\gamma\sum_{i=1}^n[\psi(X_i,\theta_0 + n^{-1/2}t) - \psi(X_i,\theta_0)] - n^{-1/2}t,\ |t| \leq B\right\}$$

to a Gaussian process in the Skorokhod J_1 topology on $\mathcal{D}[-B,B]$, and the special random change of time $t \mapsto n^{1/2}(T_n - \theta_0)$ in this convergence. The asymptotic results can be modified under weaker conditions, for vector parameters etc., but with possibly different rates of convergence and different limiting processes.

References

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