Interacting Particle Systems

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Graphs

By definition, a *graph* is a pair (Λ, E) with:

- *vertex set* Λ, a countable set.
- *edge set* E, a set whose elements are unordered pairs $\{i, j\}$ with $i, j \in \Lambda$, $i \neq j$.

Let

$$\mathcal{E} := \big\{(i,j) : \{i,j\} \in E\big\} \quad \text{and} \quad \mathcal{N}_i := \big\{j \in \Lambda : \{i,j\} \in E\big\}.$$

Example $\Lambda = \mathbb{Z}^d$. For $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$, let

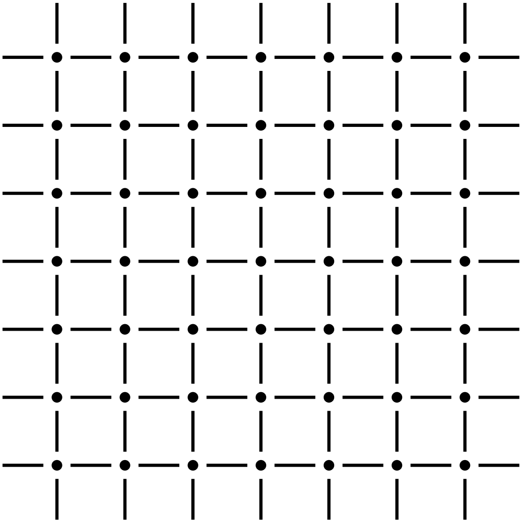
$$\|i\|_1 := \sum_{k=1}^{d} |i_k| \quad \text{and} \quad \|i\|_\infty := \max_{k=1,\ldots,d} |i_k| \qquad (i \in \mathbb{Z}^d).$$

For $R \geq 1$, we set

$$E^d := \big\{\{i,j\} : \|i-j\|_1 = 1\big\}, \quad E^d_R := \big\{\{i,j\} : 0 < \|i-j\|_\infty \leq R\big\}.$$

The square lattice with nearest neighbor edges

$(\mathbb{Z}^2, E^2)$



The one-dimensional integer lattice

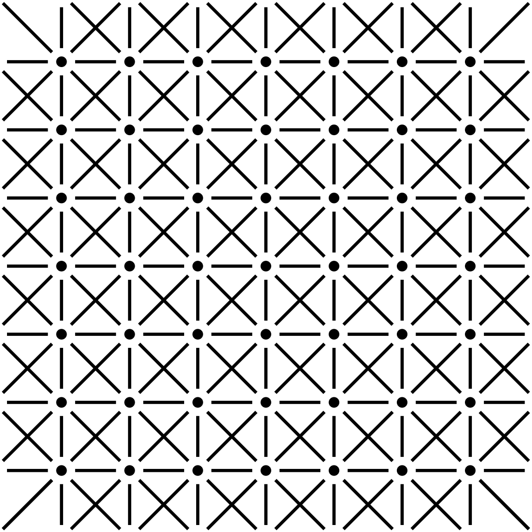
$(\mathbb{Z}^1, E^1)$

Next nearest neighbor edges

$(\mathbb{Z}^1, E_2^1)$

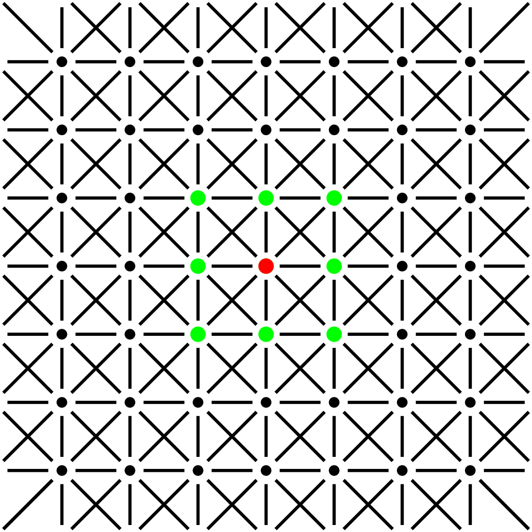
The square lattice with L_1 neighborhood

$(\mathbb{Z}^2, E_1^2)$



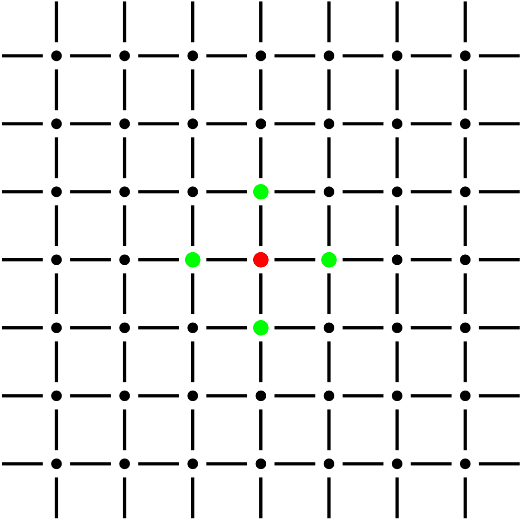
The square lattice with L_1 neighborhood

$\mathcal{N}_i$ in $(\mathbb{Z}^2, E_1^2)$



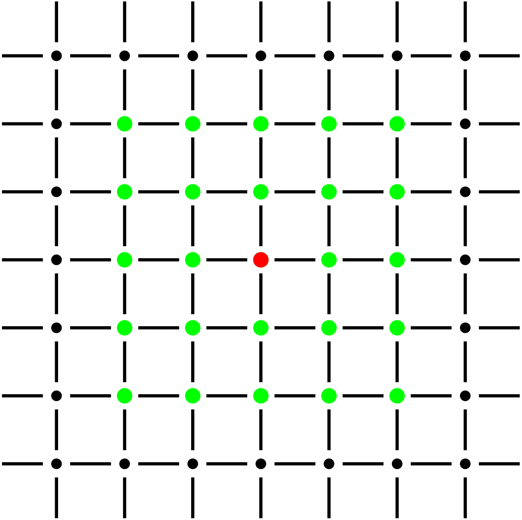
Nearest neighbor neighborhood

$\mathcal{N}_i$ in $(\mathbb{Z}^2, E^2)$



Range two neighborhood

$\mathcal{N}_i$ in $(\mathbb{Z}^2, E_2^2)$



Interacting particle systems

Λ	countable set	*lattice*
S	finite set	*local state space*
S^Λ	Carthesian product	set of *configurations* $x = \big(x(i)\big)_{i\in\Lambda}$ with $x(i) \in S \ \forall\ i \in \Lambda$.

$X = (X_t)_{t\geq 0}$ *interacting particle system*, Markov process with state space S^Λ.

$$X_t = \big(X_t(i)\big)_{i\in\Lambda} \quad \text{with} \quad X_t(i) \in S \ \forall\ i \in \Lambda.$$

$\mathcal{G}$ collection of *local maps* $m : S^\Lambda \to S^\Lambda$.
$(r_m)_{m\in\mathcal{G}}$ collection of *Poisson rates*.

$$\textit{generator} \quad Gf(x) = \sum_{m\in\mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\}.$$

Interpretation: $r_m \,\mathrm{d}t$ is the probability that the map m is applied during the time interval $(t, t + \mathrm{d}t]$.

Example: the contact process

$S = \{0,1\}$. Interpretation: $x \in \{0,1\}^\Lambda$ is a *particle configuration*

$$x(i) = 0 \text{ means the site } i \text{ is empty,}$$
$$x(i) = 1 \text{ means there is a particle at } i.$$

For each $(i,j) \in \mathcal{E}$, define a *branching map* $\mathtt{bra}_{ij} : S^\Lambda \to S^\Lambda$ by

$$\mathtt{bra}_{ij} x(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

For each $i \in \Lambda$, define a *death map* $\mathtt{death}_i : S^\Lambda \to S^\Lambda$ by

$$\mathtt{death}_i x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Rates: $r_{\mathtt{bra}_{ij}} = \lambda \quad \forall (i,j) \in \mathcal{E} \qquad r_{\mathtt{death}_i} = 1 \quad \forall i \in \Lambda.$

The contact process

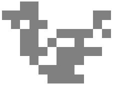
Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 0$.

The contact process

Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 1$.

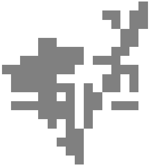
Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 2$.

The contact process



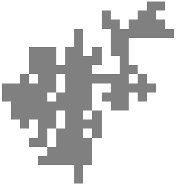
Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 3$.

The contact process



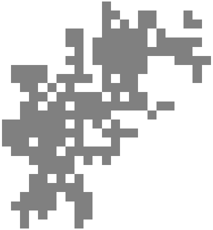
Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 4$.

The contact process



Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 5$.

The contact process



Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 6$.

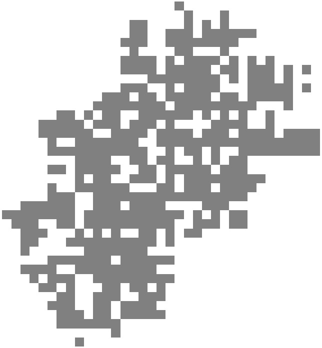
The contact process

Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 7$.

The contact process

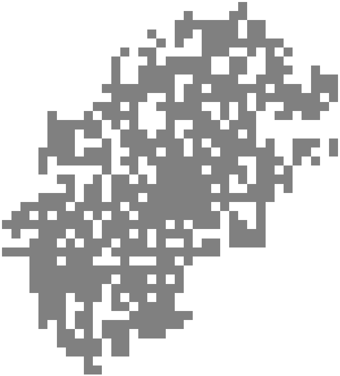
Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 8$.

The contact process



Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 9$.

The contact process



Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 10$.

The contact process

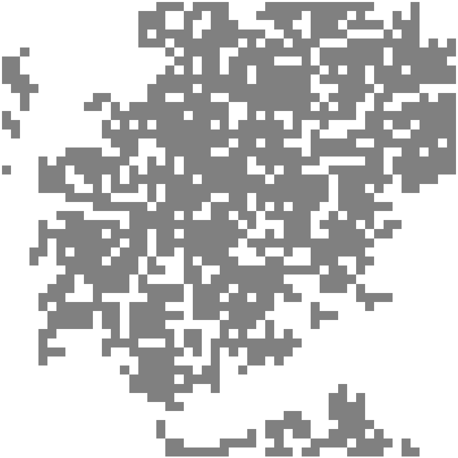
Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 11$.

Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 12$.

The contact process

Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 13$.

The contact process

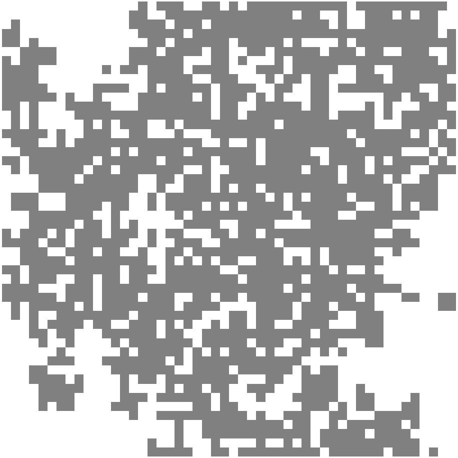


Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 14$.

The contact process

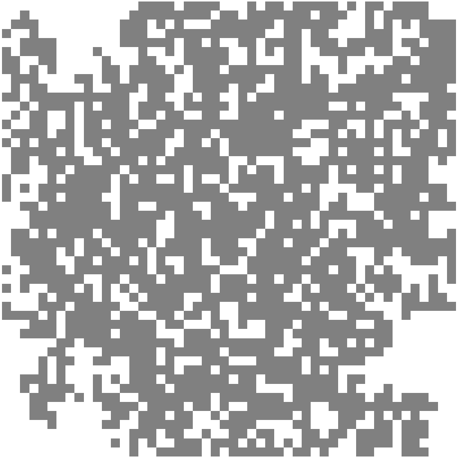
Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 15$.

The contact process

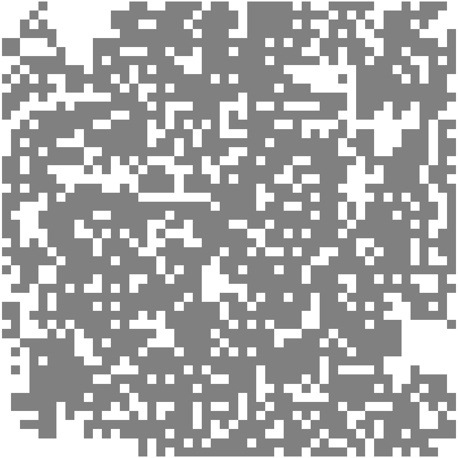


Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 16$.

The contact process

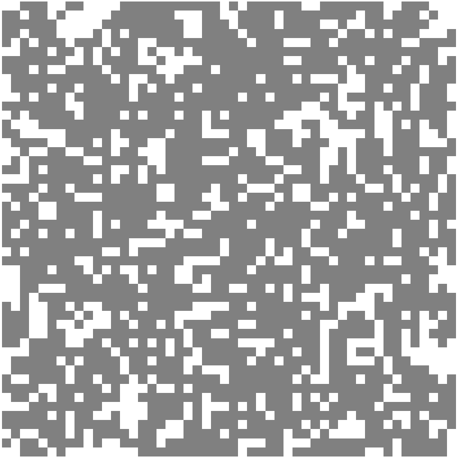


Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 17$.



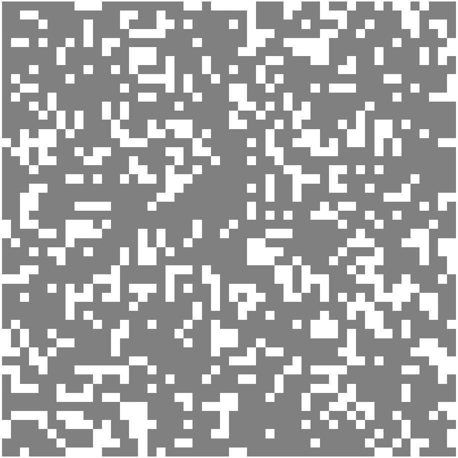
Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 18$.

The contact process



Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 19$.

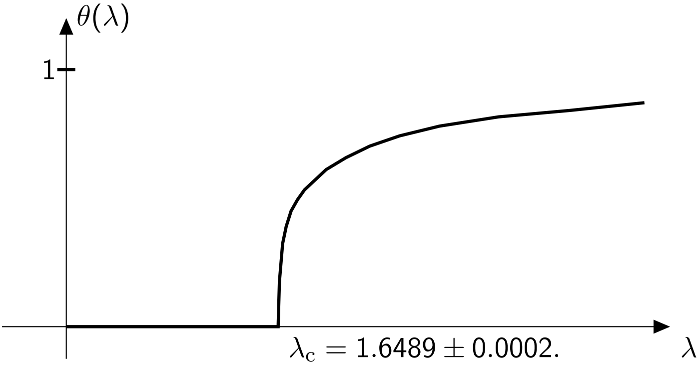
The contact process



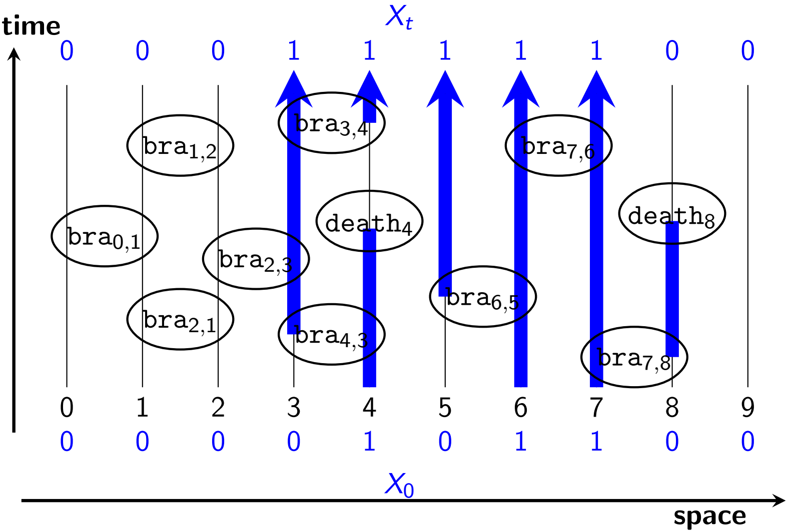
Contact process on $(\mathbb{Z}^2, E^2)$ with $\lambda = 2$.
Time $t = 20$.

Claim equilibrium density equals survival probability

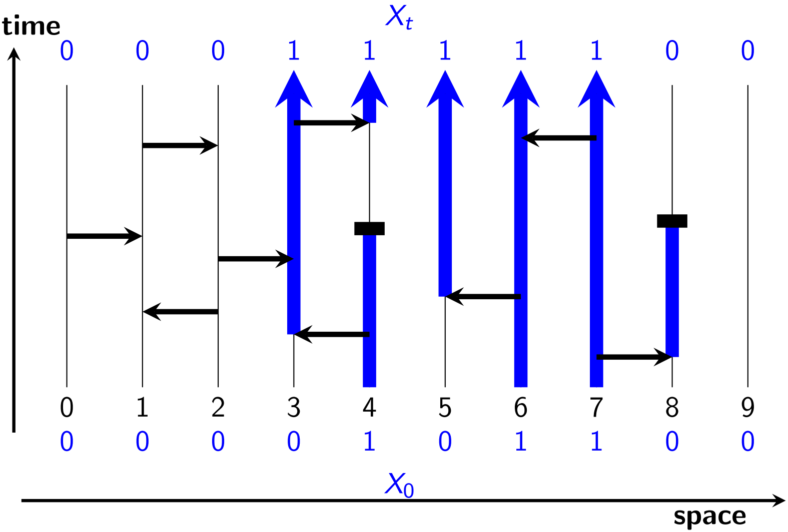
$$\theta(\lambda) := \mathbb{P}^{\underline{1}}[X_\infty(0) = 1] = \mathbb{P}^{1_{\{0\}}}[X_t \neq 0\ \forall t \geq 0].$$



Example: a finite contact process



Example: a finite contact process



Poisson construction of interacting particle systems

Idea: construct a continuous time Markov chain by applying maps $m : S \to S$ at times of a Poisson process.

Let ω be a Poisson point set on $\mathcal{G} \times \mathbb{R}$ with intensity $r_m \, \mathrm{d}t$.

Interpretation: for each $(m, t) \in \omega$, apply the map m at time t.

Let $\{t \geq 0 : (m, t) \in \omega\} = \{t_1, t_2, \ldots\}$ with $t_1 < t_2 < \cdots$
Then t_1, $t_2 - t_1$, $t_3 - t_2$,... are i.i.d. exponentially distributed with mean $1/r_m$.

Let $\mathbb{P}^x[(X_t)_{t\geq 0} \in \cdot\,] := \mathbb{P}[(X_t)_{t\geq 0} \in \cdot\,|\, X_0 = x]$.

Let $P_t(x,A) := \mathbb{P}^x[X_t \in A]$ denote the transition kernel of $(X_t)_{t\geq 0}$.
Semigroup property:

$$P_s P_t = P_{s+t} \quad \text{with} \quad P_s P_t(x,A) := \int P_s(x,\mathrm{d}y) P_t(y,A).$$

$\mathbb{E}^x[f(X_t)] = \int P_t(x,\mathrm{d}y) f(y) = f(x) + tGf(x) + O(t^2)$, where

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\}.$$

is the *generator* of the semigroup $(P_t)_{t\geq 0}$.

Voter model

S = any finite local state space.

Voter map

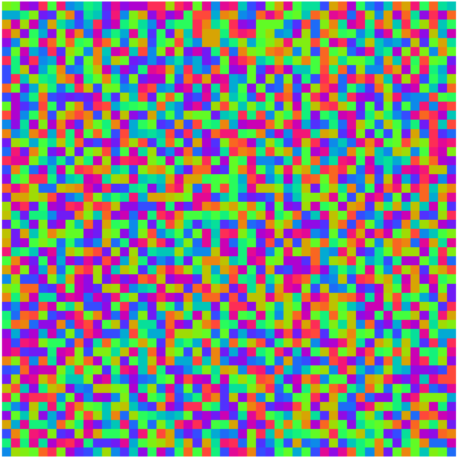
$$\mathtt{vot}_{ji}(x)(k) := \begin{cases} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise,} \end{cases}$$

Generator

$$Gf(x) = \sum_{(i,j)\in\mathcal{E}} \frac{1}{|\mathcal{N}_i|} \big\{ f\big(\mathtt{vot}_{ji}x\big) - f\big(x\big) \big\}.$$

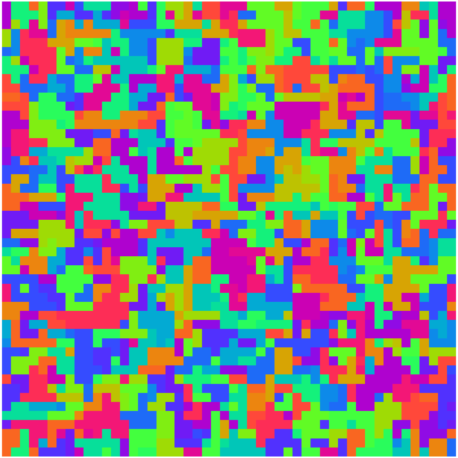
Interpretation: each site with rate 1 copies the state of a uniformly chosen neighbor.

The voter model



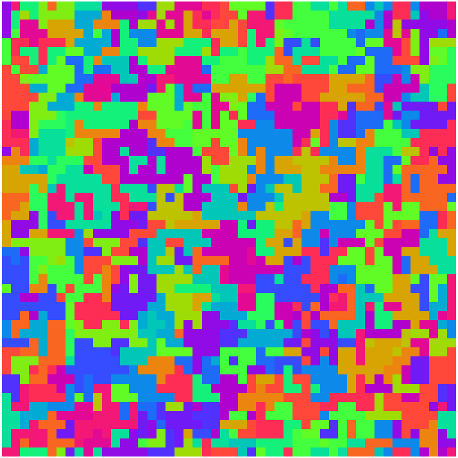
Time $t = 0$.

The voter model



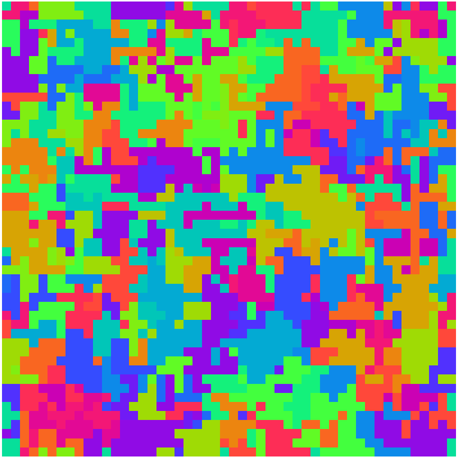
Time $t = 0.25$.

The voter model

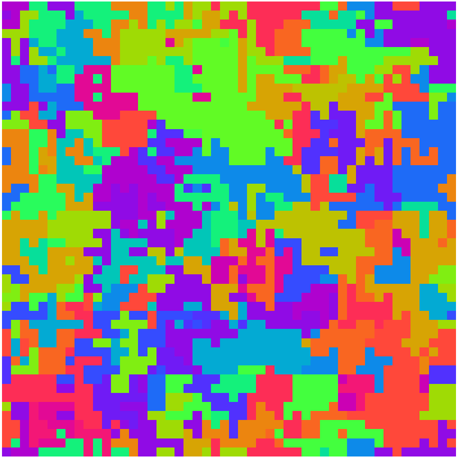


Time $t = 0.5$.

The voter model

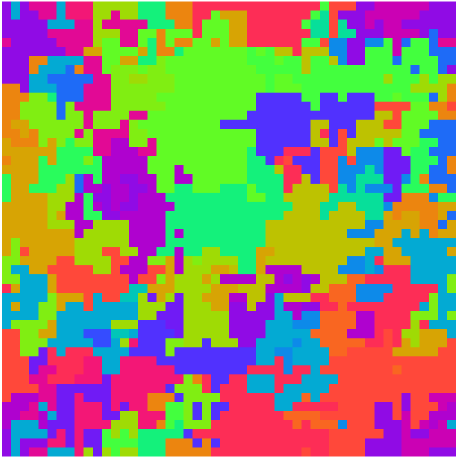


Time $t = 1$.

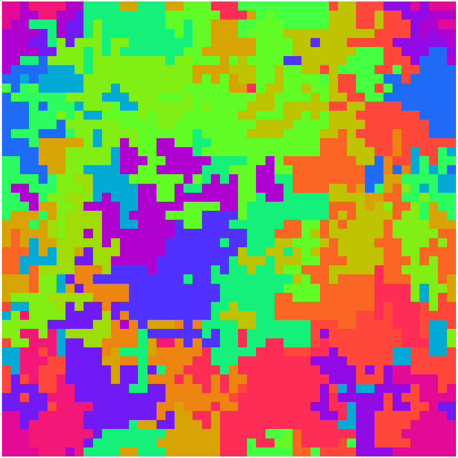


Time $t = 2$.

The voter model

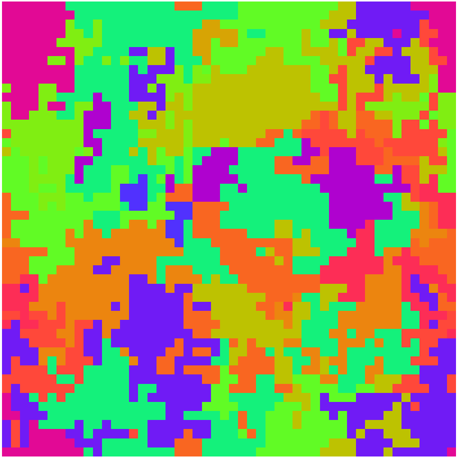


Time $t = 4$.



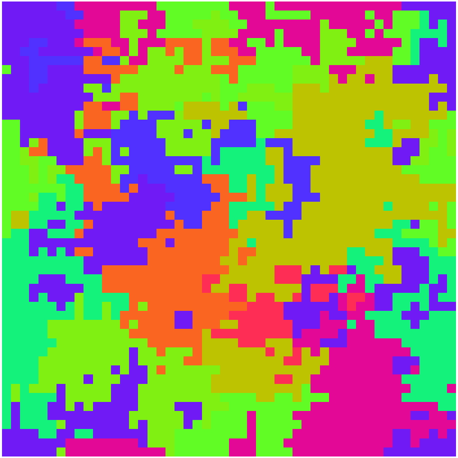
Time $t = 8$.

The voter model



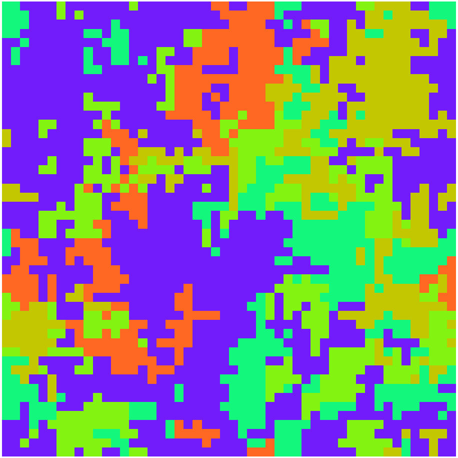
Time $t = 16$.

The voter model



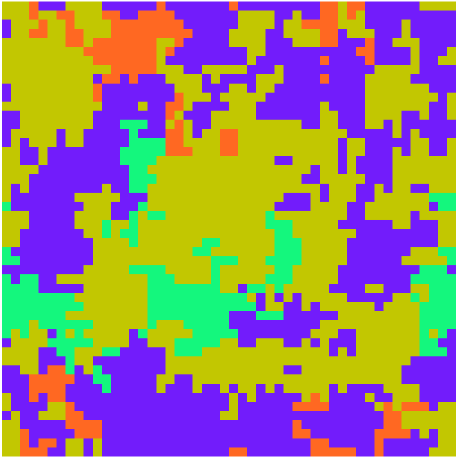
Time $t = 31.25$.

The voter model



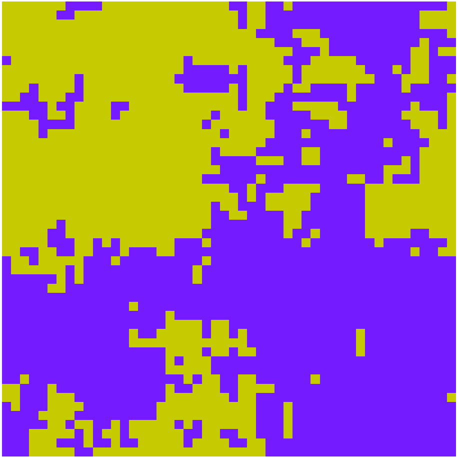
Time $t = 62.5$.

The voter model



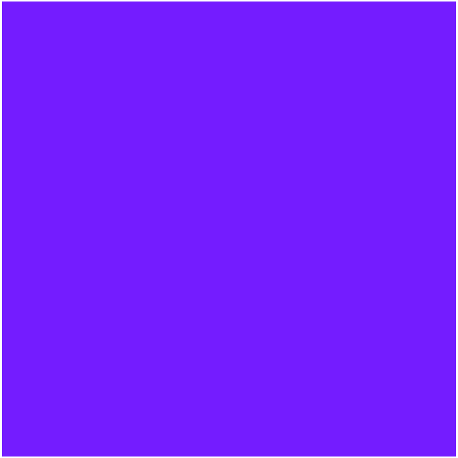
Time $t = 125$.

The voter model



Time $t = 250$.

The voter model



Time $t = 500$.

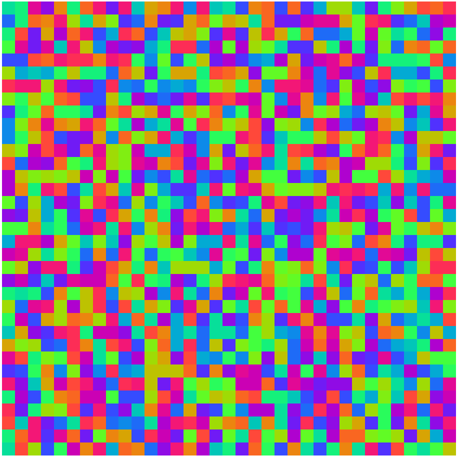
The voter model

The behavior of the voter model strongly depends on the dimension.

Clustering in dimensions $d = 1, 2$.

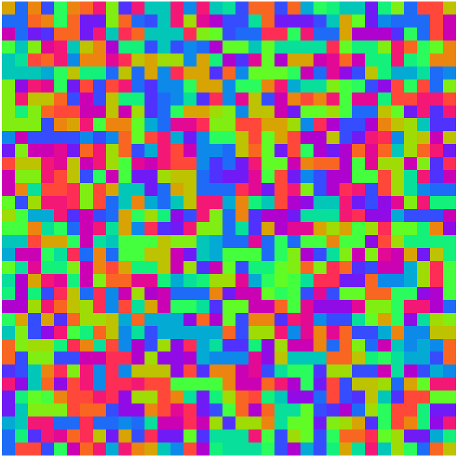
Stable behavior in dimensions $d \geq 3$.

The voter model



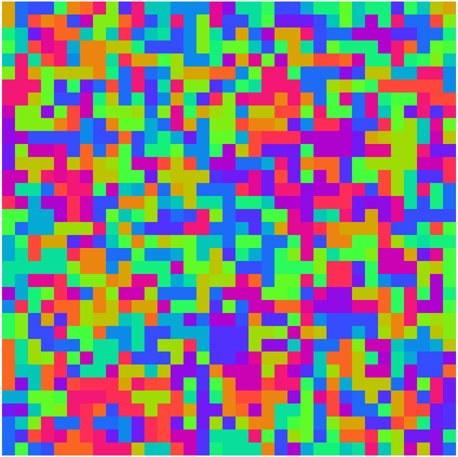
Cut of 3-dimensional model, time $t = 0$.

The voter model



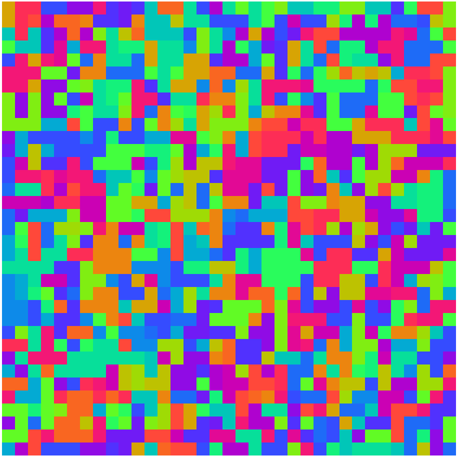
Cut of 3-dimensional model, time $t = 1$.

The voter model



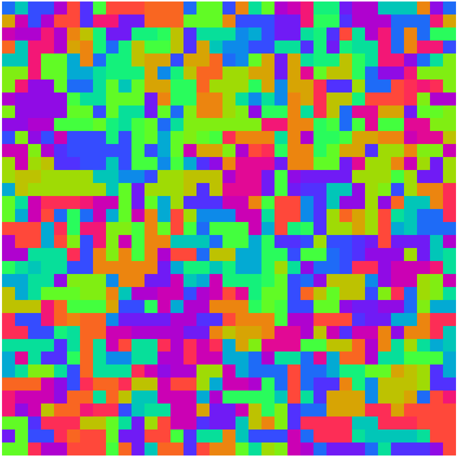
Cut of 3-dimensional model, time $t = 2$.

The voter model



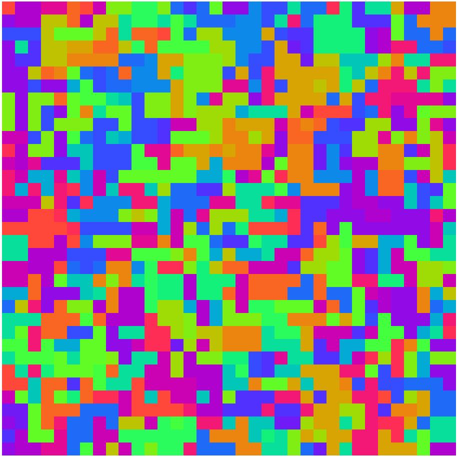
Cut of 3-dimensional model, time $t = 4$.

The voter model



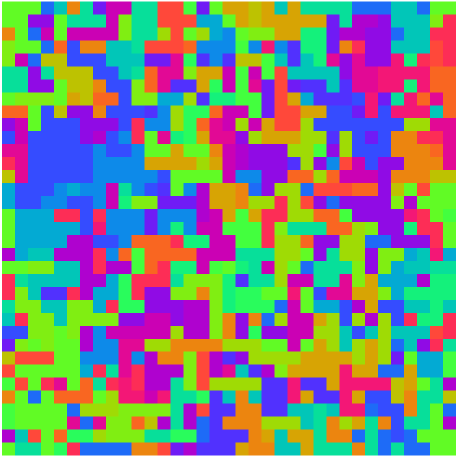
Cut of 3-dimensional model, time $t = 8$.

The voter model

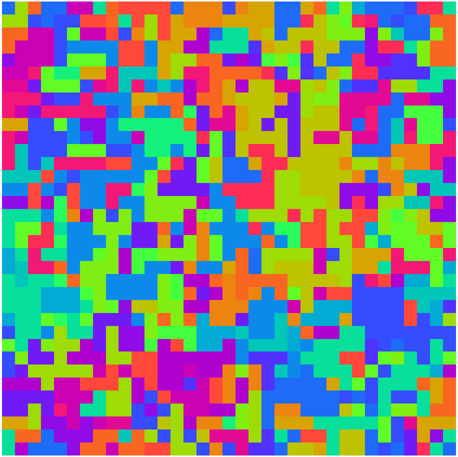


Cut of 3-dimensional model, time $t = 16$.

The voter model

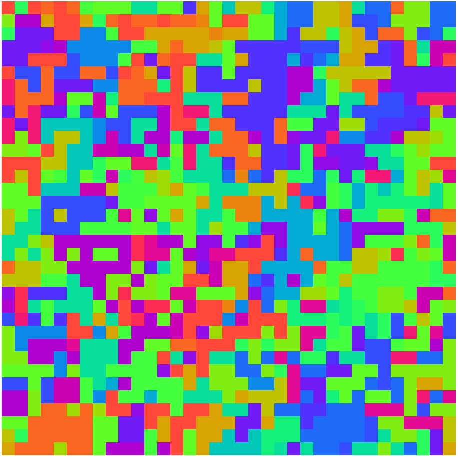


Cut of 3-dimensional model, time $t = 32$.



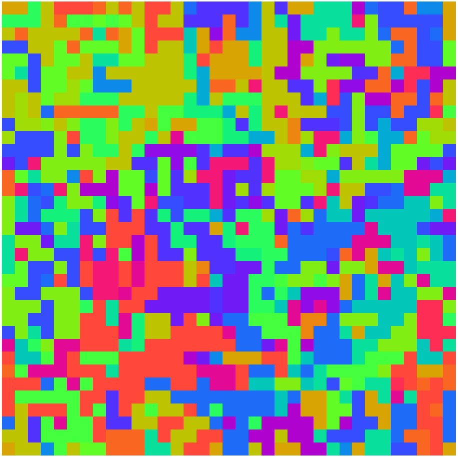
Cut of 3-dimensional model, time $t = 64$.

The voter model



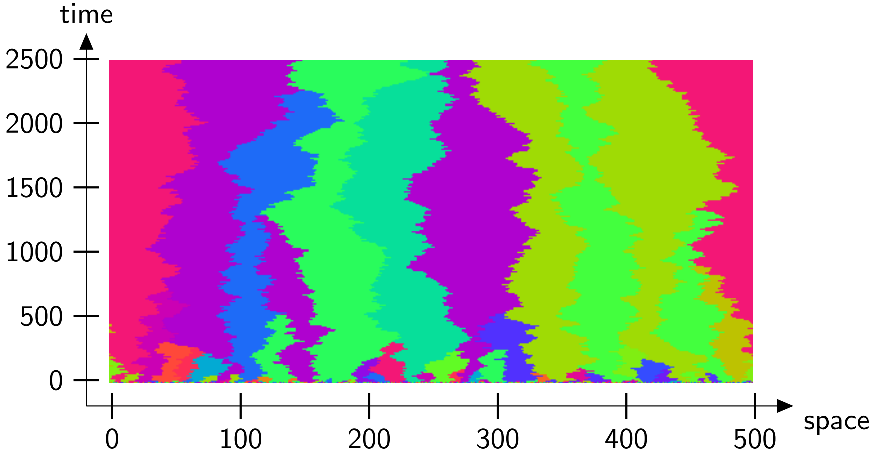
Cut of 3-dimensional model, time $t = 125$.

The voter model



Cut of 3-dimensional model, time $t = 250$.

A one-dimensional voter model



A one-dimensional voter model.

A stochastic Ising model

$S = \{-1, +1\}$ *spin*.

For any $x \in S^\Lambda$, we call

$$M_i(x) := \sum_{j \in \mathcal{N}_i} x(j)$$

the *local magnetization* around $i \in \Lambda$. Let $\kappa_i(x, \cdot\,)$ denote the law of a random variable X such that

$$\mathbb{P}[X(i) = \pm 1] = \frac{e^{\beta \pm M_i(x)}}{e^{\beta \pm M_i(x)} + e^{\beta \mp M_i(x)}},$$

and $X(j) = x(j)$ a.s. for all $j \neq i$. Then κ_i is a local probability kernel and

$$Gf(x) = \sum_{i \in \Lambda} \Big(\int \kappa_i(x, \mathrm{d}y) f(y) - f(x) \Big)$$

defines the generator of a *stochastic Ising model* with *Glauber dynamics*.

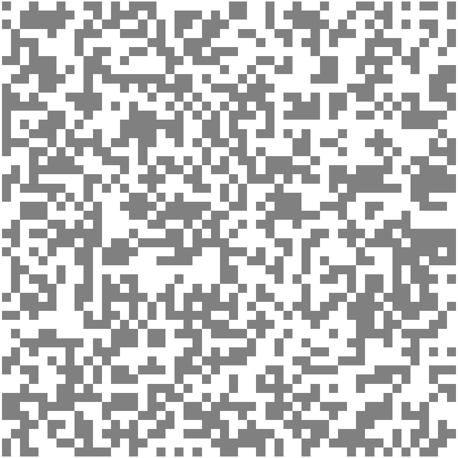
A stochastic Ising model

Interpretation: each site i with rate 1 chooses a new state according to the probability kernel κ_i.

When the parameter β is large, nearby spins like have the same sign.

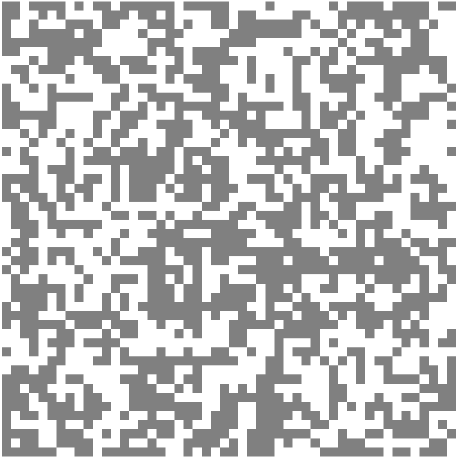
We start the process in product measure for different values of β and see what happens.

The Ising model



$\beta = 0.3$, time $t = 0$.

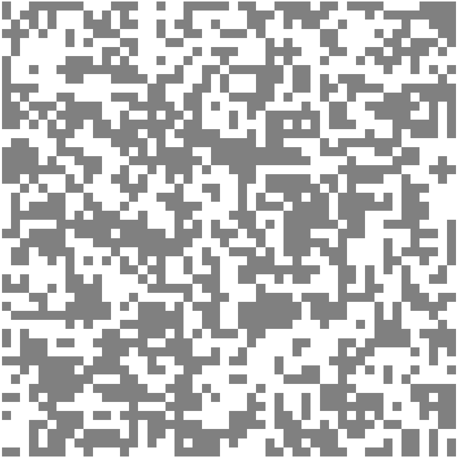
The Ising model



$\beta = 0.3$, time $t = 1$.

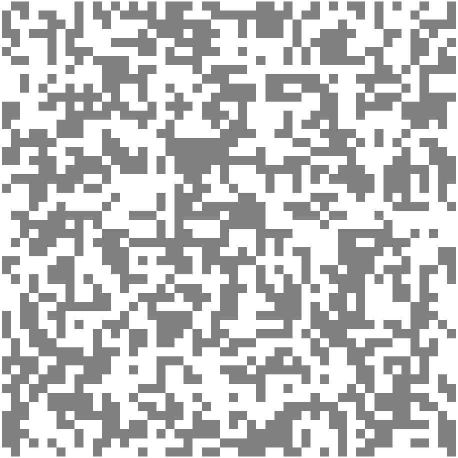
$\beta = 0.3$, time $t = 2$.

The Ising model



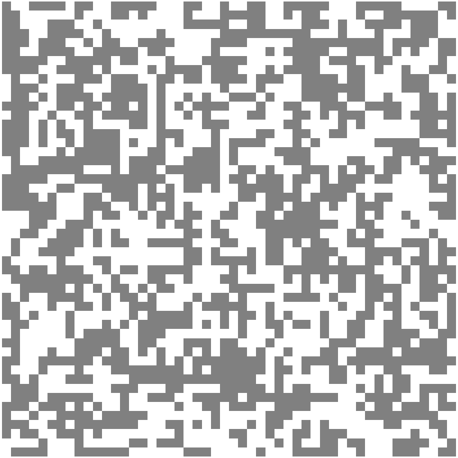
$\beta = 0.3$, time $t = 4$.

The Ising model



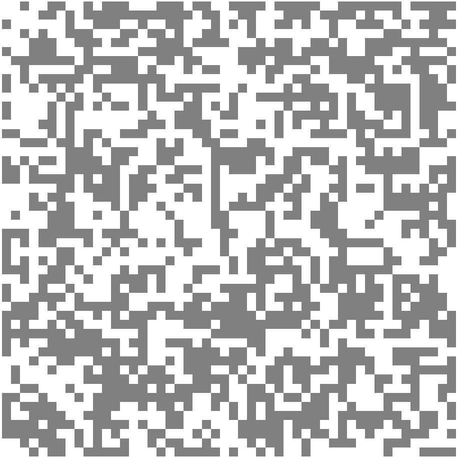
$\beta = 0.3$, time $t = 8$.

The Ising model



$\beta = 0.3$, time $t = 16$.

The Ising model

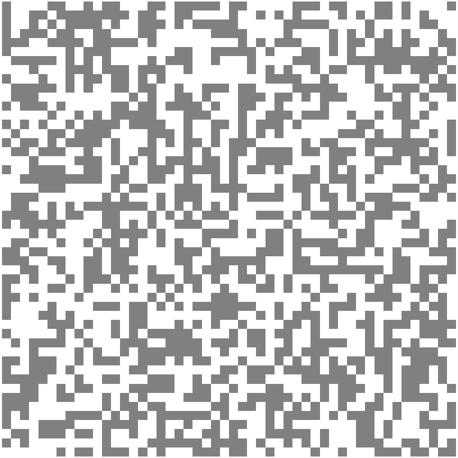


$\beta = 0.3$, time $t = 32$.

The Ising model

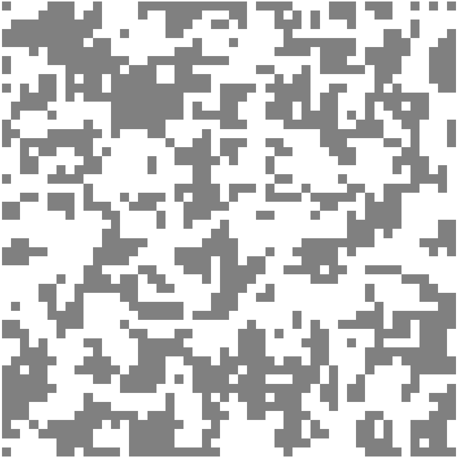
$\beta = 0.3$, time $t = 64$.

The Ising model



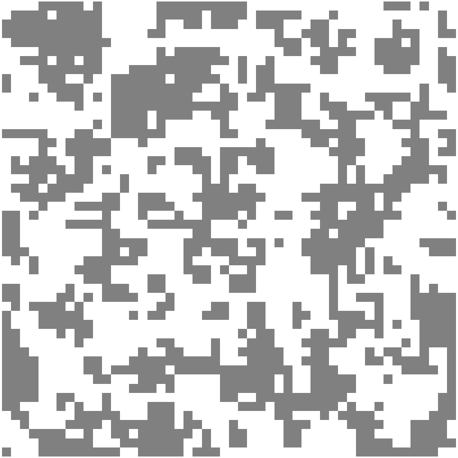
$\beta = 0.7$, time $t = 0$.

The Ising model



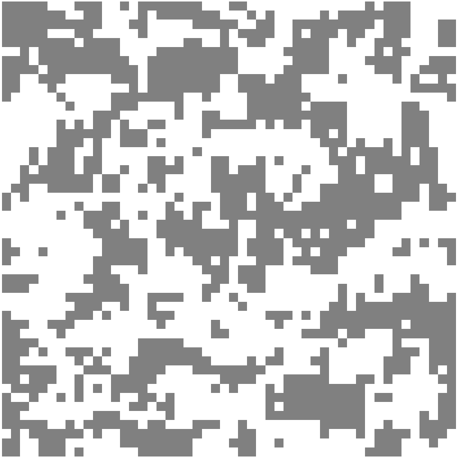
$\beta = 0.7$, time $t = 1$.

The Ising model



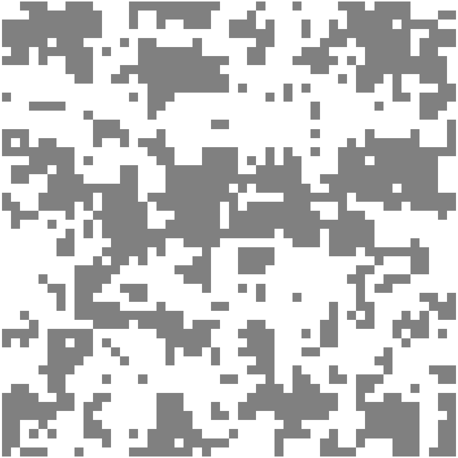
$\beta = 0.7$, time $t = 2$.

The Ising model



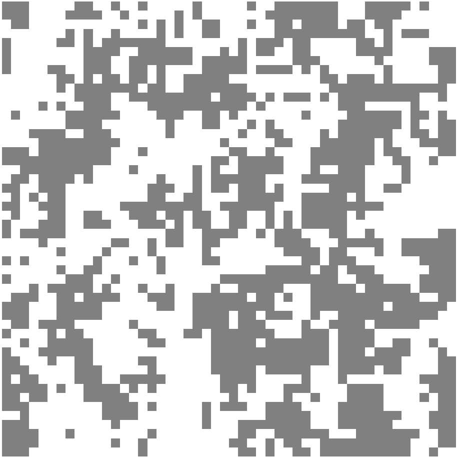
$\beta = 0.7$, time $t = 4$.

The Ising model



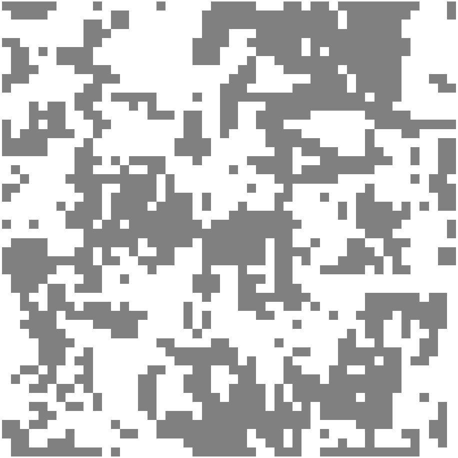
$\beta = 0.7$, time $t = 8$.

The Ising model



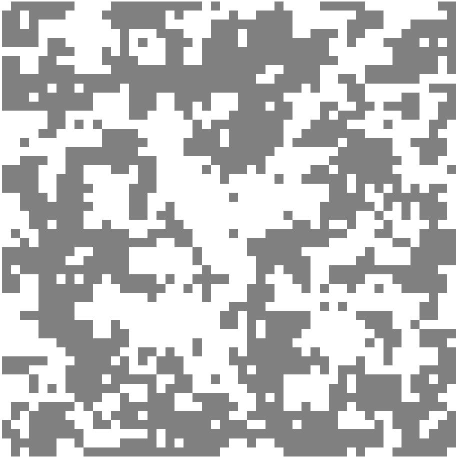
$\beta = 0.7$, time $t = 16$.

The Ising model



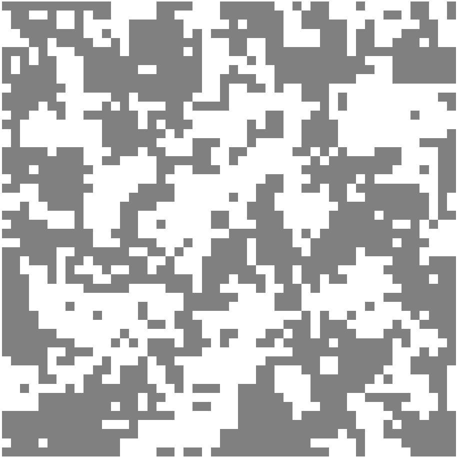
$\beta = 0.7$, time $t = 32$.

The Ising model



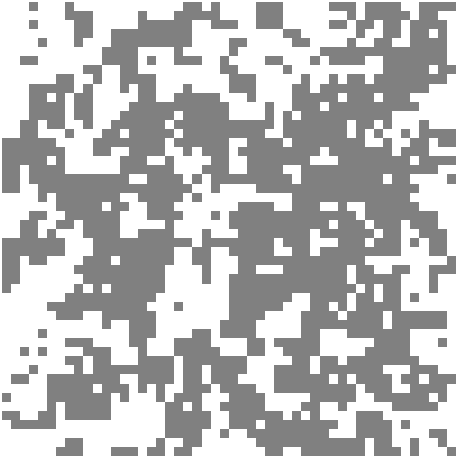
$\beta = 0.7$, time $t = 64$.

The Ising model



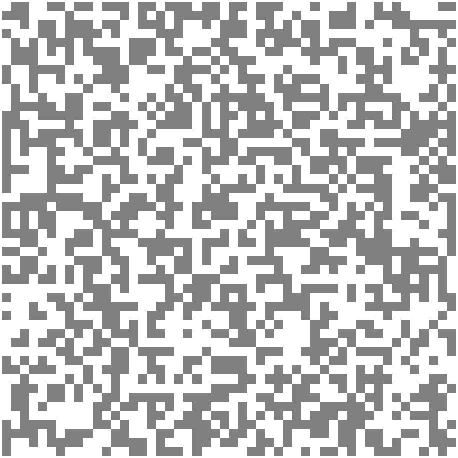
$\beta = 0.7$, time $t = 125$.

The Ising model



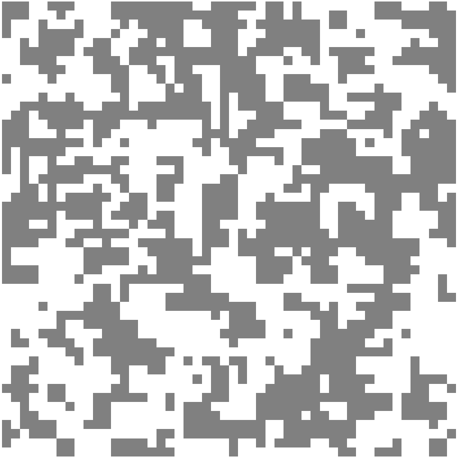
$\beta = 0.7$, time $t = 250$.

The Ising model



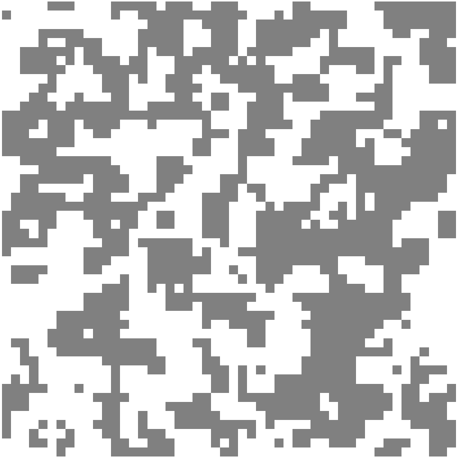
$\beta = 1$, time $t = 0$.

The Ising model



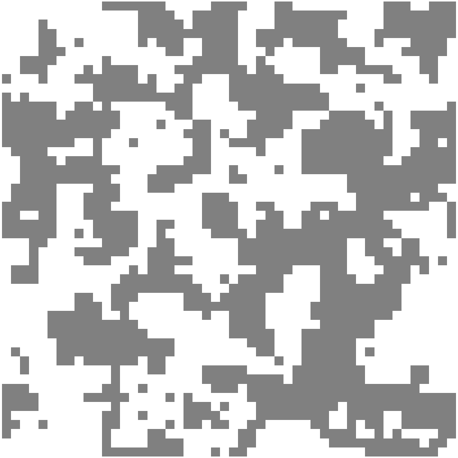
$\beta = 1$, time $t = 1$.

The Ising model



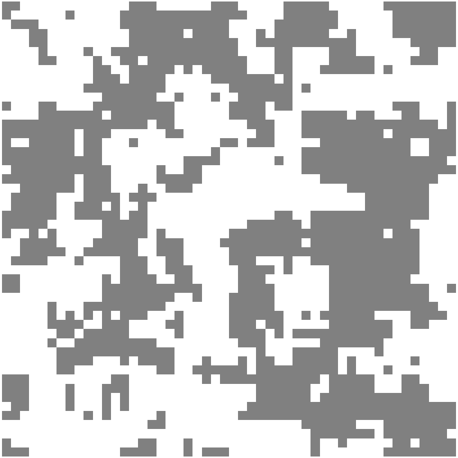
$\beta = 1$, time $t = 2$.

The Ising model



$\beta = 1$, time $t = 4$.

The Ising model

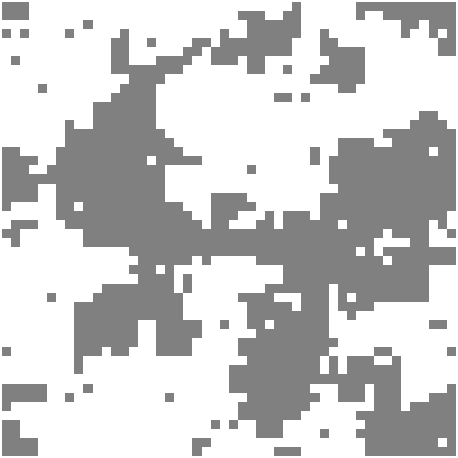


$\beta = 1$, time $t = 8$.

The Ising model

$\beta = 1$, time $t = 16$.

The Ising model

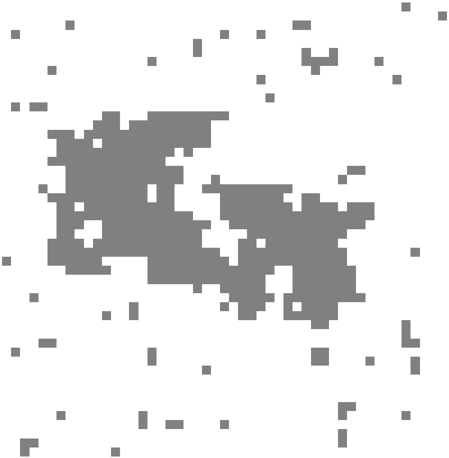


$\beta = 1$, time $t = 32$.

The Ising model

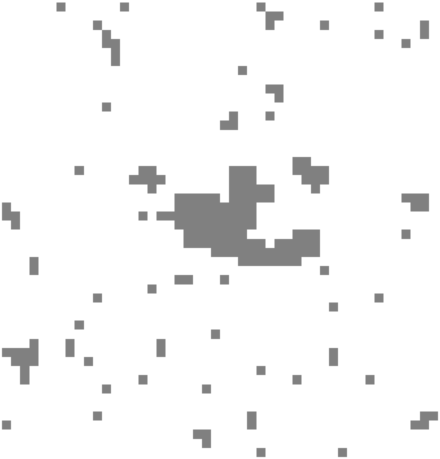
$\beta = 1$, time $t = 64$.

The Ising model



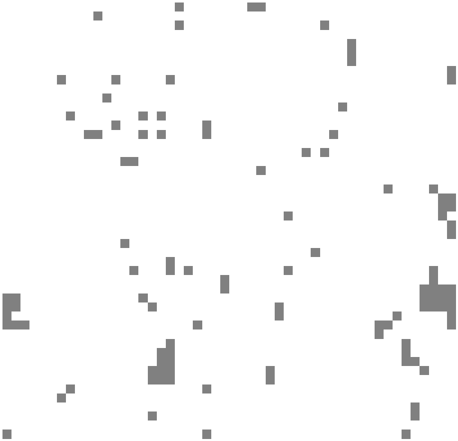
$\beta = 1$, time $t = 125$.

The Ising model



$\beta = 1$, time $t = 250$.

The Ising model



$\beta = 1$, time $t = 500$.

The spontaneous magnetization

The *spontaneous magnetization* is defined as

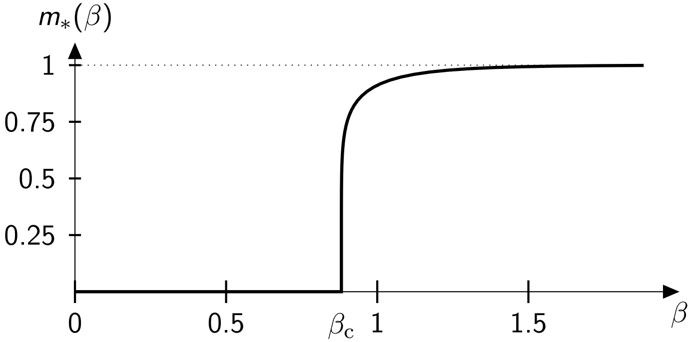
$$m_*(\beta) := \begin{array}{l} \text{the equilibrium expectation of } X_t(0) \\ \text{started from } X_0 = \underline{1}. \end{array}$$

For the model on $(\mathbb{Z}^2, E^2)$, Onsager (1944) proved

$$m_*(\beta) = \begin{cases} \big(1 - \sinh(\beta)^{-4}\big)^{1/8} & \text{for } \beta \geq \beta_c := \log(1 + \sqrt{2}), \\ 0 & \text{for } \beta \leq \beta_c. \end{cases}$$

For $\mathbb{Z}^3$, the graph of $m_*(\beta)$ looks roughly similar with $\beta_c \approx 0.442$ but no explicit formulas are known.

The spontaneous magnetization



The spontaneous magnetization of the two-dimensional Ising model.

A stochastic Potts model

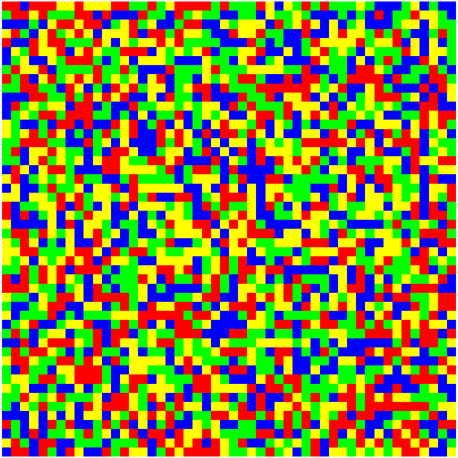
Instead of allowing only two states $-1, +1$, we can more generally allow $q \geq 2$ states $1, \ldots, q$.
Each person i chooses a new state at times of a Poisson process with rate 1.
The probability that the newly chosen state is $k \in \{1, \ldots, q\}$ equals

$$\frac{e^{\beta M_i(k)}}{\sum_{m=1}^{q} e^{\beta M_i(m)}},$$

where $M_i(k)$ denotes the number of neighbors of i that are in the state k.
Setting $q = 2$ and replacing β by 2β yields the Ising model.

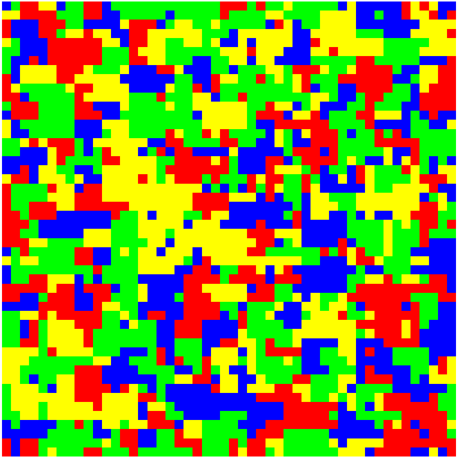
On $\mathbb{Z}^2$ for $q > 4$ the "magnetization" makes a jump at the point of the phase transition.

The Potts model



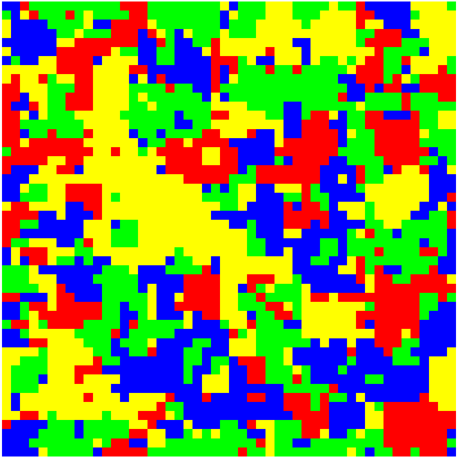
$\beta = 1.2$, time $t = 0$.

The Potts model



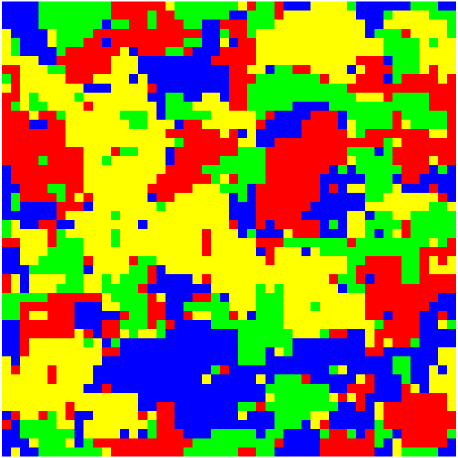
$\beta = 1.2$, time $t = 1$.

The Potts model



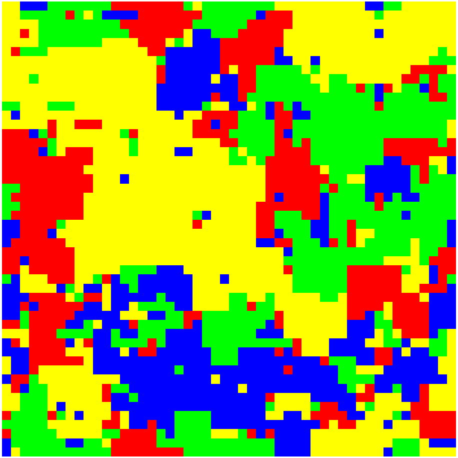
$\beta = 1.2$, time $t = 2$.

The Potts model



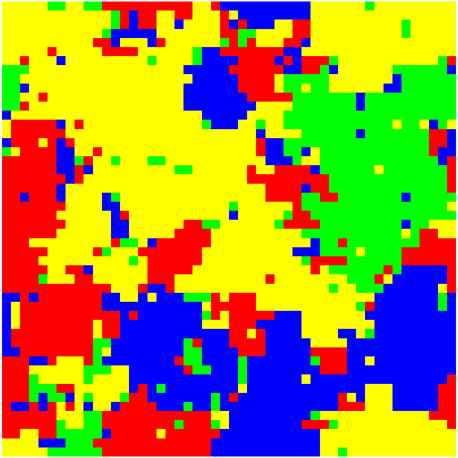
$\beta = 1.2$, time $t = 4$.

The Potts model



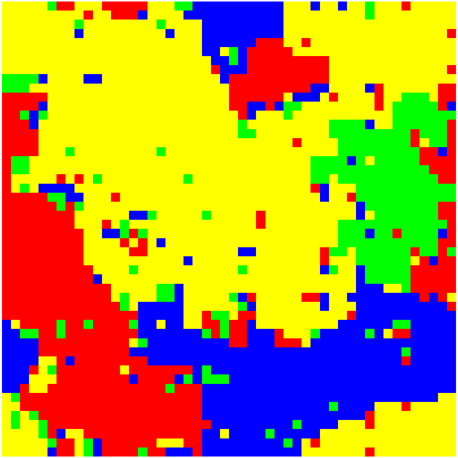
$\beta = 1.2$, time $t = 8$.

The Potts model



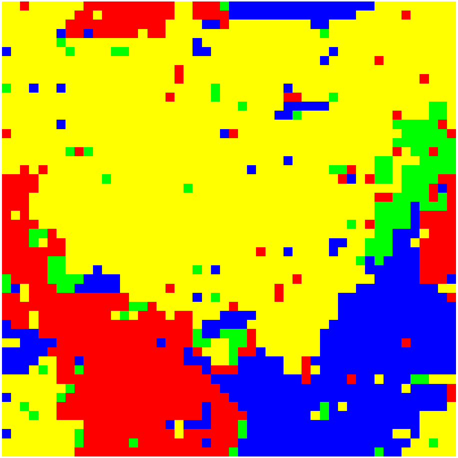
$\beta = 1.2$, time $t = 16$.

The Potts model



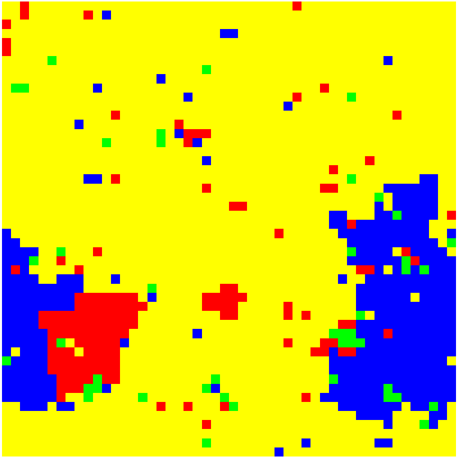
$\beta = 1.2$, time $t = 32$.

The Potts model



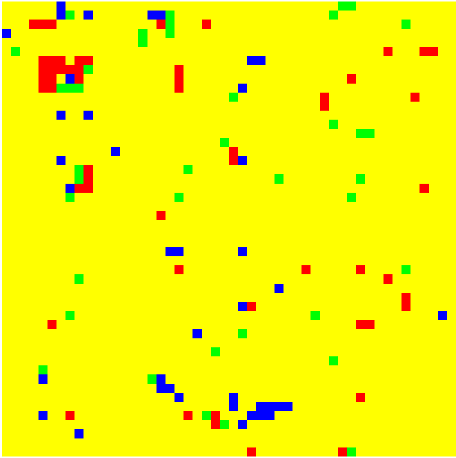
$\beta = 1.2$, time $t = 64$.

The Potts model



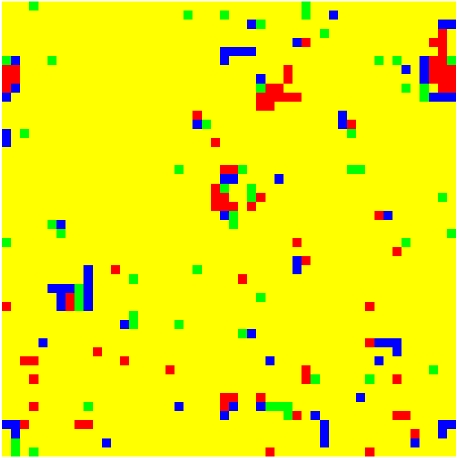
$\beta = 1.2$, time $t = 125$.

The Potts model



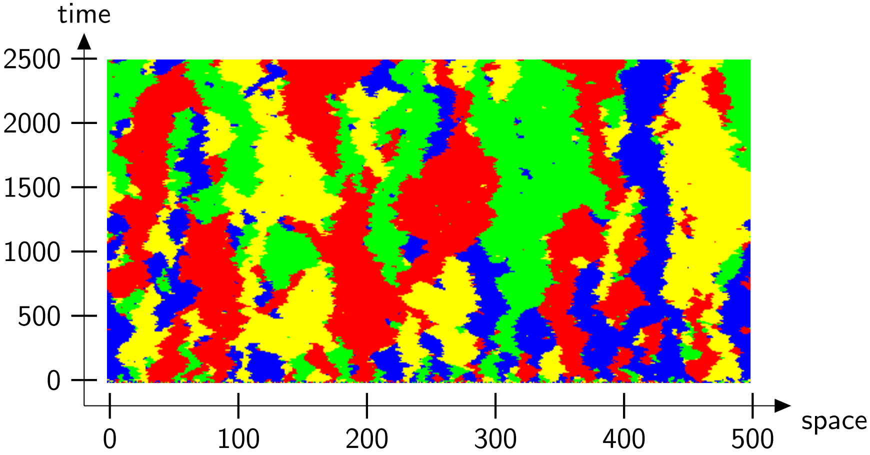
$\beta = 1.2$, time $t = 250$.

The Potts model



$\beta = 1.2$, time $t = 500$.

A one-dimensional Potts model



In one-dimensional Potts models, the cluster size remains bounded in time even at very high β ($=$ low temperature).

The biased voter model

In the *biased* voter model with two states $\{0,1\}$, each organism i changes its type $X_t(i)$ with the rates

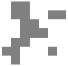
$$\begin{aligned}
0 &\mapsto 1 \qquad \text{with rate } (1+s)\cdot \text{fraction of type 1 neighbors},\\
1 &\mapsto 0 \qquad \text{with rate } 1\cdot \text{fraction of type 0 neighbors},
\end{aligned}$$

where $s>0$ gives type 1 a (small) advantage.

Contrary to the voter model, even if we start with just a single organism of type 1, there is a positive probability that type 1 never dies out.

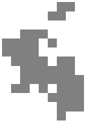
Models spread of *advantageous mutation*.

Biased voter model with $s = 0.2$. Time $t = 0$.



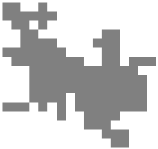
Biased voter model with $s = 0.2$. Time $t = 10$.

The biased voter model



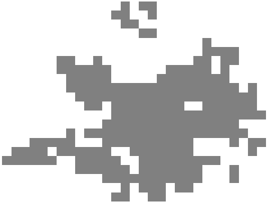
Biased voter model with $s = 0.2$. Time $t = 20$.

The biased voter model



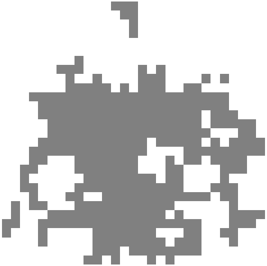
Biased voter model with $s = 0.2$. Time $t = 30$.

The biased voter model



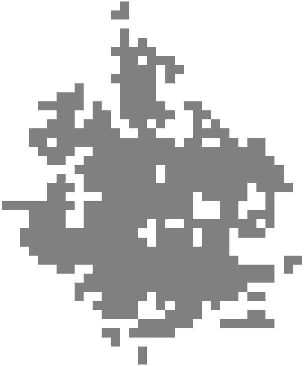
Biased voter model with $s = 0.2$. Time $t = 40$.

The biased voter model

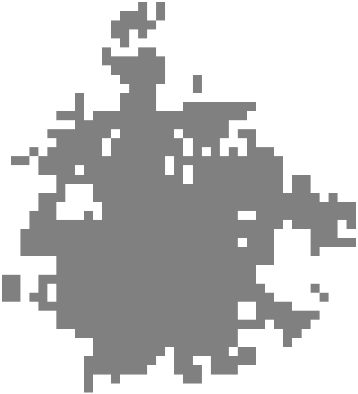


Biased voter model with $s = 0.2$. Time $t = 50$.

The biased voter model

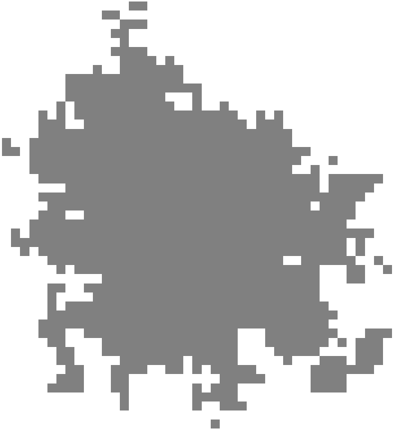


Biased voter model with $s = 0.2$. Time $t = 60$.



Biased voter model with $s = 0.2$. Time $t = 70$.

The biased voter model



Biased voter model with $s = 0.2$. Time $t = 80$.

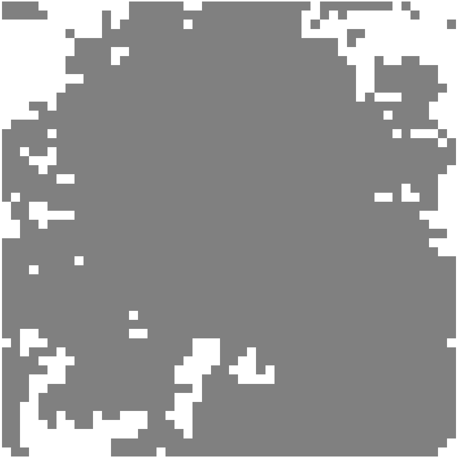
The biased voter model

Biased voter model with $s = 0.2$. Time $t = 90$.

The biased voter model

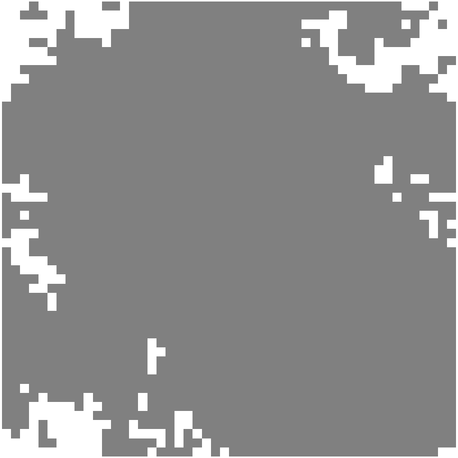
Biased voter model with $s = 0.2$. Time $t = 100$.

The biased voter model



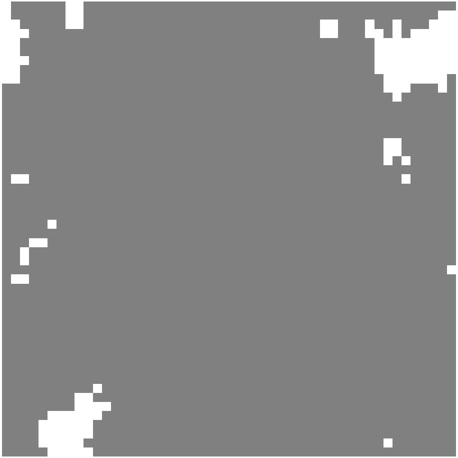
Biased voter model with $s = 0.2$. Time $t = 110$.

The biased voter model



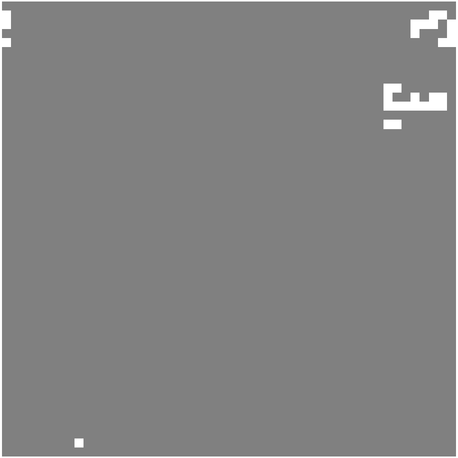
Biased voter model with $s = 0.2$. Time $t = 120$.

The biased voter model

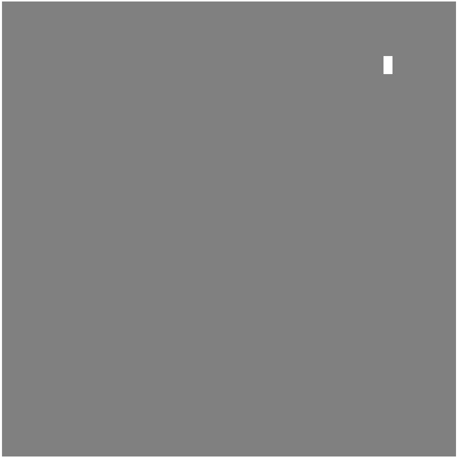


Biased voter model with $s = 0.2$. Time $t = 130$.

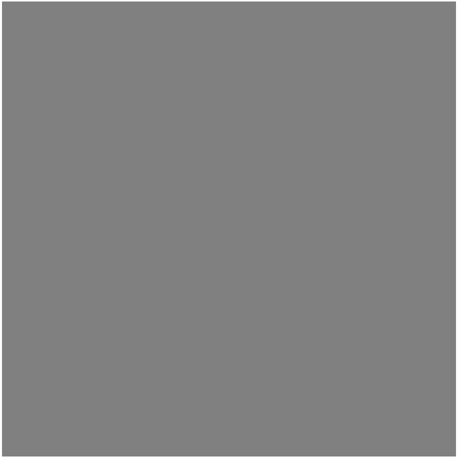
The biased voter model



Biased voter model with $s = 0.2$. Time $t = 140$.

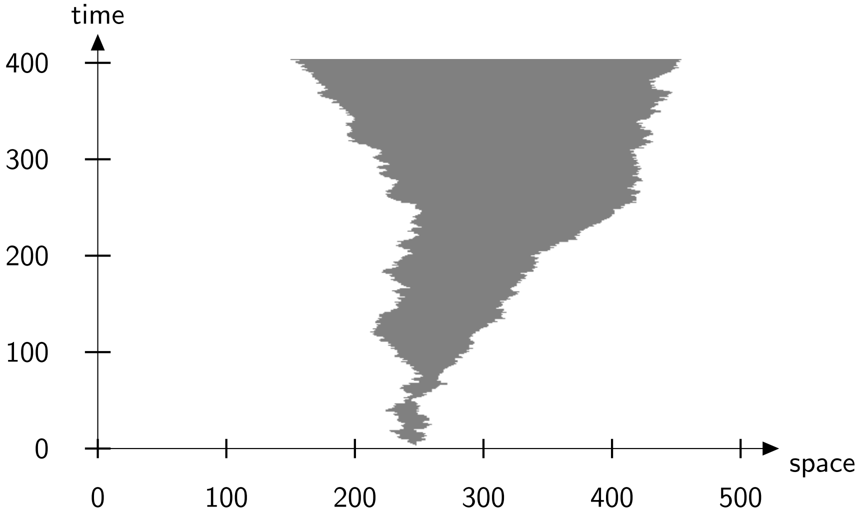


Biased voter model with $s = 0.2$. Time $t = 150$.



Biased voter model with $s = 0.2$. Time $t = 160$.

The biased voter model



A one-dimensional biased voter model with bias $s = 0.2$.

The biased voter model

We can extend the biased voter model by also allowing spontaneous jumps from 1 to 0.

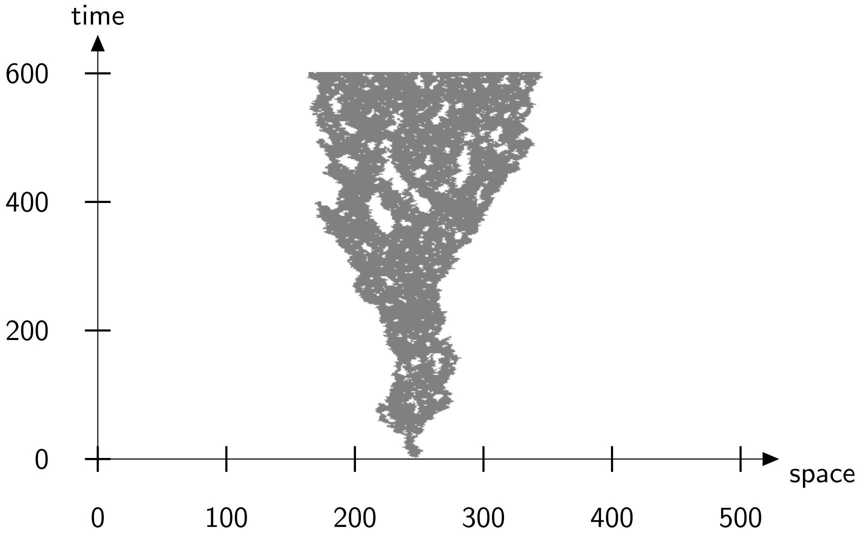
$$
\begin{aligned}
0 \mapsto 1 \qquad & \text{with rate } (1+s)\cdot \text{fraction of type 1 neighbors},\\
1 \mapsto 0 \qquad & \text{with rate } 1\cdot \text{fraction of type 0 neighbors}\\
& \qquad\qquad + d,
\end{aligned}
$$

where $s > 0$ gives type 1 an advantage and $d \geq 0$ is a *death rate*.

This models the fact that genes may become disfunctional due to deleterious mutations.

Whether 1's have a positive probability to survive now depends in a nontrivial way on s and d.

The biased voter model



Process with bias $s = 0.5$, death rate $d = 0.02$.

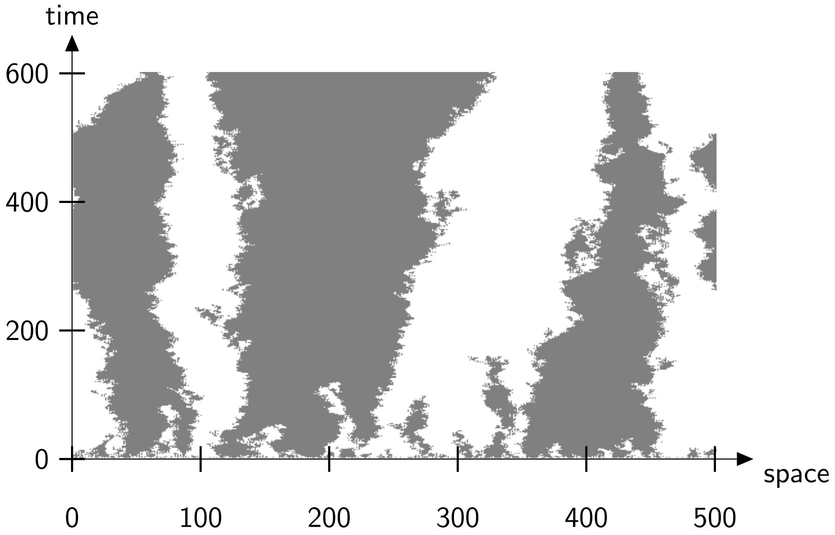
The *rebellious voter map* is defined as

$$\mathtt{rvot}_{kji}(x)(l) := \begin{cases} 1 - x(i) & \text{if } l = i \text{ and } x(k) \neq x(j), \\ x(l) & \text{otherwise.} \end{cases}$$

The *rebellious voter model* is the one-dimensional model with generator

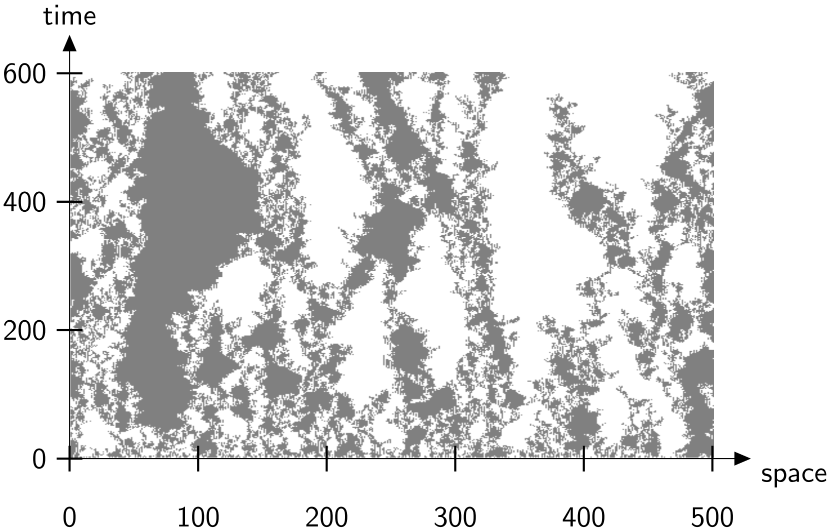
$$\begin{aligned} Gf(x) := {} & \alpha \sum_i \big\{ f\big(\mathtt{vot}_{i,i+1}(x)\big) - f\big(x\big) \big\} \\ & + \alpha \sum_i \big\{ f\big(\mathtt{vot}_{i,i-1}(x)\big) - f\big(x\big) \big\} \\ & + (1-\alpha) \sum_i \big\{ f\big(\mathtt{rvot}_{i-1,i,i+1}(x)\big) - f\big(x\big) \big\} \\ & + (1-\alpha) \sum_i \big\{ f\big(\mathtt{rvot}_{i+1,i,i-1}(x)\big) - f\big(x\big) \big\}. \end{aligned}$$

A rebellious voter model



Process with $\alpha = 0.8$ behaves more or less as a voter model.

A rebellious voter model



In the process with $\alpha = 0.3$, cluster size remains bounded in time.

Reaction diffusion models

Another rich class of models are *reaction diffusion models*.

These are systems of particles that perform independent random walks and interact when they are near to each other.

Let $X_t(i) = 1$ (resp. 0) signify the presence (resp. absence) of a particle and consider the maps $\mathtt{rw}_{ij} : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$

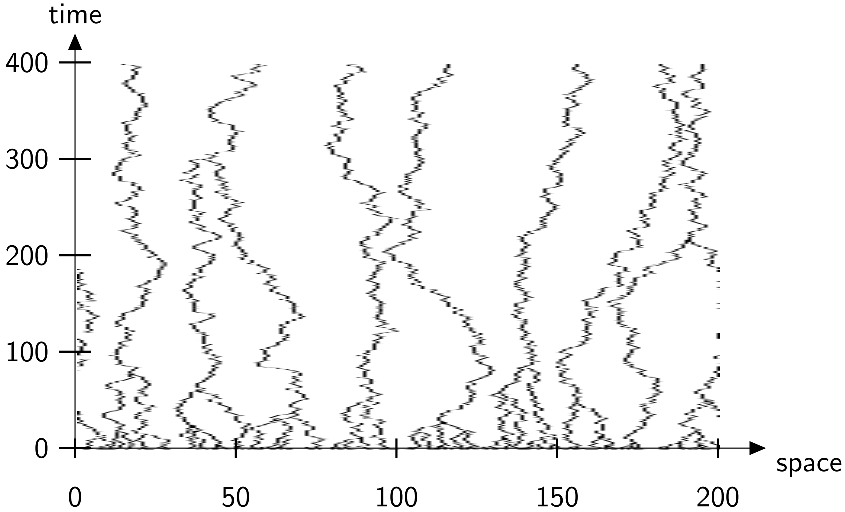
$$\mathtt{rw}_{i,j}x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

The process with generator

$$G = \tfrac{1}{2} \sum_{i \in \mathbb{Z}} \big\{ f\big(\mathtt{rw}_{i,i+1}x\big) - f\big(x\big) \big\} + \tfrac{1}{2} \sum_{i \in \mathbb{Z}} \big\{ f\big(\mathtt{rw}_{i,i-1}x\big) - f\big(x\big) \big\}$$

describes *coalescing random walks*.

Coalescing random walks



Reaction diffusion models

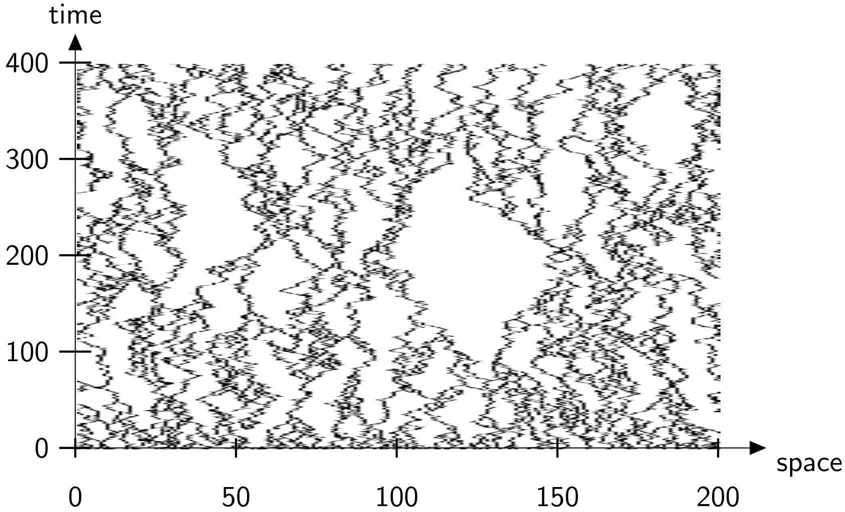
In analogy with the *branching map*

$$\mathtt{bra}_{ij}x(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

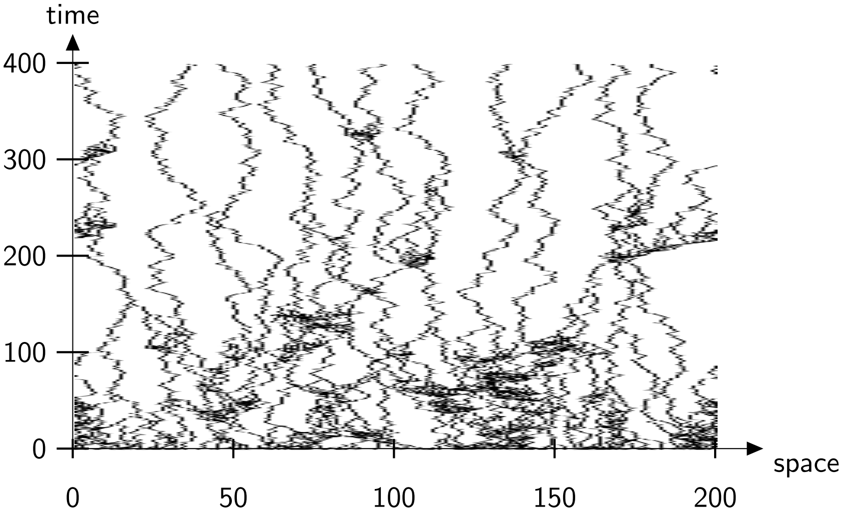
we can also define a *cooperative branching map*

$$\mathtt{coop}_{ii'j}x(k) := \begin{cases} \big(x(i) \wedge x(i')\big) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

Branching and coalescing random walks

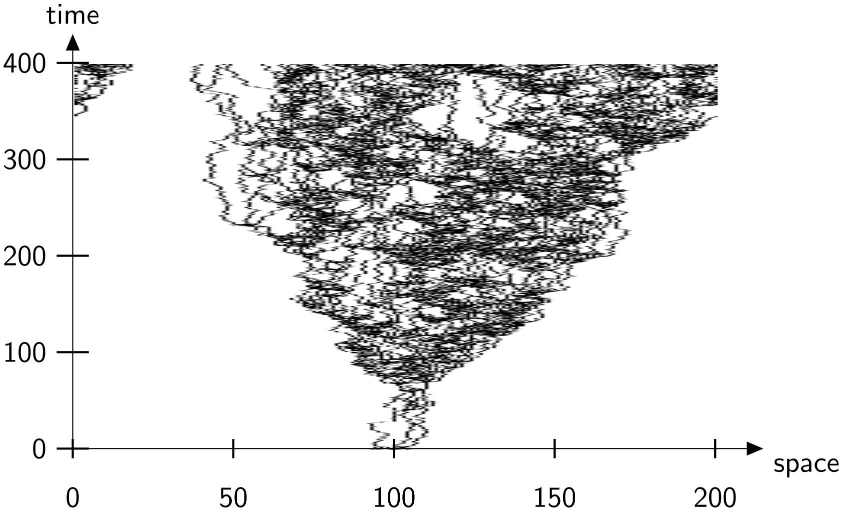


Cooperative branching and coalescence



Cooperative branching rate 2.2.

Cooperative branching



Cooperative branching rate 3.

A cancellative system

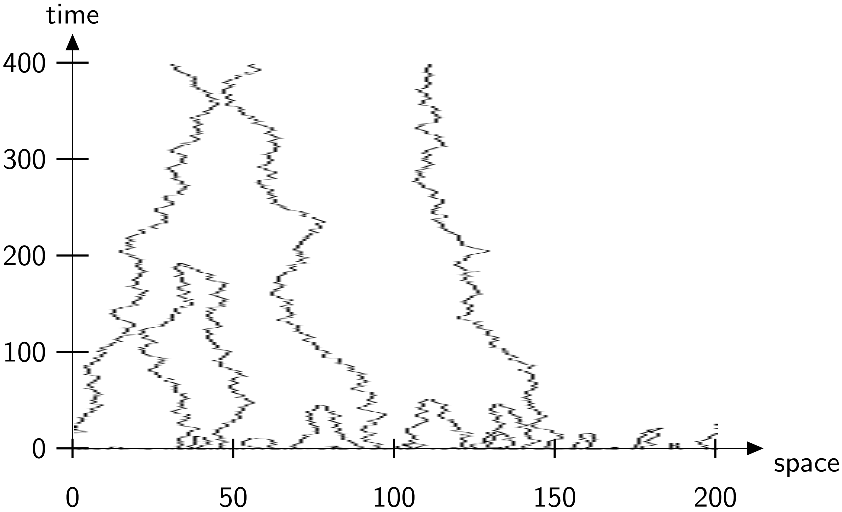
Two more maps of interest are the *annihilating random walk* map

$$\mathtt{arw}_{ij}x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(i) + x(j) \mod(2) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

and the *annihilating branching map*

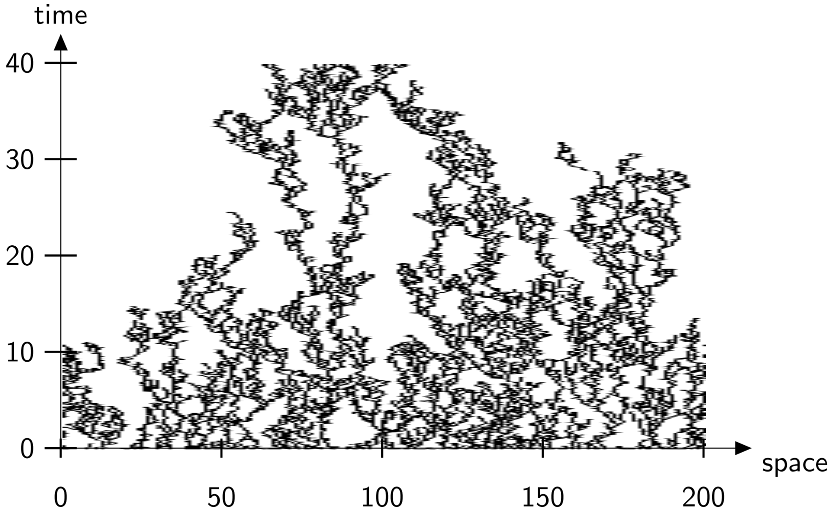
$$\mathtt{abra}_{ij}x(k) := \begin{cases} x(i) + x(j) \mod(2) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

A cancellative system



Annihilating random walks.

A cancellative system



A system of branching annihilating random walks.

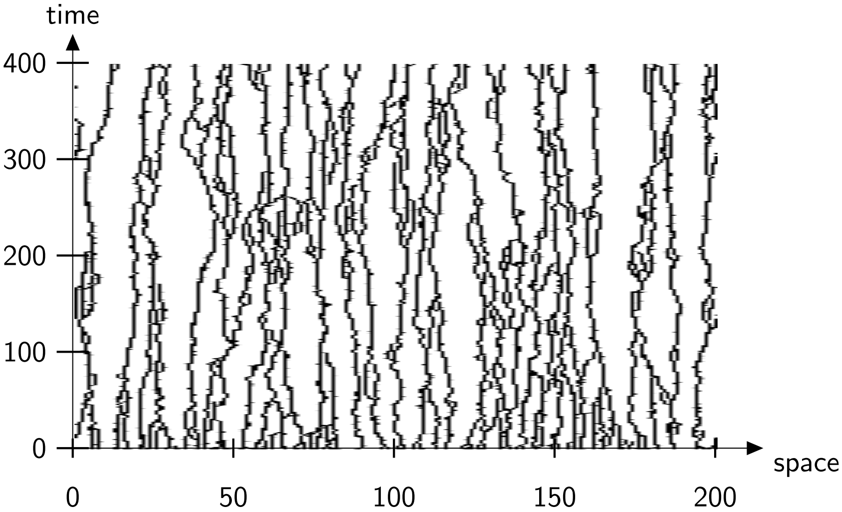
Killing

Define a killing map as

$$\texttt{kill}_{ij}x(k) := \begin{cases} \big(1 - x(i)\big) \wedge x(j) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

which says that the particle at i, if present, kills any particle at j.

Branching and killing



A system with branching and killing.