On Skorohod's topologies

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Cadlag functions

Let $[s,u]$ be a compact real interval.
Let $(\mathcal{X},d)$ be a metric space.

Def a function $f:[s,u]\to\mathcal{X}$ is *cadlag* (continue à droite, limite à gauche) if

(i) $f(t)=\lim_{r\downarrow t} f(r) \quad \forall t\in[s,u)$,

(ii) $f(t-):=\lim_{r\uparrow t} f(r)$ exists $\quad \forall t\in(s,u]$,

Def a function $f:[s,u]\to\mathcal{X}$ is *caglad* (continue à gauche, limite à droite) if

(i) $f(t+):=\lim_{r\downarrow t} f(r)$ exists $\quad \forall t\in[s,u)$,

(ii) $f(t):=\lim_{r\uparrow t} f(r) \quad \forall t\in(s,u]$.

Cadlag functions

Def $\overline{\mathcal{D}}_{[s,u]}(\mathcal{X})$ is the space of all functions

$$[s,u] \ni t \mapsto \big(f(t-), f(t+)\big)$$

that satisfy the equivalent conditions:

(i) $t \mapsto f(t+)$ is cadlag and $f(t-) = \lim_{r \uparrow t} f(r+)$ $\forall t \in (s,u]$.

(ii) $t \mapsto f(t-)$ is caglad and $f(t+) = \lim_{r \downarrow t} f(r-)$ $\forall t \in [s,u)$.

Def $\mathcal{D}_{[s,u]}(\mathcal{X})$ is the space of all functions $f \in \overline{\mathcal{D}}_{[s,u]}(\mathcal{X})$ such that $f(s-) = f(s+)$ and $f(u-) = f(u+)$.

An element of $\mathcal{D}_{[s,u]}(\mathcal{X})$ is uniquely determined by either $t \mapsto f(t-)$ or $t \mapsto f(t+)$.

We call $t \mapsto f(t-)$ the caglad modification of $t \mapsto f(t+)$.

Skorohod's J1 topology

For brevity, write $f(t) = f(t+)$.

[Skorohod 1956] There exists a metric $d_{\rm S}$ on $\mathcal{D}_{[s,u]}(\mathcal{X})$ such that $d_{\rm S}(f_n, f) \to 0$ iff there exist λ_n such that:

(i) $\lambda_n : [s,u] \to [s,u]$ is continuous and strictly increasing with $\lambda_n(s) = s$ and $\lambda_n(u) = u$

(ii) $\sup_{t\in[s,u]} \big|\lambda_n(t) - t\big| \underset{n\to\infty}{\longrightarrow} 0$,

(iii) $\sup_{t\in[s,u]} d\big(f_n(\lambda_n(t)), f(t)\big) \underset{n\to\infty}{\longrightarrow} 0$.

If $(\mathcal{X}, d)$ is separable, then so is $\mathcal{D}_{[s,u]}(\mathcal{X})$.
If $(\mathcal{X}, d)$ is complete, then $d_{\rm S}$ can be chosen complete too.
If d, d' define the same topology on $\mathcal{X}$, then
$d_{\rm S}, d'_{\rm S}$ define the same topology on $\mathcal{D}_{[s,u]}(\mathcal{X})$.

Skorohod also derived a compactness criterion.

Cadlag functions

The *split real line* is the set $\mathbb{R}_{\mathfrak{s}}$ consisting of all pairs $t\pm$ consisting of a real number $t \in \mathbb{R}$ and a sign $\pm \in \{-,+\}$.
For an element $\tau = t\pm$ of $\mathbb{R}_{\mathfrak{s}}$ we let $\underline{\tau} := t$ denote its real part and $\mathfrak{s}(\tau) := \pm$ its sign.
We equip $\mathbb{R}_{\mathfrak{s}}$ with the lexographic order, in which $\sigma \leq \tau$ if and only if $\underline{\sigma} < \underline{\tau}$ or $\underline{\sigma} = \underline{\tau}$ and $\mathfrak{s}(\sigma) \leq \mathfrak{s}(\tau)$.
We write $\sigma < \tau$ iff $\sigma \leq \tau$ and $\sigma \neq \tau$ and define intervals

$$\langle\!\langle \sigma, \rho \rangle\!\rangle := \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma < \tau < \rho\}, \qquad [\![\sigma, \rho \rangle\!\rangle := \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma \leq \tau < \rho\},$$
$$\langle\!\langle \sigma, \rho]\!] := \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma < \tau \leq \rho\}, \qquad [\![\sigma, \rho]\!] := \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma \leq \tau \leq \rho\}.$$

There is some redundancy, e.g., $\langle\!\langle s-, r+]\!] = [\![s+, r+]\!]$.

Cadlag functions

We equip the split real line $\mathbb{R}_{\mathfrak{s}}$ with the *order topology*.
A basis for the topology is formed by all open intervals $(\!(\sigma,\rho)\!)$ with $\sigma,\rho\in\mathbb{R}_{\mathfrak{s}}$, $\sigma<\rho$.

(i) $\tau_n \to t+$ iff $\underline{\tau}_n \to t$ and $\tau_n \geq t+$ for n sufficiently large.

(ii) $\tau_n \to t-$ iff $\underline{\tau}_n \to t$ and $\tau_n \leq t-$ for n sufficiently large.

Lemma $\mathbb{R}_{\mathfrak{s}}$ is first countable, Hausdorff and separable, but not second countable and not metrisable.

Lemma For $C\subset\mathbb{R}_{\mathfrak{s}}^d$, the following are equivalent:

(i) C is compact,

(ii) C is sequentially compact,

(iii) C is closed and bounded.

Cadlag functions

[Kolmogorov 1956] A function $f : [\![s-, u+]\!] \to \mathcal{X}$ is continuous iff $t \mapsto \big(f(t-), f(t+)\big)$ is an element of $\overline{\mathcal{D}}_{[s,u]}(\mathcal{X})$.

Similarly, continuous functions $f : [\![s+, u-]\!] \to \mathcal{X}$ correspond to elements of $\mathcal{D}_{[s,u]}(\mathcal{X})$.

Advantages of this approach:

- Symmetry with respect to time reversal.
- Functions in $\overline{\mathcal{D}}_{[s,u]}(\mathcal{X})$ can jump at the endpoints s and u of the interval.
- Cadlag functions of several variables.

The closed graph

The *closed graph* of a function $f \in \overline{\mathcal{D}}_{[s,u]}(\mathcal{X})$ is defined as

$$\begin{aligned}\mathcal{G}(f) &:= \big\{\big(\underline{\tau}, f(\tau)\big) : \tau \in [\![s-, u+]\!]\big\}\\ &= \big\{\big(t, f(t\pm)\big) : t \in [s,u]\big\}.\end{aligned}$$

It is easy to see that $\mathcal{G}(f) \subset [s,u] \times \mathcal{X}$ is compact.

Idea: define a metric on the space $\overline{\mathcal{D}}_{[s,u]}(\mathcal{X})$
by measuring the distance between closed graphs.

The Hausdorff metric

Let $(\mathcal{X}, d)$ be a metric space.
Let $\mathcal{K}_+(\mathcal{X})$ be the set of nonempty compact subsets of $\mathcal{X}$.
The *Hausdorff metric* $d_{\rm H}$ is defined as

$$d_{\rm H}(K_1, K_2) := \sup_{x_1 \in K_1} d(x_1, K_2) \vee \sup_{x_2 \in K_2} d(x_2, K_1),$$

where $d((x, K) := \inf_{y \in K} d(x, y)$.
A *correspondence* between two sets A_1, A_2 is a set $R \subset A_1 \times A_2$ such that

$$\forall x_1 \in A_1 \ \exists x_2 \in A_2 \text{ s.t. } (x_1, x_2) \in R,$$
$$\forall x_2 \in A_2 \ \exists x_1 \in A_1 \text{ s.t. } (x_1, x_2) \in R.$$

Let $\mathrm{Cor}(A_1, A_2)$ denote the set of all correspondences between A_1 and A_2.

$$d_{\rm H}(K_1, K_2) = \inf_{R \in \mathrm{Cor}(K_1, K_2)} \sup_{(x_1, x_2) \in R} d(x_1, x_2).$$

The Hausdorff metric

Convergence criterion $d_{\mathrm{H}}(K_n, K) \underset{n\to\infty}{\longrightarrow} 0 \quad \Leftrightarrow$

(i) $\exists$ compact C such that $K_n \subset C \ \forall n$,

(ii) $K = \{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x_n \to x\}$,

(iii) $K = \{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x \text{ is a cluster point of } (x_n)_{n\in\mathbb{N}}\}$.

Corollary If d, d' generate the same topology on $\mathcal{X}$, then $d_{\mathrm{H}}, d'_{\mathrm{H}}$ generate the same topology on $\mathcal{K}_+(\mathcal{X})$.

Note For (ii) and (iii) suffices to check

(ii)' $K \subset \{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x_n \to x\}$,

(iii)" $K \supset \{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x \text{ is a cluster point of } (x_n)_{n\in\mathbb{N}}\}$.

Lemma
If $(\mathcal{X}, d)$ is separable, then so is $(\mathcal{K}_+(\mathcal{X}), d_{\rm H})$.
If $(\mathcal{X}, d)$ is complete, then so is $(\mathcal{K}_+(\mathcal{X}), d_{\rm H})$.

Lemma
$\mathcal{A} \subset \mathcal{K}_+(\mathcal{X})$ is precompact $\quad \Leftrightarrow$
$\exists$ compact C such that $K \subset C \quad \forall K \in \mathcal{A}$.

Lemma
$d_{\rm H}(K_n, K) \underset{n\to\infty}{\longrightarrow} 0$ and K_n connected $\forall n \quad \Rightarrow \quad K$ connected.

The ordered Hausdorff metric

Let $\mathcal{X}$ be a metrisable space and let $\preceq$ be a partial order on $\mathcal{X}$.

Def $\preceq$ is *compatible with the topology* if

$$\mathcal{X}^{\langle 2\rangle} := \big\{(x,y)\in\mathcal{X}^2 : x\preceq y\big\}$$

is a closed subset of $\mathcal{X}^2$, equipped with the product topology. In other words: $x_n \preceq y_n,\ x_n \underset{n\to\infty}{\longrightarrow} x,\ y_n \underset{n\to\infty}{\longrightarrow} y \ \Rightarrow\ x\preceq y$.

Def $\mathcal{K}_{\rm part}(\mathcal{X})$ is the set of pairs $(K,\preceq)$ such that $K\in\mathcal{K}_+(\mathcal{X})$ and $\preceq$ is a partial order on K that is compatible with the induced topology from $\mathcal{X}$.

$$\mathcal{K}_{\rm tot}(\mathcal{X}) := \big\{(K,\preceq)\in\mathcal{K}_{\rm part}(\mathcal{X}) : \preceq \text{ is a total order}\big\}.$$

We often denote elements of $\mathcal{K}_{\rm part}(\mathcal{X}),\mathcal{K}_{\rm tot}(\mathcal{X})$ simply as K.

The ordered Hausdorff metric

Note $d^2\big((x_1,y_1),(x_2,y_2)\big) := d(x_1,x_2) \vee d(y_1,y_2)$ generates the product topology.

Def $d_{\rm part}(K_1,K_2) := d^2_{\rm H}(K_1^{\langle 2\rangle},K_2^{\langle 2\rangle}) \qquad \big(K_1,K_2 \in \mathcal{K}_{\rm part}(\mathcal{X})\big)$, where $d^2_{\rm H}$ is the Hausdorff metric associated with d^2.

Def ${\rm Cor}_+(K_1,K_2)$ is the set of correspondences $R \in {\rm Cor}(K_1,K_2)$ that are *monotone* in the sense that:

$$\nexists\ (x_1,x_2),(y_1,y_2) \in R \text{ such that } x_1 \prec y_1 \text{ and } y_2 \prec x_2,$$

where $x \prec y$ means $x \preceq y$ and $x \neq y$.

Def $d_{\rm tot}(K_1,K_2) := \inf_{R \in {\rm Cor}_+(K_1,K_2)} \sup_{(x_1,x_2)\in R} d(x_1,x_2)$

$\big(K_1,K_2 \in \mathcal{K}_{\rm tot}(\mathcal{X})\big)$.

The ordered Hausdorff metric

Lemma $d_{\mathrm{H}}(K_1, K_2) \le d_{\mathrm{part}}(K_1, K_2) \le d_{\mathrm{tot}}(K_1, K_2)$ $\big(K_1, K_2 \in \mathcal{K}_{\mathrm{tot}}(\mathcal{X})\big)$, but the opposite inequalities do not hold: $\forall \varepsilon > 0\ \exists K_1, K_2 \in \mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ s.t. $d_{\mathrm{part}}(K_1, K_2) \le \varepsilon d_{\mathrm{tot}}(K_1, K_2)$.

Theorem d_{part} and d_{tot} generate the same topology on $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$.

Def *mismatch modulus* $m_\varepsilon(K)$ as

$$m_\varepsilon(K) := \sup\Big\{ d(x_1, y_1) \vee d(x_2, y_2) : x_1, y_1, x_2, y_2 \in K \\ d(x_1, x_2) \vee d(y_1, y_2) \le \varepsilon,\ x_1 \preceq y_1,\ y_2 \preceq x_2 \Big\}.$$

Theorem $\mathcal{A} \subset \mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ is precompact $\Leftrightarrow$

(i) $\exists$ compact $C \subset \mathcal{X}$ s.t. $K \subset C\ \forall K \in \mathcal{A}$,

(ii) $\lim_{\varepsilon \to 0} \sup_{K \in \mathcal{A}} m_\varepsilon(K) = 0$.

The ordered Hausdorff metric

Recall that a topological space $\mathcal{X}$ is Polish if:

(i) $\mathcal{X}$ is separable,

(ii) there exists a complete metric generating the topology on $\mathcal{X}$.

Note There are in general also many noncomplete metrics generating the same topology, unless $\mathcal{X}$ is compact.

Theorem If $\mathcal{X}$ is a Polish space, then so is $\mathcal{K}_{\rm tot}(\mathcal{X})$, equipped with the ordered Hausdorff topology.

Skorohod's J1 topology

Recall: the closed graph of a function $f \in \overline{\mathcal{D}}_{[s,u]}(\mathcal{X})$ is defined as

$$\begin{aligned}\mathcal{G}(f) &:= \big\{\big(\underline{\tau}, f(\tau)\big) : \tau \in [\![s-, u+]\!]\big\}\\ &= \big\{\big(t, f(t\pm)\big) : t \in [s,u]\big\}.\end{aligned}$$

We equip $\mathcal{G}(f)$ with a total order such that

$$\big(\underline{\sigma}, f(\sigma)\big) \preceq \big(\underline{\tau}, f(\tau)\big) \quad \Leftrightarrow \quad \sigma \leq \tau.$$

Then $\mathcal{G}(f) \in \mathcal{K}_{\rm tot}\big(\mathbb{R} \times \mathcal{X}\big)$, and

$$\begin{aligned}d^{\rm S}_{\rm part}(f,g) &:= d_{\rm part}\big(\mathcal{G}(f), \mathcal{G}(g)\big),\\ d^{\rm S}_{\rm tot}(f,g) &:= d_{\rm tot}\big(\mathcal{G}(f), \mathcal{G}(g)\big)\end{aligned}$$

both generate Skorohod's J1 topology on $\mathcal{D}_{[s,u]}(\mathcal{X})$.

To see this, equip $\mathbb{R} \times \mathcal{X}$ with the metric

$$\rho\big((t_1, x_1), (t_2, x_2)\big) := d(x_1, x_2) + |t_1 - t_2|.$$

Let Λ_+ denote the space of continuous increasing functions $\lambda : [s, u] \to [s, u]$ with $\lambda(s) = s$ and $\lambda(u) = u$.

Lemma For $f, g \in \mathcal{D}_{[s,u]}(\mathcal{X})$, one has:

$$d^{\mathrm{S}}_{\mathrm{tot}}(f, g) = \inf_{\lambda \in \Lambda_+} \sup_{t \in [s,u]} \big\{ d\big(f(\lambda(t)), g(t)\big) + \big|\lambda(t) - t\big| \big\}.$$

Proof idea The closure of

$$\big\{\big((\lambda(t), f(\lambda(t))), (t, g(t))\big) : t \in [s, t]\big\}$$

is a monotone correspondence between $\mathcal{G}(f)$ and $\mathcal{G}(g)$, and every monotone correspondence can be approximated by monotone correspondences of this form. ∎

Skorohod's J1 topology

Remark 1 The previous lemma holds only for $f, g \in \mathcal{D}_{[s,u]}(\mathcal{X})$, but $d^{\rm S}_{\rm tot}$ is well-defined on $\overline{\mathcal{D}}_{[s,u]}(\mathcal{X})$.

Remark 2 The topology generated by $d^{\rm S}_{\rm tot}$ depends only on the topology on $\mathcal{X}$ and not on the choice of the metric d on $\mathcal{X}$.

Theorem If $\mathcal{X}$ is a Polish space, then so is $\overline{\mathcal{D}}_{[s,u]}(\mathcal{X})$, equipped with the J1 topology.

Proposition The space $\mathcal{C}_{[s,u]}(\mathcal{X})$ of continuous functions $f : [s, u] \to \mathcal{X}$ is a closed subset of $\overline{\mathcal{D}}_{[s,u]}(\mathcal{X})$.

Proof $f \in \mathcal{C}_{[s,u]}(\mathcal{X}) \quad \Leftrightarrow \quad \mathcal{G}(f)$ connected, and convergence in the Hausdorff metric preserves connectedness. ∎

Compactness criterion

For each $\delta > 0$, the *Skorohod modulus of continuity* is defined as

$$m^{\mathrm{S}}_\delta(f) := \sup_{\substack{\tau_1 \leq \tau_2 \leq \tau_3 \\ \tau_3 - \tau_1 \leq \delta}} d\big(f(\tau_2), \{f(\tau_1), f(\tau_3)\}\big).$$

Theorem $\mathcal{A} \subset \overline{\mathcal{D}}_{[s,u]}(\mathcal{X})$ is precompact $\Leftrightarrow$

(i) $\exists$ compact $C \subset \mathcal{X}$ s.t. $f(t\pm) \in C \ \forall f \in \mathcal{A},\ t \in [s,u]$,

(ii) $\lim_{\delta \to 0} \sup_{f \in \mathcal{A}} m^{\mathrm{S}}_\delta(f) = 0$.

For precompactness in $\mathcal{D}_{[s,u]}(\mathcal{X})$, one in addition needs

(iii) $\lim_{\delta \to 0} \sup_{f \in \mathcal{A}} \sup_{t \leq s+\delta} d\big(f(s), f(t)\big) = 0$,

(iv) $\lim_{\delta \to 0} \sup_{f \in \mathcal{A}} \sup_{t \geq u-\delta} d\big(f(t), f(u)\big) = 0$,

as proved by Skorohod (1956).

Unbounded time intervals

Billingsley (1968), Ethier & Kurtz (1986), and Whitt (2002) have extended the J1 topology to $\mathcal{D}_{[0,\infty)}(\mathcal{X})$ by showing that there exists a metric $d'_{\rm S}$ such that $d'_{\rm S}(f_n, f) \underset{n\to\infty}{\longrightarrow} 0 \;\Leftrightarrow$

$$d_{\rm S}\big(f_n\big|_{[0,t]}, f\big|_{[0,t]}\big) \underset{n\to\infty}{\longrightarrow} 0 \quad \forall t > 0 \text{ s.t. } f(t-) = f(t),$$

where $f\big|_{[0,t]}$ denotes the restriction of f to $[0,t]$.

Note The map $f \mapsto f\big|_{[0,t]}$ is not continuous.

Def *squeezed space* $\mathcal{R}(\mathcal{X}) := (\mathbb{R} \times \mathcal{X}) \cup \big\{(-\infty, *), (\infty, *)\big\}$.

Lemma There exists a metric $d_{\rm sqz}$ on $\mathcal{R}(\mathcal{X})$ such that
$d\big((t_n, x_n), (t, x)\big) \underset{n\to\infty}{\longrightarrow} 0 \quad \Leftrightarrow$

(i) $t_n \to t$ in the topology on $\overline{\mathbb{R}}$,

(ii) if $t \in \mathbb{R}$, then also $x_n \to x$ in the topology on $\mathcal{X}$.

Proof Let $d_{\overline{\mathbb{R}}}$ generate the topology on $\overline{\mathbb{R}} = [-\infty, \infty]$.
Let $\varphi : \overline{\mathbb{R}} \to [0, \infty)$ satisfy $\varphi(t) > 0 \ \Leftrightarrow \ t \in \mathbb{R}$.

Then $d_{\rm sqz}\big((s, x), (t, y)\big) :=$
$\big(\varphi(s) \wedge \varphi(t)\big)\big(d(x, y) \wedge 1\big) + \big|\varphi(s) - \varphi(t)\big| + d_{\overline{\mathbb{R}}}(s, t)$
does the trick. ∎

Idea: care less about spatial distances
when the time coordinates are large.

Squeezed space

Lemma
If $(\mathcal{X}, d)$ is separable, then so is $(\mathcal{R}(\mathcal{X}), d_{\mathrm{sqz}})$.
If $(\mathcal{X}, d)$ is complete, then so is $(\mathcal{R}(\mathcal{X}), d_{\mathrm{sqz}})$.

Lemma
$A \subset \mathcal{R}(\mathcal{X})$ is precompact $\quad \Leftrightarrow$
$\forall T < \infty \ \exists$ compact $C \subset \mathcal{X}$
such that $x \in C \quad \forall (t, x) \in A, \ t \in [-T, T]$.

Path space

For each $I \subset \mathbb{R}$ set $I_{\mathfrak{s}} := \{t\pm : t \in I\}$.

By definition, a *path* is a pair (I, f), where $I \subset \mathbb{R}$ is closed and $f : I_{\mathfrak{s}} \to \mathcal{X}$ is continuous. For brevity, write $f = (I, f)$ and $I(f) = I$.

Def The *closed graph* of a path is

$$\mathcal{G}(f) := \big\{(t,x) : t \in I(f),\ x \in \{f(t-), f(t+)\}\big\} \\ \cup\big\{(-\infty,*), (\infty,*)\big\}.$$

Naturally $\mathcal{G}(f) \in \mathcal{K}_{\rm tot}\big(\mathcal{R}(\mathcal{X})\big)$.

Equip the *path space* $\Pi(\mathcal{X})$ with the metric

$$d^{\rm S}_{\rm tot}(f,g) := d_{\rm tot}\big(\mathcal{G}(f), \mathcal{G}(g)\big).$$

Proposition Restricted to $\mathcal{D}_{[0,\infty)}(\mathcal{X})$, this generates the topology of Billingsley, Ethier & Kurtz, and Whitt.

Path space

Advantages of this approach:

- Comparison of functions with different domains.
- No need to interpolate.
- No need to extrapolate.

Example Let $(X_n)_{n\geq 0}$ be a random walk in the domain of attraction of an α-stable Lévy process $L=(L_t)_{t\geq 0}$.
Define $X^\varepsilon \in \Pi(\mathbb{R})$ by

$$X^\varepsilon_{\varepsilon^\alpha n} := \varepsilon X_n \quad \text{with domain} \quad \{\varepsilon^\alpha n : n \in \mathbb{N}\}.$$

Then

$$\mathbb{P}\big[X^\varepsilon \in \cdot\,\big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\big[L \in \cdot\,\big],$$

where $\Rightarrow$ denotes weak convergence of probability laws on $\Pi(\mathbb{R})$.

Theorem If $\mathcal{X}$ is a Polish space, then so is $\Pi(\mathcal{X})$, equipped with the J1 topology.

For each $T < \infty$ and $\delta > 0$, define

$$m^{\rm S}_{T,\delta}(f) := \sup_{\substack{\tau_1 \leq \tau_2 \leq \tau_3 \\ -T \leq \tau_1,\ \tau_3 \leq T \\ \tau_3 - \tau_1 \leq \delta}} d\big(f(\tau_2), \{f(\tau_1), f(\tau_3)\}\big).$$

Theorem $\mathcal{A} \subset \Pi(\mathcal{X})$ is precompact $\Leftrightarrow$

(i) $\forall T < \infty \ \exists$ compact $C \subset \mathcal{X}$ s.t. $f(t\pm) \in C$ $\forall f \in \mathcal{A},\ t \in I(f) \cap [-T, T]$,

(ii) $\lim_{\delta \to 0} \sup_{f \in \mathcal{A}} m^{\rm S}_{T,\delta}(f) = 0 \quad \forall T < \infty$.

Betweenness

Def a *betweenness* on $\mathcal{X}$ is a a function that assigns to each pair x, z of elements of $\mathcal{X}$ a subset $\langle x, z\rangle$ of $\mathcal{X}$, such that:

(i) $\langle x, z\rangle = \langle z, x\rangle$,

(ii) $x \in \langle x, z\rangle$,

(iii) $y \in \langle x, z\rangle \;\Rightarrow\; \langle x, y\rangle \cap \langle y, z\rangle = \{y\}$,

(iv) $y \in \langle x, z\rangle \;\Rightarrow\; \langle x, y\rangle \cup \langle y, z\rangle = \langle x, z\rangle$.

Def total order $\leq$ on the *segment* $\langle x, z\rangle$ by

$$y \leq y' \quad \Leftrightarrow \quad \langle x, y\rangle \subset \langle x, y'\rangle.$$

Def a betweenness is *compatible with the topology* if $\langle x, z\rangle$ is compact and $\mathcal{X}^2 \ni (x, z) \mapsto \langle x, z\rangle \in \mathcal{K}_{\rm tot}(\mathcal{X})$ is continuous.

Betweenness

Linear betweenness If $\mathcal{X}$ is a topological vector space, then $\langle x, z\rangle := \big\{(1-p)x + pz : p \in [0,1]\big\}$ is a betweenness that is compatible with the topology.

Trivial betweenness $\langle x, z\rangle := \{x, z\}$ is always a betweenness that is compatible with the topology.

Geodesic betweenness If $(\mathcal{X}, d)$ has unique geodesics, then letting $\langle x, z\rangle$ denote the geodesic with endpoints x, z defines a betweenness. If closed balls are compact, then this betweennness is compatible with the topology.

Order betweenness If $\mathcal{X} \subset \mathbb{R}$ is closed, then
$\langle x, z\rangle := \big\{y : x \leq y \leq z \text{ or } z \leq y \leq x\big\}$
is a betweenness that is compatible with the topology.

The M1 topology

Assume that $\mathcal{X}$ is equipped with a betweenness that is compatible with the topology.

Def The *filled graph* of a path is

$$\mathcal{G}(f) := \big\{(t,x) : t \in I(f),\ x \in \langle f(t-), f(t+)\rangle\big\} \cup \big\{(-\infty,*),(\infty,*)\big\}.$$

Naturally $\mathcal{G}(f) \in \mathcal{K}_{\rm tot}\big(\mathcal{R}(\mathcal{X})\big)$.

Equip the path space $\Pi(\mathcal{X})$ with the metric

$$d^{\rm S}_{\rm tot}(f,g) := d_{\rm tot}\big(\mathcal{G}(f),\mathcal{G}(g)\big).$$

For the trivial betweenness, this yields the J1 topology.
For the linear betweenness, this yields the M1 topology.

Path space

Fix a betweenness on $\mathcal{X}$ that is compatible with the topology and equip $\Pi(\mathcal{X})$ with the associated Skorohod topology. Then:

Theorem If $\mathcal{X}$ is a Polish space, then so is $\Pi(\mathcal{X})$.

For each $T < \infty$ and $\delta > 0$, define

$$m^{\mathrm{S}}_{T,\delta}(f) := \sup_{\substack{\tau_1 \leq \tau_2 \leq \tau_3 \\ -T \leq \tau_1,\ \tau_3 \leq T \\ \tau_3 - \tau_1 \leq \delta}} d\big(f(\tau_2), \langle f(\tau_1), f(\tau_3)\rangle\big).$$

Theorem $\mathcal{A} \subset \Pi(\mathcal{X})$ is precompact $\Leftrightarrow$

(i) $\forall T < \infty\ \exists$ compact $C \subset \mathcal{X}$ s.t. $f(t\pm) \in C$ $\forall f \in \mathcal{A},\ t \in I(f) \cap [-T, T]$,

(ii) $\lim_{\delta \to 0} \sup_{f \in \mathcal{A}} m^{\mathrm{S}}_{T,\delta}(f) = 0 \quad \forall T < \infty$.