Gibbs measures and finite systems

Let Λ and S be finite sets and let $H : S^\Lambda \to \mathbb{R}$ be a function. By definition, the *Gibbs measure* belonging to the *Hamiltonian* (or *energy function*) H and *inverse temperature* β is the probability measure on S^Λ given by

$$\mu(\{x\}) = \frac{1}{Z_\beta} e^{-\beta H(x)} \qquad \big(x \in S^\Lambda\big), \tag{1}$$

where

$$Z_\beta := \sum_{x \in S^\Lambda} e^{-\beta H(x)} \tag{2}$$

is a normalization constant, also called the *partition sum.* Note that if H, H' are two energy functions that differ only by a constant, then the associated Gibbs measures are the same. Indeed, if $H(x) = H'(x) + c$ and μ, μ' are the associated Gibbs measures, then all probabilities in μ' get an extra factor $e^{-\beta c}$, but this disappears in the normalization. Indeed, we make the following simple observation.

Lemma 1 (Relative probabilities)
If Λ is a finite set and μ is the Gibbs measure on S^Λ with Hamiltonian H and inverse temperature β, then

$$\frac{\mu(\{x'\})}{\mu(\{x\})} = e^{-\beta(H(x') - H(x))} \qquad \big(x, x' \in S^\Lambda\big). \tag{3}$$

Conversely, if μ is a probability measure on S^Λ and (3) holds for all x, x' that differ only at a single site $i \in \Lambda$, then μ is the Gibbs measure on S^Λ associated with H and β.

Proof If μ is the Gibbs measure associated with H and β, then it is obvious that (3) holds. To prove the converse, we note that for each $x, x' \in S^\Lambda$ we can find $x_0, \ldots, x_n$ such that $x = x_0$, $x' = x_n$, and x_k differs only in one site from x_{k-1} $(k = 1, \ldots, n)$. In view of this, if (3) holds for all x, x' that differ only at a single site, then it actually holds for all x, x'. Choosing some arbitrary reference state x', we see that (3) determines all probabilities up to an overall multiplicative constant, which follows from the normalization. ∎

Instead of looking at the proportion of two probabilities, it is often more natural to look at conditional probabilities. For our next lemma we need to introduce some notation. If Λ_1, Λ_2 are disjoint sets, $x \in S^{\Lambda_1}$, and $y \in S^{\Lambda_2}$, then we define $x\&y \in S^{\Lambda_1 \cup \Lambda_2}$ as $(x\&y)(i) := x(i)$ if $i \in \Lambda_1$ and $(x\&y)(i) := y(i)$ if $i \in \Lambda_2$. If

$H : S^\Lambda \to \mathbb{R}$ is a function and $\Delta \subset \Lambda$, then for all $y \in S^{\Lambda\backslash\Delta}$ we define $H^\Delta_y : S^\Delta \to \mathbb{R}$ by

$$H^\Delta_y(x) := H(x\&y) \qquad \big(x \in S^\Delta,\ y \in S^{\Lambda\backslash\Delta}\big). \tag{4}$$

Lemma 2 (Conditional distributions)
Let $(X(i))_{i\in\Lambda}$ be random variables whose joint law is the the Gibbs measure on S^Λ with Hamiltonian H and inverse temperature β. Then for each $\Delta \subset \Lambda$ and $y \in S^{\Lambda\backslash\Delta}$, one has

$$\mathbb{P}\big[(X(i))_{i\in\Delta} \in \cdot \,\big|\, (X(i))_{i\in\Lambda\backslash\Delta} = y\big] = \mu^{\Delta,\beta}_y, \tag{5}$$

where $\mu^{\Delta,\beta}_y$ is the Gibbs measure on S^Δ with Hamiltonian H^Δ_y and inverse temperature β. Conversely, if (5) holds for all Δ that consist of a single element, then the law of $(X(i))_{i\in\Lambda}$ is the Gibbs measure associated with H and β.

Proof We observe that

$$\begin{aligned}
&\frac{\mathbb{P}\big[(X(i))_{i\in\Delta} = x' \,\big|\, (X(i))_{i\in\Lambda\backslash\Delta} = y\big]}{\mathbb{P}\big[(X(i))_{i\in\Delta} = x \,\big|\, (X(i))_{i\in\Lambda\backslash\Delta} = y\big]} \\
&\quad = \frac{\mathbb{P}\big[(X(i))_{i\in\Delta} = x',\ (X(i))_{i\in\Lambda\backslash\Delta} = y\big]}{\mathbb{P}\big[(X(i))_{i\in\Delta} = x,\ (X(i))_{i\in\Lambda\backslash\Delta} = y\big]} = \frac{e^{-\beta H(x'\&y)}}{e^{-\beta H(x\&y)}} = e^{-\beta(H^\Delta_y(x') - H^\Delta_y(x))}.
\end{aligned}$$

In view of this, the statements follow from Lemma 1. ∎

Remark By our earlier remarks, in Lemma 2, we can replace $H^\Delta_y(x)$ by $H^\Delta_y(x)+c^\Delta_y$, where c^Δ_y is a constant that may depend on Δ and y but not on x.