The Algebraic Approach to Duality: An Introduction

Anja Sturm [*] Jan M. Swart[†] Florian Völlering [‡]

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Abstract

This survey article gives an elementary introduction to the algebraic approach to Markov process duality, as opposed to the pathwise approach. In the algebraic approach, a Markov generator is written as the sum of products of simpler operators, which each have a dual with respect to some duality function. We discuss at length the recent suggestion by Giardinà, Redig, and others, that it may be a good idea to choose these simpler operators in such a way that they form an irreducible representation of some known Lie algebra. In particular, we collect the necessary background on representations of Lie algebras that is crucial for this approach. We also discuss older work by Lloyd and Sudbury on duality functions of product form and the relation between intertwining and duality.

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[*]Institute for Mathematical Stochastics, Georg-August-Universität Göttingen, Goldschmidtstr. 7, 37077 Göttingen, Germany; asturm@math.uni-goettingen.de

[†]The Czech Academy of Sciences, Institute of Information Theory and Automation. Pod vodárenskou věží 4, 18208 Prague 8, Czech Republic; swart@utia.cas.cz

[‡]Department of Mathematics, University of Bath, Claverton Down, Bath BA2 7AY, United Kingdom; f.m.vollering@bath.ac.uk

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1 Introduction

1.1 Outline

The aim of the present text is to give an introduction to the algebraic approach to the theory of duality of Markov processes. In particular, we present some of the pioneering work done by Lloyd and Sudbury [LS95, LS97, Sud00] and spend a lot of time explaining the more recent work of Giardinà, Redig, and others [GKRV09, CGGR15]. The algebraic approach differs fundamentally from the pathwise approach propagated in e.g., [JK14, SS16]. In principle, the algebraic approach is able to find a wider class of dualities, but the price we pay for this is that it may suggest dual operators that turn out not to be Markov generators.

In the remainder of this section, we quickly introduce the basic ideas behind the algebraic approach. In Subsection 1.2, we explain how Markov process duality can algebraically be viewed as an intertwining relation between the generator of one Markov process and the adjoint of the generator of another Markov process. As explained in Subsection 1.3, it is then natural to view a Markov generator as being built up out of sums and products of other, simpler operators. If all these building blocks have duals with respect to a duality function, then so has the original Markov generator.

A central idea of of Giardinà, Redig, et al. [GKRV09, CGGR15] is to choose these building blocks so that they form a representation of a Lie algebra. To understand why that may be a good idea, one needs quite a bit of background on Lie algebras. Since probabilists may not be familiar with this, after a small detour to pathwise duality in Subsection 1.4, we devote all of Section 2 to providing this background.

The study of Lie algebras and their representations is a huge subject with a venerable history. Although there exist good introductory texts, we will need some theory that is considered too advanced for the usual textbooks. In particular, this refers to the representation theory of non-compact Lie groups like SU(1,1) or the Heisenberg group. In order to squeeze the essential facts that we need for our purposes into little over 10 pages, we had to cut some corners and in some cases resign on full mathematical rigour. We also leave out a lot of background material (e.g., Lie groups, as opposed to Lie algebras, stay almost completely out of the picture). To partly compensate for this, we have added Appendix A which gives a somewhat more complete, but still sketchy picture.

After our little excursion into Lie algebras, in Section 3, we come to the core of our text. In Subsections 3.1, 3.3, and 3.4 we demonstrate the approach via Lie algebras on three examples, which are based on representation theory for the Heisenberg algebra, SU(2), and SU(1,1), respectively. In Subsection 3.1, we formulate a general principle and apply it to discover a self-duality of the Wright-Fisher diffusion from (1.8). After Subsection 3.2, which is needed to deal with infinite state spaces, in Subsection 3.3, we use the well-known representation theory of SU(2) to derive a duality for the symmetric exclusion process. This duality is not very interesting on its own, but serves as a preparation for the symmetric inclusion process in Section 3.4 which turns out to be very similar to the former, except that SU(2) is replaced by SU(1,1).

In Sections 3.5–3.7 we present results of Lloyd and Sudbury [LS95, LS97, Sud00] that do not require knowledge of Lie algebras, but do use some facts about tensor products from Section 2.6. In particular, in Section 3.5 we discuss duality functions of product form, including q-duality, while in Section 3.6 we discuss intertwining of Markov processes, and in particular thinning relations which are closely connected to q-duality.

In Sections 3.8 and 3.9, finally, we discuss another observation from [GKRV09], who show that nontrivial dualities can sometimes be found by starting from a "trivial" duality which is based on reversibility, and then using a symmetry of the model to transform such a duality into a nontrivial one. Although Lie algebras are not strictly needed in this approach, writing generators in terms of the basis elements of a representation of a Lie algebra can help finding suitable symmetries.

1.2 Markov duality and intertwining

In Section 1, for technical simplicity, we mostly restrict ourselves to Markov processes with finite state spaces. As we will see in Section 3, many of the basic ideas discussed here can with some care be made to work also in infinite dimensional settings. How to do this is in part discussed in Section 3.2, but for brevity, we will not always go into the technical details and sometimes use the calculations of the present section merely as an inspiration.

The generator of a continuous-time Markov process with finite state space Ω is a matrix L such that

$$L(x,y) \geq 0 \quad (x \neq y) \quad \text{and} \quad \sum_y L(x,y) = 0. \tag{1.1}$$

Equivalently, we can identify L with the linear operator $L : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ defined by

$$Lf(x) := \sum_{y \in \Omega} L(x,y) f(y) \qquad (x \in \Omega). \tag{1.2}$$

A linear operator $L : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ is a Markov generator (i.e., satisfies (1.1)) if and only if the semigroup[1] of operators $(P_t)_{t\geq 0}$ defined by

$$P_t := e^{tL} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n L^n$$

is a Markov semigroup, i.e., P_t is a probability kernel for each $t \geq 0$. If L is a Markov generator, then $(P_t)_{t\geq 0}$ are the transition kernels of some Ω-valued Markov process $(X_t)_{t\geq 0}$.

Let Ω and $\hat{\Omega}$ be finite sets. We can view a function $D : \Omega \times \hat{\Omega} \to \mathbb{R}$ as a matrix

$$(D(x, y))_{x\in\Omega,\ y\in\hat{\Omega}}$$

that as in (1.2) corresponds to a linear operator $D : \mathbb{R}^{\hat{\Omega}} \to \mathbb{R}^\Omega$.

Let L and $\hat{L}$ be generators of Markov processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ with state spaces Ω and $\hat{\Omega}$ and semigroups $(P_t)_{t\geq 0}$ and $(\hat{P}_t)_{t\geq 0}$, and let $D : \Omega\times\hat{\Omega} \to \mathbb{R}$ be a function. We make the following simple observation. Below, we let $A^\dagger(x, y) := A(y, x)$ (or $A^\dagger(x, y) := \overline{A(y, x)}$ for matrices over the complex numbers) denote the *adjoint* of a matrix A.[2]

Lemma 1 (Duality) *The following conditions are equivalent.*

(i) $LD = D\hat{L}^\dagger$,

(ii) $P_t D = D\hat{P}_t^\dagger$ *for all* $t \geq 0$,

(iii) $\mathbb{E}^x[D(X_t, y)] = \mathbb{E}^y[D(x, Y_t)]$ *for all* $x \in \Omega$, $y \in \hat{\Omega}$, *and* $t \geq 0$.

Proof If (i) holds for L, then it also holds for any linear combination of powers of L. In particular, filling in the definition of P_t, we see that (i) implies (ii). Conversely, differentiating with respect to t, we see that (ii) implies (i). Condition (iii) is just a rewrite of (ii). ∎

If the conditions of Lemma 1 are satisfied, then we say that $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are *dual* with *duality function* D. If $L = \hat{L}$, then we speak of *self-duality*. Condition (i) can also be written as

$$LD(\,\cdot\,, y)(x) = \hat{L}D(x,\,\cdot\,)(y) \qquad (x \in \Omega,\ y \in \hat{\Omega}). \tag{1.3}$$

Under suitable assumptions, the equivalence of (iii) and (1.3) can often also be established for Markov processes with infinite state space.

[1] The semigroup property says that $P_0 = I$ and $P_s P_t = P_{s+t}$.

[2] In other words, $\langle A^\dagger f|g\rangle := \langle f|Ag\rangle$ where $\langle f|g\rangle := \sum_{x\in\Omega} \overline{f(x)}g(x)$ denotes the usual inner product on $\mathbb{C}^\Omega$. For adjoints with respect to a general inner product on finite or infinite dimensional spaces we write A^*.

An algebraic relation of the form $AB = BC$ is called an *intertwining relation* between operators A and C. The operator B is called the *intertwiner*. Thus, Lemma 1 says that two Markov processes are dual if and only if there exists an intertwiner between the generator of one Markov process, and the adjoint of the generator of another Markov process. Note that if L is dual to $\hat{L}$ with duality function D, then $\hat{L}$ is dual to L with duality function $D^\dagger$. Thus, duality is a symmetric concept.

Closely related to Markov process duality is the concept of *intertwining of Markov processes*, which has a more narrow meaning than the algebraic concept of intertwining. Let, again, L and $\hat{L}$ be generators of Markov processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ with state spaces Ω and $\hat{\Omega}$ and semigroups $(P_t)_{t\geq 0}$ and $(\hat{P}_t)_{t\geq 0}$. Let $K : \Omega \times \hat{\Omega} \to \mathbb{R}$ be a function. In what follows, we assume that K is a probability kernel, i.e., $K(x, y) \geq 0\ \forall x, y$ and $\sum_y K(x, y) = 1$ for each x.

Lemma 2 (Intertwining of Markov processes) *The the following conditions are equivalent.*

(i) $LK = K\hat{L}$.

(ii) $P_t K = K\hat{P}_t \qquad (t \geq 0)$.

(iii) $\mu_0 K = \nu_0$ *implies* $\mu_0 P_t K = \nu_0 \hat{P}_t \qquad (t \geq 0)$.

Proof The equivalence of (i) and (ii) follows by the same argument as in Lemma 1. Condition (ii) implies $\mu_0 P_t K = (\mu_0 K)\hat{P}_t$ $(t \geq 0)$. Setting $\mu_0 = \delta_x$ we see that (iii) implies (ii). ∎

In condition (iii), note that $\mu_0 P_t$ and $\nu_0 \hat{P}_t$ describe the laws at time t of the Markov processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ started in initial laws μ_0 and ν_0, respectively. If the conditions of Lemma 2 are satisfied, then we say that the Markov processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are *intertwined*.

If K is invertible as a matrix, then $LK = K\hat{L}$ implies $\hat{L}K^{-1} = K^{-1}L$; however, K^{-1} will in general not be a probability kernel. In view of this, in an intertwining relation between Markov processes, the two processes do not play symmetric roles. To stress the different roles of X and Y, following [Swa13], it is convenient to say that Y is an intertwined Markov process *on top* of X.

If the conditions of Lemma 2 are satisfied, then the Markov processes X and Y can actually be coupled such that $(X_t, Y_t)_{t\geq 0}$ is a Markov process and

$$\mathbb{P}[Y_t \in \cdot\,|(X_s)_{0\leq s\leq t}] = K(X_s, \cdot\,) \quad \text{a.s.} \quad (t \geq 0),$$

see [Fil92, Swa13]. Note that this strengthens condition (iii) of Lemma 2.

1.3 The algebraic approach

We make the following simple observation. Below, $\mathbb{R}^\Omega$ denotes the space of all functions $f : \Omega \to \mathbb{R}$.

Lemma 3 (Duality of building blocks) *Let $\Omega, \hat{\Omega}$ be finite spaces and let $A_i : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$, $B_i : \mathbb{R}^{\hat{\Omega}} \to \mathbb{R}^{\hat{\Omega}}$ $(i = 1, 2)$, and $D : \mathbb{R}^{\hat{\Omega}} \to \mathbb{R}^\Omega$ be linear operators such that*

$$A_i D = D B_i^\dagger \qquad (i = 1, 2). \tag{1.4}$$

Then

$$(r_1 A_1 + r_2 A_2) D = D(r_1 B_1 + r_2 B_2)^\dagger \quad \textit{and} \quad (A_1 A_2) D = D(B_2 B_1)^\dagger. \tag{1.5}$$

Lemma 3 implies that if we can write a Markov generator L as a linear combination of products of "simpler" operators A_i, for example, (denoting the identity operator by I),

$$L = r_\emptyset I + r_1 A_1 + r_{23} A_2 A_3 + r_{113} A_1^2 A_3, \tag{1.6}$$

and these "building blocks" satisfy $A_i D = D B_i^\dagger$ for some duality function D, the L will be dual to the operator

$$\hat{L} = r_\emptyset I + r_1 B_1 + r_{23} B_3 B_2 + r_{113} B_3 B_1^2. \tag{1.7}$$

Note that in each term, we have not only replaced A_i by B_i but also reversed the order of the factors. If we are lucky, $\hat{L}$ is a Markov generator and we have discovered a Markov duality.

We demonstrate this approach on the Wright-Fisher diffusion with selection parameter $s \in \mathbb{R}$, which is the diffusion in $[0, 1]$ with generator

$$Lf(x) = x(1-x)\tfrac{\partial^2}{\partial x^2} + sx(1-x)\tfrac{\partial}{\partial x}. \tag{1.8}$$

We are immediately cheating here, since L is not a linear operator acting on a finite dimensional space. Ignoring the difficulties associated with infinite dimension, we can write L in terms of simpler "building blocks" as follows. We set

$$A^- f(x) := (1-x)f(x) \quad \text{and} \quad A^+ f(x) := \tfrac{\partial}{\partial x} f(x), \tag{1.9}$$

and we write L in terms of these building blocks as

$$L = A^-(I - A^-)A^+(sI + A^+). \tag{1.10}$$

As our dual space, we choose $\mathbb{N} = \{0, 1, \ldots\}$ and as our duality function we choose the function $D : [0, 1] \times \mathbb{N} \to \mathbb{R}$ given by

$$D(x, n) := (1-x)^n \qquad (x \in [0, 1],\ n \in \mathbb{N}). \tag{1.11}$$

Let $B^\pm$ be operators acting on functions $f:\mathbb{N}\to\mathbb{R}$ as

$$B^- f(n) := f(n+1) \quad \text{and} \quad B^+ f(n) := -nf(n-1). \tag{1.12}$$

Then $B^\pm$ are dual to $A^\pm$ in the sense of (1.3), i.e.,

$$A^\pm D(\,\cdot\,,n)(x) = B^\pm D(x,\,\cdot\,)(n) \qquad (x\in[0,1],\ n\in\mathbb{N}). \tag{1.13}$$

Therefore, in view of Lemma 3, the following operator should be dual to L:

$$\hat{L} = (sI + B^+)B^+(I - B^-)B^-. \tag{1.14}$$

(Note that we have replaced $A^\pm$ by $B^\pm$ and reversed the order of the factors.) A little calculation reveals that

$$\hat{L}f(n) = n(n-1)\big\{f(n-1) - f(n)\big\} + sn\big\{f(n+1) - f(n)\big\}. \tag{1.15}$$

This is not, in general, a Markov generator. For $s\geq 0$, however, it is the generator of a Markov process in $\mathbb{N}$ that jumps from n to $n-1$ with rate $n(n-1)$ and from n to $n+1$ with rate sn.

Recall that the *commutator* of two operators A, B is defined as $[A,B] := AB - BA$. For our operators $A^\pm$, it is easy to check that

$$[A^-, A^+] = I. \tag{1.16}$$

This is similar to the commutation relation between the position and momentum operators in quantum physics. Indeed, the operators $A^\pm$ can be used to define a representation of the Heisenberg algebra, which is a particular Lie algebra. The connection to Lie algebras can help us to choose good building blocks and can sometimes also suggest duality functions. To explain this, we need some theory about representations of Lie algebras, which will be presented in the next section.

1.4 The pathwise approach

In the remainder of this section, we point out some differences and similarities between the algebraic and pathwise approaches to Markov process duality. A *random mapping representation* of a probability kernel K is a random map M such that

$$K(x,\mathrm{d}y) = \mathbb{P}[M(x)\in\mathrm{d}y]. \tag{1.17}$$

A *stochastic flow* is a collection $(\mathbf{X}_{s,u})_{s\leq u}$ of random maps $\mathbf{X}_{s,u}:\Omega\to\Omega$ such that $\mathbf{X}_{s,s} = I$ and $\mathbf{X}_{t,u}\circ\mathbf{X}_{s,t} = \mathbf{X}_{s,u}$. We say that $(\mathbf{X}_{s,u})_{s\leq u}$ has *independent increments* if

$$\mathbf{X}_{t_1,t_2},\ldots,\mathbf{X}_{t_{n-1},t_n} \tag{1.18}$$

are independent for any $t_1 < \cdots < t_n$. If $(\mathbf{X}_{s,u})_{s \leq u}$ is a stochastic flow with independent increments such that the law of $\mathbf{X}_{s,u}$ depends only on the difference $u - s$, and X_0 is an independent Ω-valued random variable, then setting

$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0) \tag{1.19}$$

defines a Markov process with transition kernels

$$P_{u-s}(x, \mathrm{d}y) := \mathbb{P}[\mathbf{X}_{s,u}(x) \in \mathrm{d}y] \qquad (s \leq u). \tag{1.20}$$

Note that this formula says that $\mathbf{X}_{s,u}$ is a random mapping representation of P_{u-s}.

Markov processes can often be constructed from stochastic flows. For example, if a stochastic differential equation has unique strong solutions, then these solutions (for different initial states) define a stochastic flow with independent increments that can be used to construct a diffusion process. If L is the generator of a Markov process with finite state space Ω, then L can always be written in the form

$$Lf(x) = \sum_{m \in \mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\}, \tag{1.21}$$

where $\mathcal{G}$ is a finite collection of maps $m : \Omega \to \Omega$. We say that two maps $m, \hat{m}$ are *dual* with respect to a duality function D if

$$D\big(m(x), y\big) = D\big(x, \hat{m}(y)\big) \qquad (x \in \Omega,\ y \in \hat{\Omega}). \tag{1.22}$$

Two stochastic flows $(\mathbf{X}_{s,u})_{s \leq u}$ and $(\mathbf{Y}_{s,u})_{s \leq u}$ are dual[3] if for each $s \leq u$, a.s., $\mathbf{Y}_{-u,-s}$ is dual to $\mathbf{X}_{s,u}$. If two stochastic flows are dual, then we say that their associated Markov processes are *pathwise dual*. It is easy to see that this implies Markov process duality.

We recall that in the algebraic approach, there may be many ways in which a given Markov generator can be written in terms of more elementary "building blocks" as in (1.6). Similarly, in the pathwise approach, there are usually many different ways in which a Markov generator can be written in terms of maps as in (1.21). In the algebraic approach we have seen that if all building blocks have duals with respect to a given duality function, then a Markov generator built up from these building blocks also has a dual $\hat{L}$. Similarly, in the pathwise approach, if all maps m occurring in (1.21)

[3]The definition of duality for stochastic flows that we give here is a weak one. It is often natural to give a somewhat stronger definition, see [SS16].

have duals $\hat{m}$ with respect to some duality function D, then the process with generator L is pathwise dual to the process with generator

$$\hat{L}f(x) := \sum_{m\in\mathcal{G}} r_m\big\{f\big(\hat{m}(x)\big) - f\big(x\big)\big\}. \tag{1.23}$$

An advantage of the pathwise approach is that an operator $\hat{L}$ of this form is guaranteed to me a Markov generator. On the other hand, not all dualities can be constructed as pathwise dualities, so the algebraic approach is more general. Nevertheless, many known dualities, including the duality for the Wright-Fisher diffusion discussed in the previous subsection, can be obtained in a pathwise way or as limits of such pathwise dualities, see [Swa06, AH07].

There are more analogies between the algebraic and pathwise approaches. In Subsection 3.8, we will see that in the algebraic approach, nontrivial dualities can sometimes be found by starting with a "trivial" duality obtained from reversibility and then applying a symmetry transformation. In [SS16], it is shown that nontrivial pathwise dualities can be found by starting with a "trivial" duality to the inverse image map and then looking for invariant subspaces of the dual process.

2 Representations of Lie algebras

2.1 Lie algebras

A complex[4] (resp. real) *Lie algebra* is a finite-dimensional linear space $\mathfrak{g}$ over $\mathbb{C}$ (resp. $\mathbb{R}$) together with a map $[\,\cdot\,,\,\cdot\,] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ called *Lie bracket* such that

(i) $(\mathbf{x},\mathbf{y})\mapsto[\mathbf{x},\mathbf{y}]$ is bilinear,

(ii) $[\mathbf{x},\mathbf{y}] = -[\mathbf{y},\mathbf{x}]$ (skew symmetry),

(iii) $[\mathbf{x},[\mathbf{y},\mathbf{z}]] + [\mathbf{y},[\mathbf{z},\mathbf{x}]] + [\mathbf{z},[\mathbf{x},\mathbf{y}]] = 0$ (Jacobi identity).

An *adjoint operation* on a Lie algebra $\mathfrak{g}$ is a map $\mathbf{x}\mapsto\mathbf{x}^*$ such that

(i) $\mathbf{x}\mapsto\mathbf{x}^*$ is conjugate linear,

(ii) $(\mathbf{x}^*)^* = \mathbf{x}$,

[4] In this section, we mostly focus on complex Lie algebras. Some results stated in the present section (in particular, part (b) of Schur's lemma) are true for complex Lie algebras only. See Appendix A for a more detailed discussion.

(iii) $[\mathbf{x}^*, \mathbf{y}^*] = [\mathbf{y}, \mathbf{x}]^*$.

If $\mathfrak{g}$ is a complex Lie algebra, then the space of its skew symmetric elements $\mathfrak{h} := \{\mathbf{x} \in \mathfrak{g} : \mathbf{x}^* = -\mathbf{x}\}$ forms a real Lie algebra. Conversely, starting from a real Lie algebra $\mathfrak{h}$, we can always find a complex Lie algebra $\mathfrak{g}$ equipped with a adjoint operation such that $\mathfrak{h}$ is the space of skew symmetric elements of $\mathfrak{g}$. Then $\mathfrak{g}$ is called the *complexification* of $\mathfrak{h}$.

If $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis for $\mathfrak{g}$, then the Lie bracket on $\mathfrak{g}$ is uniquely characterized by the *commutation relations*

$$[\mathbf{x}_i, \mathbf{x}_j] = \sum_{k=1}^{n} c_{ijk} \mathbf{x}_k \qquad (i < j).$$

The constants c_{ijk} are called the *structure constants*. If $\mathfrak{g}$ is equipped with an adjoint operation, then the latter is uniquely characterized by the *adjoint relations*

$$\mathbf{x}_i^* = \sum_j d_{ij} \mathbf{x}_j.$$

Example Let V be a finite dimensional complex linear space, let $\mathcal{L}(V)$ denote the space of all linear operators $A : V \to V$, and let $\mathrm{tr}(A)$ denote the trace of an operator A. Then

$$\mathfrak{g} := \{A \in \mathcal{L}(V) : \mathrm{tr}(A) = 0\} \quad \text{with} \quad [A, B] := AB - BA$$

is a Lie algebra. Note that $\mathrm{tr}([A, B]) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$ by the basic property of the trace, which shows that $[A, B] \in \mathfrak{g}$ for all $A, B \in \mathfrak{g}$. Note also that $\mathfrak{g}$ is in general not an algebra, i.e., $A, B \in \mathfrak{g}$ does not imply $AB \in \mathfrak{g}$. If V is equipped with an inner product $\langle \cdot | \cdot \rangle$ (which we always take conjugate linear in its first argument and linear in its second argument) and A^* denotes the adjoint of A with respect to this inner product, i.e.,

$$\langle A^* v | w \rangle := \langle v | A w \rangle,$$

then one can check that $A \mapsto A^*$ is an adjoint operation on $\mathfrak{g}$.

By definition, a *Lie algebra homomorphism* is a map $\phi : \mathfrak{g} \to \mathfrak{h}$ from one Lie algebra into another that preserves the structure of the Lie algebra, i.e., ϕ is linear and

$$\phi([A, B]) = [\phi(A), \phi(B)].$$

If ϕ is invertible, then its inverse is also a Lie algebra homomorphism. In this case we call ϕ a *Lie algebra isomorphism*. We say that a Lie algebra homomorphism ϕ is *unitary* if it moreover preserves the structure of the adjoint operation, i.e.,

$$\phi(A^*) = \phi(A)^*.$$

If $\mathfrak{g}$ is a Lie algebra, then we can define a *conjugate* of $\mathfrak{g}$, which is a Lie algebra $\overline{\mathfrak{g}}$ together with a conjugate linear bijection $\mathfrak{g} \ni \mathbf{x} \mapsto \overline{\mathbf{x}} \in \overline{\mathfrak{g}}$ such that

$$[\overline{\mathbf{x}}, \overline{\mathbf{y}}] = [\mathbf{y}, \mathbf{x}].$$

It is easy to see that such a conjugate Lie algebra is unique up to natural isomorphisms, and that the $\overline{\overline{\mathfrak{g}}}$ is naturally isomorphic to $\mathfrak{g}$. If $\mathfrak{g}$ is equipped with an adjoint operation, then we can define an adjoint operation on $\overline{\mathfrak{g}}$ by $\overline{\mathbf{x}}^* := \overline{(\mathbf{x}^*)}$.

Example Let V be a complex linear space on which an inner product is defined and let $\mathfrak{g} \subset \mathcal{L}(V)$ be a linear subspace such that $A, B \in \mathfrak{g}$ implies $[A, B] \in \mathfrak{g}$. Then $\mathfrak{g}$ is a sub-Lie-algebra of $\mathcal{L}(V)$. Now $\overline{\mathfrak{g}} := \{A^* : A \in \mathfrak{g}\}$, together with the map $\overline{A} := A^*$ is a realization of the conjugate Lie algebra of $\mathfrak{g}$.

2.2 Representations

If V is a finite dimensional linear space, then the space $\mathcal{L}(V)$ of linear operators $A : V \to V$, equipped with the *commutator*

$$[A, B] := AB - BA$$

is a Lie algebra. By definition, a *representation* of a complex Lie algebra $\mathfrak{g}$ is a pair (V, π) where V is a complex linear space of dimension $\dim(V) \geq 1$ and $\pi : \mathfrak{g} \to \mathcal{L}(V)$ is a Lie algebra homomorphism. A representation is *unitary* if this homomorphism is unitary and *faithful* if π is an isomorphism to its image $\pi(\mathfrak{g}) := \{\pi(\mathbf{x}) : \mathbf{x} \in \mathfrak{g}\}$.

There is another way of looking at representations that is often useful. If (V, π) is a representation, then we can define a map

$$\mathfrak{g} \times V \ni (\mathbf{x}, v) \mapsto \mathbf{x}v \in V$$

by $\mathbf{x}v := \pi(\mathbf{x})v$. Such a map satisfies

(i) $(\mathbf{x}, v) \mapsto Av$ is bilinear (i.e., linear in both arguments),

(ii) $[\mathbf{x}, \mathbf{y}]v = \mathbf{x}(\mathbf{y}v) - \mathbf{y}(\mathbf{x}v)$.

Any map with these properties is called a *left action* of $\mathfrak{g}$ on V. It is easy to see that if V is a complex linear space that is equipped with a left action of $\mathfrak{g}$, then setting $\pi(\mathbf{x})v := \mathbf{x}v$ defines a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathcal{L}(V)$. Thus, we can view representations as linear spaces on which a left action of $\mathfrak{g}$ is defined.

Example For any Lie algebra, we may set $V := \mathfrak{g}$. Then, using the Jacobi identity, one can verify that the map $(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}, \mathbf{y}]$ is a left action of $\mathfrak{g}$ on V. (See Lemma 15 in the appendix.) In this way, every Lie algebra can be represented on itself. This representation is not always faithful, but for many Lie algebras of interest, it is.

Yet another way to look at representations is in terms of commutation relations. Let $\mathfrak{g}$ be a Lie algebra with basis elements $\mathbf{x}_1, \ldots, \mathbf{x}_n$, which satisfy the commutation relations

$$[\mathbf{x}_i, \mathbf{x}_j] = \sum_{k=1}^{n} c_{ijk} \mathbf{x}_k \qquad (i < j).$$

Let V be a complex linear space with $\dim(V) \geq 1$ and let $X_1, \ldots, X_n \in \mathcal{L}(V)$ satisfy

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k \qquad (i < j).$$

Then there exists a unique Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathcal{L}(V)$ such that $\pi(\mathbf{x}_i) = X_i$ $(i = 1, \ldots, n)$. Thus, any collection of linear operators that satisfies the commutation relations of $\mathfrak{g}$ defines a representation of $\mathfrak{g}$. Such a representation is faithful if and only if $X_1, \ldots, X_n$ are linearly independent. If $\mathfrak{g}$ is equipped with an adjoint operation and V is equipped with an inner product, then the representation (V, π) is unitary if and only if $X_1, \ldots, X_n$ satisfy the adjoint relations of $\mathfrak{g}$, i.e.,

$$\mathbf{x}_i^* = \sum_j d_{ij} \mathbf{x}_j \quad \text{and} \quad X_i^* = \sum_j d_{ij} X_j.$$

Let V be a representation of a Lie algebra $\mathfrak{g}$. By definition, an *invariant subspace* of V is a linear subspace $W \subset V$ such that $\mathbf{x}w \in W$ for all $w \in W$ and $\mathbf{x} \in \mathfrak{g}$. A representation is *irreducible* if its only invariant subspaces are $W = \{0\}$ and $W = V$.

Let V, W be two representations of the same Lie algebra $\mathfrak{g}$. By definition, an *intertwiner* of representations is a linear map $\phi : V \to W$ that preserves the structure of a representation, i.e.,

$$\phi(\mathbf{x}v) = \mathbf{x}\phi(v).$$

If ϕ is a bijection then its inverse is also an intertwiner. In this case we call ϕ an *isomorphism* and say that the representations are *equivalent* (or *isomorphic*).

The following result can be found in, e.g., [Hal03, Thm 4.29]. Below and in what follows, we let $I \in \mathcal{L}(V)$ denote the *identity operator* $Iv := v$.

Proposition 4 (Schur's lemma)

(a) *Let V and W be irreducible representations of the same Lie algebra and let $\phi : V \to W$ be an intertwiner. Then either $\phi = 0$ or ϕ is an isomorphism.*

(b) *Let V be an irreducible representation of a Lie algebra and let $\phi : V \to V$ be an intertwiner. Then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$.*

For us, the following simple consequence of Schur's lemma will be important.

Corollary 5 (Unique intertwiner) *Let (V, π_V) and (W, π_W) be equivalent irreducible representations of some Lie algebra. Then there exists an intertwiner $\phi : V \to W$ that is unique up to a multiplicative constant, such that*

$$\phi\pi_V(\mathbf{x}) = \pi_W(\mathbf{x})\phi.$$

Proof By assumption, V and W are equivalent, so there exists an isomorphism $\phi : V \to W$. Assume that $\psi : V \to W$ is another intertwiner. Then $\phi^{-1} \circ \psi$ is an intertwiner from V into itself, so by part (b) of Schur's lemma, $\phi^{-1} \circ \psi = \lambda I$ and hence $\psi = \lambda\phi$. ∎

If V is a complex linear space, then we can define a *conjugate* of V, which is a complex linear space $\overline{V}$ together with a conjugate linear bijection $\phi \mapsto \overline{\phi}$.

Example Let V be a complex linear space with inner product $\langle \cdot | \cdot \rangle$. Let V' denote the dual space of V, i.e., the space of all linear forms $l : V \to \mathbb{C}$. For any $v \in V$, we can define a linear form $\langle v| \in V'$ by $\langle v|w := \langle v|w\rangle$. Then V', together with the map $v \mapsto \langle v|$, is a realization of the conjugate of V.

If (V, π) is a representation of a Lie algebra $\mathfrak{g}$, then we can equip the conjugate space $\overline{V}$ with the structure of a representation of the conjugate Lie algebra $\overline{\mathfrak{g}}$ by putting

$$\overline{\mathbf{x}}\,\overline{v} := \overline{\mathbf{x}v}.$$

It is easy to see that this defines a left action of $\overline{\mathfrak{g}}$ on $\overline{V}$. We call $\overline{V}$, equipped with this left action of $\overline{\mathfrak{g}}$, the *conjugate* of the representation V.

There is a close relation between Lie algebras and Lie groups. Roughly speaking, a Lie group is a smooth differentiable manifold that is equipped with a group structure. In particular, a matrix Lie group G is a group whose elements are invertible linear operators acting on some finite dimensional linear space V. The Lie algebra of G is then defined as

$$\mathfrak{h} := \{A \in \mathcal{L}(V) : e^{tA} \in G \ \forall t \geq 0\}.$$

In general, this is a real Lie algebra. More generally, one can associate a Lie algebra to each Lie group (not necessarily a matrix Lie group) and prove that each Lie algebra is the Lie algebra of some Lie group. Under a certain condition (simple connectedness), the Lie algebra determines its associated Lie group uniquely. A finite dimensional representation of a Lie group G is a pair (V, Π) where V is a finite dimensional linear space and $\Pi : G \to \mathcal{L}(V)$ is a group homomorphism. Each representation (V, π) of a real Lie algebra $\mathfrak{h}$ gives rise to a representation (V, Π) of the associated Lie group such that $\Pi(e^{tA}) = e^{t\pi(A)}$. If $\mathfrak{g}$ is the complexification of $\mathfrak{h}$ and (V, π) is a unitary representation of $\mathfrak{g}$, then (V, Π) is a unitary representation of G in the sense that $\Pi(A)$ is a unitary operator for each $A \in G$. All his is explained in more detail in Appendix A.

2.3 The Lie algebra SU(2)

The Lie algebra $\mathfrak{su}(2)$ is the three dimensional complex Lie algebra defined by the commutation relations between its basis elements

$$[\mathbf{s}_{\mathrm{x}}, \mathbf{s}_{\mathrm{y}}] = 2i\mathbf{s}_{\mathrm{z}}, \quad [\mathbf{s}_{\mathrm{y}}, \mathbf{s}_{\mathrm{z}}] = 2i\mathbf{s}_{\mathrm{x}}, \quad [\mathbf{s}_{\mathrm{z}}, \mathbf{s}_{\mathrm{x}}] = 2i\mathbf{s}_{\mathrm{y}}. \tag{2.1}$$

It is customary to equip $\mathfrak{su}(2)$ with an adjoint operation that is defined by

$$\mathbf{s}_{\mathrm{x}}^* = \mathbf{s}_{\mathrm{x}}, \quad \mathbf{s}_{\mathrm{y}}^* = \mathbf{s}_{\mathrm{y}}, \quad \mathbf{s}_{\mathrm{z}}^* = \mathbf{s}_{\mathrm{z}}. \tag{2.2}$$

A faithful unitary representation of $\mathfrak{su}(2)$ is defined by the *Pauli matrices*

$$S_{\mathrm{x}} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_{\mathrm{y}} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad S_{\mathrm{z}} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

It is straightforward to check that these matrices are linearly independent and satisfy the commutation and adjoint relations (2.1) and (2.2). In particular, this shows that $\mathfrak{su}(2)$ is well-defined.[5]

[5]Not every set of commutation relations that one can write down defines a bona fide Lie algebra. By linearity and skew symmetry, specifying $[\mathbf{x}_i, \mathbf{x}_j]$ for all $i < j$ uniquely defines a bilinear map $[\,\cdot\,,\,\cdot\,]$, but such a map may fail to satisfy the Jacobi identity. Similarly, it is not a priori clear that (2.2) defines a bona fide adjoint operation, but the faithful unitary representation defined by the Pauli matrices shows that it does.

In general, if $S_\mathrm{x}, S_\mathrm{y}, S_\mathrm{z}$ are linear operators on some complex linear space V that satisfy the commutation relations (2.1), and hence define a representation (V,π) of $\mathfrak{su}(2)$, then the so-called *Casimir operator* is defined as

$$C := S_\mathrm{x}^2 + S_\mathrm{y}^2 + S_\mathrm{z}^2.$$

The operator C is in general not an element of $\{\pi(\mathbf{x}) : \mathbf{x} \in \mathfrak{su}(2)\}$, i.e., C does not correspond to an element of the Lie algebra $\mathfrak{su}(2)$. It does correspond, however, to an element of the so-called *universal enveloping algebra* of $\mathfrak{su}(2)$; see Appendix A.4 below.

The finite-dimensional irreducible representations of $\mathfrak{su}(2)$ are well understood. Part (a) of the following proposition follows from Theorem 22 in the appendix, using the compactness of the Lie group SU(2). Parts (b) and (c), and also Proposition 7 below, follow from [Hal03, Thm 4.32] and a calculation of the Casimir operator for the representation in Proposition 7.

Proposition 6 (Irreducible representations of $\mathfrak{su}(2)$) *Let $S_\mathrm{x}, S_\mathrm{y}, S_\mathrm{z}$ be linear operators on a finite dimensional complex linear space V, that satisfy the commutation relations (2.1) and hence define a representation (V,π) of $\mathfrak{su}(2)$. Then:*

- **(a)** *There exists an inner product $\langle\,\cdot\,|\,\cdot\,\rangle$ on V, which is unique up to a multiplicative constant, such that with respect to this inner product the representation (V,π) is unitary.*
- **(b)** *If the representation (V,π) is irreducible, then there exists an integer $n \geq 1$, which we call the* index *of (V,π), such that the Casimir operator C is given by $C = n(n+2)I$.*
- **(c)** *Two irreducible representations V, W of $\mathfrak{su}(2)$ are equivalent if and only if they have the same index.*

Proposition 6 says that the finite dimensional irreducible representations of $\mathfrak{su}(2)$, up to isomorphism, can be labeled by their index n, which is a natural number $n \geq 1$. We next describe what an irreducible representation with index n looks like. In spite of the beautiful symmetry of the commutation relations (2.1), it will be useful to work with a different, less symmetric basis $\{\mathbf{j}^-, \mathbf{j}^+, \mathbf{j}^0\}$ defined as

$$\mathbf{j}^- := \tfrac{1}{2}(\mathbf{s}_\mathrm{x} - i\mathbf{s}_\mathrm{y}), \quad \mathbf{j}^+ := \tfrac{1}{2}(\mathbf{s}_\mathrm{x} + i\mathbf{s}_\mathrm{y}), \quad \text{and} \quad \mathbf{j}^0 := \tfrac{1}{2}\mathbf{s}_\mathrm{z}, \tag{2.4}$$

which satisfies the commutation and adjoint relations:

$$[\mathbf{j}^0, \mathbf{j}^\pm] = \pm\mathbf{j}^\pm, \quad [\mathbf{j}^-, \mathbf{j}^+] = -2\mathbf{j}^0, \quad (\mathbf{j}^-)^* = \mathbf{j}^+, \quad (\mathbf{j}^0)^* = \mathbf{j}^0. \tag{2.5}$$

The next proposition describes what an irreducible representation of $\mathfrak{su}(2)$ with index n looks like.

Proposition 7 (Raising and lowering operators) *Let V be a finite dimensional complex linear space that is equipped with an inner product and let $J^{\pm}, J^0$ be linear operators on V that satisfy the commutation and adjoint relations (2.5) and hence define a unitary representation (V, π) of $\mathfrak{su}(2)$. Assume that (V, π) is irreducible and has index n. Then V has dimension $n+1$ and there exists an orthonormal basis*

$$\{\phi(-n/2), \phi(-n/2+1), \ldots, \phi(n/2)\}$$

such that

$$\begin{aligned} J^0\phi(k) &= k\phi(k), \\ J^-\phi(k) &= \sqrt{(n/2-k+1)(n/2+k)}\phi(k-1), \\ J^+\phi(k) &= \sqrt{(n/2-k)(n/2+k+1)}\phi(k+1) \end{aligned} \tag{2.6}$$

for $k = -n/2, -n/2+1, \ldots, n/2$, with the conventions $J^-\phi(-n/2) := 0$ and $J^+\phi(n/2) := 0$.

We see from (2.6) that $\phi(k)$ is an eigenvector of J^0 with eigenvalue k, and that the operators $J^{\pm}$ maps such an eigenvector into an eigenvector with eigenvalue $k \pm 1$, respectively. In view of this, $J^{\pm}$ are called *raising* and *lowering* operators, or also *creation* and *annihilation* operators. It is instructive to see how this property of $J^{\pm}$ follows rather easily from the commutation relations (2.5). Indeed, if $\phi(k)$ is an eigenvector of J^0 with eigenvalue k, then the commutation relations imply that

$$J^0J^+\phi(k) = \big(J^+J^0 + [J^0, J^+]\big)\phi(k) = \big(J^+J^0 + J^+\big)\phi(k) = (k+1)J^+\phi(k),$$

which shows that $J^+\phi(k)$ is a (possibly zero) multiple of $\phi(k+1)$. The concept of raising and lowering operators can be generalized to other Lie algebras.

2.4 The Lie algebra SU(1,1)

The Lie algebra $\mathfrak{su}(1,1)$ is defined by the commutation relations

$$[\mathbf{t}_{\mathrm{x}}, \mathbf{t}_{\mathrm{y}}] = 2i\mathbf{t}_{\mathrm{z}}, \quad [\mathbf{t}_{\mathrm{y}}, \mathbf{t}_{\mathrm{z}}] = -2i\mathbf{t}_{\mathrm{x}}, \quad [\mathbf{t}_{\mathrm{z}}, \mathbf{t}_{\mathrm{x}}] = 2i\mathbf{t}_{\mathrm{y}}. \tag{2.7}$$

Note that this is the same as (2.1) except for the minus sign in the second equality. A faithful representation is defined by the matrices

$$T_{\mathrm{x}} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_{\mathrm{y}} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T_{\mathrm{z}} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.8}$$

It is customary to equip $\mathfrak{su}(1,1)$ with an adjoint operation such that

$$\mathbf{t}_{\mathrm{x}}^* = \mathbf{t}_{\mathrm{x}}, \quad \mathbf{t}_{\mathrm{y}}^* = \mathbf{t}_{\mathrm{y}}, \quad \mathbf{t}_{\mathrm{z}}^* = \mathbf{t}_{\mathrm{z}}. \tag{2.9}$$

Note however, that the matrices in (2.8) are not self-adjoint and hence do not define a unitary representation of $\mathfrak{su}(1,1)$. In fact, all unitary irreducible representations of $\mathfrak{su}(1,1)$ are infinite dimensional. [6] In a given representation of $\mathfrak{su}(1,1)$, the *Casimir operator* is defined as

$$C := (\tfrac{1}{2}T_{\mathrm{x}})^2 - (\tfrac{1}{2}T_{\mathrm{y}})^2 - (\tfrac{1}{2}T_{\mathrm{z}})^2. \tag{2.10}$$

Again, it is useful to introduce raising and lowering operators, defined as

$$\mathbf{k}^0 := \tfrac{1}{2}\mathbf{t}_{\mathrm{x}} \quad \text{and} \quad \mathbf{k}^\pm := \tfrac{1}{2}(\mathbf{t}_{\mathrm{y}} \pm i\mathbf{t}_{\mathrm{z}}),$$

which satisfy the commutation and adjoint relations

$$[\mathbf{k}^0, \mathbf{k}^\pm] = \pm\mathbf{k}^\pm, \quad [\mathbf{k}^-, \mathbf{k}^+] = 2\mathbf{k}^0, \quad (\mathbf{k}^-)^* = \mathbf{k}^+, \quad (\mathbf{k}^0)^* = \mathbf{k}^0, \tag{2.11}$$

The following proposition is rewritten from [Nov04, formulas (8) and (9)], where this is stated without proof or reference. The constant $r > 0$ below is called the *Bargmann index* [Bar47, Bar61].

Proposition 8 (Representations of $\mathfrak{su}(1,1)$) *For each real constant $r > 0$, there exists an irreducible unitary representation of $\mathfrak{su}(1,1)$ on a Hilbert space with orthonormal basis $\{\phi(0), \phi(1), \ldots\}$ on which the operators $K^0, K^\pm$ act as*

$$\begin{aligned} K^0\phi(k) &= (k + r)\phi(k), \\ K^-\phi(k) &= 1_{\{k\geq 1\}}\sqrt{k(k-1+2r)}\,\phi(k-1), \\ K^+\phi(k) &= \sqrt{(k+1)(k+2r)}\,\phi(k+1). \end{aligned} \tag{2.12}$$

In this representation, the Casimir operator is given by $C = r(r-1)I$.

In what follows, we will need one more representation of $\mathfrak{su}(1,1)$, as well as a representation of its conjugate Lie algebra. Fix $\alpha > 0$ and consider the following operators acting on smooth functions $f : [0,\infty) \to \mathbb{R}$:

$$\begin{aligned} \mathcal{K}^- f(z) &= z\tfrac{\partial^2}{\partial z^2} f(z) + \alpha\tfrac{\partial}{\partial z} f(z), \\ \mathcal{K}^+ f(z) &= z f(z), \\ \mathcal{K}^0 f(z) &= z\tfrac{\partial}{\partial z} f(z) + \tfrac{1}{2}\alpha f(z). \end{aligned} \tag{2.13}$$

[6] Since $\mathfrak{su}(1,1)$ is simple, all representations are faithful. As explained in Subsection A.3, each Lie algebra is the Lie algebra of a unique simply connected Lie group. In the case of $\mathfrak{su}(1,1)$, this is the universal cover H of the Lie group $\mathrm{SU}(1,1)$ (the latter itself not being simply connected). By Theorem 18 in the appendix, each representation of $\mathfrak{su}(1,1)$ gives rise to a representation of H. Since H is not compact, the existence of a finite dimensional unitary representation would now contradict Lemma 23 in the appendix.

One can check that these operators satisfy the commutation relations (2.11) of the Lie algebra $\mathfrak{su}(1,1)_{\mathbb{C}}$, i.e.,

$$[\mathcal{K}^0, \mathcal{K}^\pm] = \pm\mathcal{K}^\pm \quad \text{and} \quad [\mathcal{K}^-, \mathcal{K}^+] = 2\mathcal{K}^0, \tag{2.14}$$

and hence define a representation of $\mathfrak{su}(1,1)$. One can check that the Casimir operator (2.10) for this representation is $C = \frac{\alpha}{2}(\frac{\alpha}{2} - 1)I$ and hence the Bargmann index is $r = \alpha/2$.

Next, fix again $\alpha > 0$ and consider the following operators acting on functions $f : \mathbb{N} \to \mathbb{R}$:

$$\begin{aligned} K^- f(x) &= x f(x-1), \\ K^+ f(x) &= (\alpha + x) f(x+1), \\ K^0 f(x) &= (\tfrac{1}{2}\alpha + x) f(x). \end{aligned} \tag{2.15}$$

One can check that these operators satisfy the commutation relations

$$[K^\pm, K^0] = \pm K^\pm \quad \text{and} \quad [K^+, K^-] = 2K^0. \tag{2.16}$$

This is similar to (2.11), except that the order of the elements inside the commutator is reversed. In view of the remarks at the end of Section 2.1, this means that the operators $K^0, K^\pm$ define a representation of the conjugate Lie algebra associated with $\mathfrak{su}(1,1)$. We will see in Section 3.4 below that the conjugate of the representation in (2.15) is equivalent to the representation in (2.13), provided we choose for both the same α.

A complete classification of all irreducible representations of $\mathfrak{su}(1,1)$, including infinite dimensional ones, is described in the book [VK91].[7]

2.5 The Heisenberg algebra

The *Heisenberg algebra* $\mathfrak{h}$ is the three dimensional complex Lie algebra defined by the commutation relations

$$[\mathbf{a}^-, \mathbf{a}^+] = \mathbf{a}^0, \quad [\mathbf{a}^-, \mathbf{a}^0] = 0, \quad [\mathbf{a}^+, \mathbf{a}^0] = 0. \tag{2.17}$$

It is customary to equip $\mathfrak{h}$ with an adjoint operation that is defined by

$$(\mathbf{a}^\pm)^* = \pm\mathbf{a}^\pm, \quad (\mathbf{a}^0)^* = \mathbf{a}^0. \tag{2.18}$$

[7]The monumental encyclopedic book [VK91] is written in a style that some readers may need to get used to, since it does not use the usual theorem-proof layout but rather states an enormous amount of facts in the main text while leaving a lot of detail to be filled in by the reader.

The *Schrödinger representation* of $\mathfrak{h}$ is defined by

$$A^- f(x) = \tfrac{\partial}{\partial x} f(x), \quad A^+ f(x) = x f(x), \quad A^0 f(x) = f(x), \tag{2.19}$$

which are interpreted as operators on the Hilbert space $L^2(\mathbb{R}, \mathrm{d}x)$ of complex functions on $\mathbb{R}$ that are square integrable with respect to the Lebesgue measure. Note in this representation, A^0 is the identity operator. Any representation of $\mathfrak{h}$ with this property is called a *central* representation.[8] The Schrödinger representation is a unitary representation, i.e., A^- is skew symmetric and A^+ and A^0 are self-adjoint, viewed as linear operators on the Hilbert space $L^2(\mathbb{R}, \mathrm{d}x)$.

Since iA^- and A^+ are self-adjoint, by Stone's theorem, one can define collections of unitary operators $(U_t^-)_{t\in\mathbb{R}}$ and $(U_t^+)_{t\in\mathbb{R}}$ by

$$U_s^- := e^{tA^-} \quad \text{and} \quad U_t^+ := e^{itA^+}. \tag{2.20}$$

These operators form one-parameter groups in the sense that $U_0^\pm = I$ and $U_s^\pm U_t^\pm = U_{s+t}^\pm$ $(s, t \in \mathbb{R})$. Note that we have a factor i in the definition of U_t^+ but not in the definition of U_s^-, because A^+ is self-adjoint but A^- is skew symmetric. The commutation relations (2.17) lead, at least formally, to the following commutation relation between U_s^- and U_t^+

$$U_s^- U_t^+ = e^{ist} U_t^+ U_s^- \qquad (s, t \in \mathbb{R}). \tag{2.21}$$

Indeed, for small ε, we have

$$\begin{aligned}
&U_{\varepsilon s}^- U_{\varepsilon t}^+ \\
&= \big(I + \varepsilon s A^- + \tfrac{1}{2}\varepsilon^2 s^2 (A^-)^2 + O(\varepsilon^3)\big)\big(I + i\varepsilon t A^+ - \tfrac{1}{2}\varepsilon^2 t^2 (A^+)^2 + O(\varepsilon^3)\big) \\
&= I + \varepsilon s A^- + \tfrac{1}{2}\varepsilon^2 s^2 (A^-)^2 + i\varepsilon t A^+ - \tfrac{1}{2}\varepsilon^2 t^2 (A^+)^2 + i\varepsilon^2 st A^- A^+ + O(\varepsilon^3) \\
&= I + \varepsilon s A^- + \tfrac{1}{2}\varepsilon^2 s^2 (A^-)^2 + i\varepsilon t A^+ - \tfrac{1}{2}\varepsilon^2 t^2 (A^+)^2 + i\varepsilon^2 st A^+ A^- \\
&\qquad\qquad\qquad\qquad\qquad\qquad + i\varepsilon^2 st [A^-, A^+] + O(\varepsilon^3) \\
&= \big(1 + i\varepsilon^2 st + O(\varepsilon^3)\big) U_{\varepsilon t}^+ U_{\varepsilon s}^- + O(\varepsilon^3).
\end{aligned} \tag{2.22}$$

The commutation relation (2.21) then follows formally by writing

$$\begin{aligned}
U_s^- U_t^+ &= (U_{s/n}^-)^n (U_{t/n}^+)^n \\
&= \big(1 + i n^{-2} st + O(n^{-3})\big)^{n^2} (U_{t/n}^+)^n (U_{s/n}^-)^n \underset{n\to\infty}{\longrightarrow} e^{ist} U_t^+ U_s^-.
\end{aligned} \tag{2.23}$$

[8] More generally, the *center* of a Lie algebra $\mathfrak{g}$ is the linear space $\mathfrak{c} := \{\mathbf{c} \in \mathfrak{g} : [\mathbf{x}, \mathbf{c}] = 0 \ \forall \mathbf{x} \in \mathfrak{g}\}$. A *central* representation of a Lie algebra is then a representation (V, π) such that for each $\mathbf{c} \in \mathfrak{c}$, there exists a $c \in \mathbb{C}$ such that $\pi(\mathbf{c}) = cI$. Note that with this definition, if (V, π) is a faithful central representation of $\mathfrak{h}$, then we can always "normalize" it by multiplying π with a constant so that $\pi(\mathbf{a}^0) = I$.

The *Stone-von Neumann theorem* states that all unitary, central representations of the Heisenberg algebra that satisfy (2.21) are equivalent [Ros04].

2.6 The direct sum and the tensor product

If V is a linear space and $V_1, \ldots, V_n$ are linear subspaces of V such that every element $v \in V$ can uniquely be written as

$$v = v_1 + \cdots + v_n$$

with $v_i \in V_i$, then we say that V is the *direct sum* of $V_1, \ldots, V_n$ and write $V = V_1 \oplus \cdots \oplus V_n$. If Ω_1, Ω_2 are finite sets and $\mathbb{C}^{\Omega_1}$ denotes the linear space of all functions $f : \Omega_i \to \mathbb{C}$, then we have the natural isomorphism

$$\mathbb{C}^{\Omega_1 \uplus \Omega_2} \cong \mathbb{C}^{\Omega_1} \oplus \mathbb{C}^{\Omega_2},$$

where $\Omega_1 \uplus \Omega_2$ denotes the disjoint union of Ω_1 and Ω_2.

If $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ are Lie algebras, then we equip $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ with the structure of a Lie algebra by putting, for $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{g}_i$,

$$\big[\mathbf{x}_1 + \cdots + \mathbf{x}_n, \mathbf{y}_1 + \cdots + \mathbf{y}_n\big] := [\mathbf{x}_i, \mathbf{y}_i] + \cdots + [\mathbf{x}_n, \mathbf{y}_n]. \tag{2.24}$$

Note that this has the effect that elements of diffent Lie algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ mutually commute. In particular, if $\{\mathbf{x}_1^1, \mathbf{x}_1^2, \mathbf{x}_1^3\}$ and $\{\mathbf{x}_2^1, \mathbf{x}_2^2, \mathbf{x}_2^3\}$ are bases for $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, then

$$\{\mathbf{x}_1^1, \mathbf{x}_1^2, \mathbf{x}_1^3, \mathbf{x}_2^1, \mathbf{x}_2^2, \mathbf{x}_2^3\}$$

is a basis for $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $[\mathbf{x}_i^k, \mathbf{x}_j^l] = 0$ whenever $i \neq j$.

By definition, a *bilinear* map of two variables is a function that is linear in each of its arguments. If V and W are finite dimensional linear spaces, then their *tensor product* is a linear space $V \otimes W$ together with a bilinear map

$$V \times W \ni (v, w) \mapsto v \otimes w \in V \otimes W$$

that has the property:

> If F is another linear space and $b : V \times W \to F$ is bilinear, then there exists a unique linear map $\overline{b} : V \otimes W \to F$ such that
>
> $$\overline{b}(v \otimes w) = b(v, w) \qquad (v \in V,\ w \in W).$$

The tensor product of three or more spaces is defined similarly. One can show that all realizations of the tensor product are naturally isomorphic. If $\{e(1), \ldots, e(n)\}$ and $\{f(1), \ldots, f(m)\}$ are bases for V and W, then one can prove that

$$\big\{e(i) \otimes f(j) : 1 \leq i \leq n,\ 1 \leq j \leq m\big\} \tag{2.25}$$

is a basis for $V \otimes W$. In particular, this means that one has the natural isomorphism

$$\mathbb{C}^{\Omega_1 \times \Omega_2} \cong \mathbb{C}^{\Omega_1} \otimes \mathbb{C}^{\Omega_2}. \tag{2.26}$$

If $A \in \mathcal{L}(V)$ and $B \in \mathcal{L}(V)$, then one defines $A \otimes B \in \mathcal{L}(V \otimes W)$ by

$$(A \otimes B)(v \otimes w) := (Av) \otimes (Bw). \tag{2.27}$$

We note that not every element of $V \otimes W$ is of the form $v \otimes w$ for some $v \in V$ and $w \in W$. Nevertheless, since the right-hand side of (2.27) is bilinear in v and w, the defining property of the tensor product tells us that this formula unambiguously defines a linear operator on $V \otimes W$.

One can check that the notation $A \otimes B$ is good notation in the sense that the space $\mathcal{L}(V \otimes W)$ together with the bilinear map $(A, B) \mapsto A \otimes B$ is a realization of the tensor product $\mathcal{L}(V) \otimes \mathcal{L}(W)$. Thus, one has the natural isomorphism

$$\mathcal{L}(V \otimes W) \cong \mathcal{L}(V) \otimes \mathcal{L}(W).$$

If V and W are equipped with inner products, then we equip $V \otimes W$ with an inner product by putting

$$\langle v \otimes w | \eta \otimes \xi \rangle := \langle v | \eta \rangle \langle w | \xi \rangle, \tag{2.28}$$

which has the effect that if $\{e(1), \ldots, e(n)\}$ and $\{f(1), \ldots, f(m)\}$ are orthonormal bases for V and W, then the basis for $V \otimes W$ in (2.25) is also orthonormal. Again, one needs the defining property of the tensor product to see that (2.28) is a good definition.

If V, W are representations of Lie algebras $\mathfrak{g}, \mathfrak{h}$, respectively, then we can naturally equip the tensor product $V \otimes W$ with the structure of a representation of $\mathfrak{g} \oplus \mathfrak{h}$ by putting

$$(\mathbf{x} + \mathbf{y})(v \otimes w) := (\mathbf{x}v) \otimes (\mathbf{y}w). \tag{2.29}$$

Again, since the right-hand side is bilinear, using the defining property of the tensor product, one can see that this is a good definition.

Let V_1, V_2 be representations of some Lie algebra $\mathfrak{g}$, and let W_1, W_2 be representations of another Lie algebra $\mathfrak{h}$. Let $\phi : V_1 \to V_2$ and $\psi : W_1 \to W_2$ be intertwiners. Then one can check that

$$\phi \otimes \psi : V_1 \otimes W_1 \to V_2 \otimes W_2 \tag{2.30}$$

is also an intertwiner.

If $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ are n copies of the Heisenberg algebra, and $\mathbf{a}_i^-, \mathbf{a}_i^+, \mathbf{a}_i^0$ are basis elements of $\mathfrak{h}_i$ that satisfy the commutation relations (2.17), then a basis for $\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n$ is formed by all elements $\mathbf{a}_i^\pm, \mathbf{a}_i^0$ with $i = 1, \ldots, n$, and these satisfy

$$[\mathbf{a}_i^- \mathbf{a}_j^+] = \delta_{ij}\mathbf{a}_i^0 \quad \text{and} \quad [\mathbf{a}_i^\pm, \mathbf{a}_j^0] = 0.$$

Since the center of $\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n$ is spanned by the elements $\mathbf{a}_i^0$ with $i = 1, \ldots, n$, a central representation of $\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n$ must map all these elements to multiples of the identity. In particular, a central representation of $\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n$ is never faithful (unless $n = 1$). The Lie algebra $\mathfrak{h}(n)$ is the $2n+1$ dimensional Lie algebra with basis elements $\mathbf{a}_i^\pm$ $(i = 1, \ldots, n)$ and $\mathbf{a}^0$, which satisfy the commutation relations

$$[\mathbf{a}_i^- \mathbf{a}_j^+] = \delta_{ij}\mathbf{a}^0 \quad \text{and} \quad [\mathbf{a}_i^\pm, \mathbf{a}^0] = 0.$$

A *central* representation of $\mathfrak{h}(n)$ is a representation (V, π) such that $\pi(\mathbf{a}^0) = I$. The Schrödinger representation of the "n-dimensional" Heisenberg algebra is the central representation of $\mathfrak{h}(n)$ on $L^2(\mathbb{R}^n, \mathrm{d}x)$ given by

$$A^- f(x) = \tfrac{\partial}{\partial x_i} f(x) \quad \text{and} \quad A^+ f(x) := x_i f(x). \tag{2.31}$$

3 The algebraic approach to duality

After our excursion into the theory of Lie algebras, we return to our main topic, which is the algebraic approach to Markov process duality. We recall from Lemma 3 that if a Markov generator L can be written in terms of "building blocks" A_i that each have a dual B_i with respect to some duality function D, then also L has a dual $\hat{L}$ with respect to D. As mentioned at the end of Section 1.3, it may be a good idea to choose the A_i's so that they define a representation of some Lie algebra. The next proposition says that in such a situation, other, equivalent representations of the same Lie algebra may lead to dual Markov processes.

Recall the definition of a conjugate Lie algebra $\overline{\mathfrak{g}}$ from Section 2.1. If $Y_1, \ldots, Y_n$ are matrices that define a representation of $\overline{\mathfrak{g}}$, then their adjoints $Y_1^\dagger, \ldots, Y_n^\dagger$ define a representation of the original Lie algebra $\mathfrak{g}$.

Proposition 9 (Intertwiners as duality functions) *Let L be the generator of a Markov process with finite state space Ω. Let $X_1, \ldots, X_n$ be linear operators on $\mathbb{C}^\Omega$ that form a representation of some Lie algebra $\mathfrak{g}$. Assume that L can be written as a linear combination of products of the operators*

$X_1, \ldots, X_n$

$$L = \sum_{(i_1,\ldots,i_k)\in\mathcal{I}} r_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k}, \tag{3.1}$$

where $\mathcal{I}$ is some finite set whose elements are sequences $(i_1, \ldots, i_k)$ with $k \geq 0$ and $1 \leq i_m \leq n$ for each m. Assume that $Y_1, \ldots, Y_n$ are linear operators on $\mathbb{C}^{\hat{\Omega}}$ that define a representation of the conjugate Lie algebra $\overline{\mathfrak{g}}$. Assume that the representations of $\mathfrak{g}$ defined by $Y_1^\dagger, \ldots, Y_n^\dagger$ and $X_1, \ldots, X_n$ are equivalent. Then there is a bijective intertwiner D, i.e., $X_i D = D Y_i^\dagger$ for each i, and L is dual w.r.t. the duality function D to the operator

$$\hat{L} := \sum_{(i_1,\ldots,i_k)\in\mathcal{I}} r_{i_1,\ldots,i_k} Y_{i_k} \cdots Y_{i_1}. \tag{3.2}$$

Proof By definition, two representations are equivalent if and only if there exists a bijective intertwiner. The fact that L is dual to the operator in (3.2) is then immediate from Lemma 3. ∎

At first sight, it may seem unlikely that Proposition 9 could be of much use. Even if we can write a generator in terms of a basis of a representation of some Lie algebra $\mathfrak{g}$, and we also find some representation of the conjugate Lie algebra $\overline{\mathfrak{g}}$, we still have to be lucky in the sense that the representations of $\mathfrak{g}$ defined by $Y_1^\dagger, \ldots, Y_n^\dagger$ and $X_1, \ldots, X_n$ are equivalent, and there is no guarantee that the operator in (3.2) is a Markov generator. Nevertheless, in what follows, we will see that Proposition 9 can help us find nontrivial dualities. In the next subsection, we demonstrate this on the operator L from (1.8), which is the generator of a Wright-Fisher diffusion with selection.

3.1 Self-duality of the Wright-Fisher diffusion

In Subsection 1.3, we have seen that the operator L from (1.8) can as in (1.10) be written in terms of the "building blocks" $A^\pm$ from (1.9). As we have seen in (1.16), these operators satisfy

$$[A^-, A^+] = I, \tag{3.3}$$

and hence define a central representation of the Heisenberg algebra $\mathfrak{h}$, as defined in Subsection 2.5.

It will be convenient to find a way of writing L in a more symmetric way than in (1.10). To this aim, we change the definitions of $A^\pm$ to

$$A^- f(x) := \frac{-1}{\sqrt{s}} \tfrac{\partial}{\partial x} f(x) \quad \text{and} \quad A^+ f(x) := \sqrt{s}\, x f(x), \tag{3.4}$$

which again satisfy (3.3), and we write L in terms of these new building blocks as

$$L = -A^{+}(\sqrt{s} - A^{+})A^{-}(\sqrt{s} - A^{-}). \tag{3.5}$$

We observe from (3.3) that setting $B^{-} := A^{+}$ and $B^{+} := A^{-}$ defines operators such that $[B^{-}, B^{+}] = -I$, i.e., B^{-}, B^{+} define a central representation of the conjugate Heisenberg algebra $\overline{\mathfrak{h}}$.

We recall from Section 2.5 that the Stone-von Neumann theorem states that, more or less, all central representations of the Heisenberg algebra are equivalent. In view of this and Proposition 9, we may expect that the operator

$$\hat{L} = -(\sqrt{s} - B^{-})B^{-}(\sqrt{s} - B^{+})B^{+} \tag{3.6}$$

is dual to L with respect to some (so far unknown) duality function D. (Here (3.6) is obtained from (3.5) by replacing $A^{\pm}$ by $B^{\pm}$ and reversing the order of the factors.) Since $B^{\pm} = A^{\mp}$, we observe that in fact $\hat{L} = L$, so our calculations lead us to suspect that the Wright-Fisher diffusion with selection parameter $s > 0$ should be self-dual.

We still need to find the duality function D. This function must satisfy

$$\frac{-1}{\sqrt{s}}\tfrac{\partial}{\partial x}D(x,y) = A^{-}D(\,\cdot\,,y)(x) = B^{-}D(x,\,\cdot\,)(y) = \sqrt{s}yD(x,y), \tag{3.7}$$

which says that $\frac{\partial}{\partial x}D(x,y) = -syD(x,y)$ and leads to the requirement that $D(x,y) = D(0,y)e^{-syx}$. In a similar way, the requirement $A^{+}D = DB^{+}$ yields $D(x,y) = D(x,0)e^{-syx}$ and in particular $D(0,y) = D(0,0)$. Thus, we find that up to a multiplicative constant, there is a unique duality function, which is given by

$$D(x,y) = e^{-sxy} \qquad \big(x,y \in [0,1]\big), \tag{3.8}$$

and we conclude that the Wright-Fisher diffusion with selection parameter $s > 0$ is self-dual with this duality function.

The argument above was heuristic, but quite smooth. What is remarkable about it is that while usually, the discovery of a duality starts with a clever choice for the duality function, here, the duality function came at the very end. Hidden behind this is the Stone-von Neumann theorem which says that two "good" representations of the Heisenberg algebra must necessarily be equivalent. We did not check the conditions of this theorem in detail (this is why the argument is only heuristic), but rather used it as an inspiration. A priori, there was no guarantee that the operator in (3.6) would be a Markov generator, but since $\hat{L} = L$ and L is a Markov generator, this turned out right as well.

Remark It is possible to "discover" the moment dual (1.15) of the Wright-Fisher duality along similar lines as we have discovered its self-duality here, by considering a suitable representation of the conjugate Heisenberg algebra $\overline{\mathfrak{h}}$ on functions $f : \mathbb{N} \to \mathbb{R}$ and applying Propositions 9 and 10. Such a derivation is less natural, however, since it requires choosing a rather peculiar representation of $\overline{\mathfrak{h}}$ that more or less has the duality function from (1.11) tacitly built into it.

3.2 Intertwiners and duality functions

In the previous subsection, just before (3.6) we appealed to Proposition 9. In doing so, we cheated in the sense that the operators $A^{\pm}$ from (3.4) do not act on a finite-dimensional space. The most obvious consequence of this is that it is not clear how the adjoint operators $B_i^{\dagger}$ from Proposition 9 should be defined. Closely related to this is that in the infinite dimensional setting, it is not immediately clear that duality functions define intertwiners and vice versa. In this subsection we show that these difficulties can be resolved by introducing a suitable inner product on the spaces of complex functions on Ω and $\hat{\Omega}$, respectively.

Assume that $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are linear operators on L^2-spaces $L^2(\Omega, \mu)$ and $L^2(\hat{\Omega}, \nu)$, respectively, that define representations of a Lie algebra $\mathfrak{g}$ and its conjugate $\overline{\mathfrak{g}}$. Let Y_i^* denote the adjoint of Y_i with respect to the inner product on $L^2(\hat{\Omega}, \nu)$. Assume that $\Phi : L^2(\hat{\Omega}, \nu) \to L^2(\Omega, \mu)$ is a linear operator of the form

$$\Phi g(x) = \int g(y) D(x, y) \nu(\mathrm{d}y), \tag{3.9}$$

for some function $D : \Omega \times \hat{\Omega} \to \mathbb{C}$ such that the expressions in (3.10) below are well-defined.

Proposition 10 (Intertwiners and duality functions) *The operator Φ is an intertwiner of the representations defined by $X_1, \ldots, X_n$ and $Y_1^*, \ldots, Y_n^*$, i.e.,*

$$X_i \Phi = \Phi Y_i^* \quad (i = 1, \ldots, n),$$

if and only if D is a duality function, in the sense that

$$X_i D(\,\cdot\,, y)(x) = Y_i D(x, \,\cdot\,)(y) \qquad (i = 1, \ldots, n) \tag{3.10}$$

for a.e. x, y with respect to the product measure $\mu \otimes \nu$.

Proof We observe that

$$\begin{aligned}
&\int \overline{f(x)}\mu(\mathrm{d}x)\int g(y)\nu(\mathrm{d}y)X_iD(\,\cdot\,,y)(x) = \int g(y)\nu(\mathrm{d}y)\langle f|X_iD(\,\cdot\,,y)\rangle_\mu \\
&\quad= \int g(y)\nu(\mathrm{d}y)\langle X_i^*f|D(\,\cdot\,,y)\rangle_\mu = \int \overline{X_i^*f(x)}\mu(\mathrm{d}x)\int g(y)\nu(\mathrm{d}y)D(x,y) \\
&\quad= \langle X_i^*f|\Phi g\rangle_\mu = \langle f|X_i\Phi g\rangle_\mu
\end{aligned}$$

and

$$\begin{aligned}
&\int \overline{f(x)}\mu(\mathrm{d}x)\int g(y)\nu(\mathrm{d}y)Y_iD(x,\,\cdot\,)(y) = \int \overline{f(x)}\mu(\mathrm{d}x)\langle g|Y_iD(x,\,\cdot\,)\rangle_\nu \\
&\quad= \int \overline{f(x)}\mu(\mathrm{d}x)\langle Y_i^*g|D(x,\,\cdot\,)\rangle_\nu = \int \overline{f(x)}\mu(\mathrm{d}x)\int Y_i^*g(y)\nu(\mathrm{d}y)D(x,y) \\
&\quad= \langle f|\Phi Y_i^*g\rangle_\mu.
\end{aligned}$$

Since this holds for all f, g, the statement follows. ∎

Remark Proposition 10 allows us to obtain an intertwiner from a duality function. Conversely, if $\Phi : L^2(\hat{\Omega},\nu) \to L^2(\Omega,\mu)$ is a bounded linear operator, then setting

$$\Delta(f\otimes g) := \int f(x)\Phi g(x)\,\mu(\mathrm{d}x)$$

defines a linear form on the linear span of all functions of the form $f\otimes g$. If Δ is bounded,[9] then it can uniquely be extended to a bounded linear form on

$$L^2(\Omega,\mu)\otimes L^2(\hat{\Omega},\nu) \cong L^2(\Omega\times\hat{\Omega},\mu\otimes\nu),$$

so that by the Riesz representation theorem there exists a $D \in L^2(\Omega\times\hat{\Omega},\mu\otimes\nu)$ such that

$$\Delta(f\otimes g) := \int f(x)D(x,y)g(x)\,\mu(\mathrm{d}x)\nu(\mathrm{d}y),$$

proving that Φ is of the form (3.9) (although there is no guarantee that $D(\,\cdot\,,y)$ and $D(x,\,\cdot\,)$ are in the domains of X_i and Y_i, resp., if these are unbounded operators).

[9] Using Cauchy-Schwarz, it is easy to see that $|\Delta(f\otimes g)| \leq \|\Phi\|\,\|f\otimes g\|$, proving that Δ is bounded on functions of the form $f\otimes g$. Nevertheless, Δ may fail to be bounded on the linear span of such functions. A counterexample is $\Omega = \hat{\Omega} = [0,1]$, $\mu = \nu =$ Lebesgue measure, and Φ the identity map, which gives $\Delta(F) = \int_0^1 F(x,x)\,\mathrm{d}x$. Since the Lebesgue measure on the diagonal $\{(x,y) : x = y\}$ does not have a density w.r.t. $\mu\otimes\nu$, this does not correspond to a bounded linear form on $L^2(\Omega\times\hat{\Omega},\mu\otimes\nu)$.

3.3 The symmetric exclusion process

In this subsection, we demonstrate Proposition 9 on a simple example, which involves the simple exclusion process and the Lie algebra $\mathfrak{su}(2)$. In the end, we find a self-duality that is not entirely trivial, but also not very useful. The present subsection serves mainly as a warm-up for Subsection 3.4 where we will replace $\mathfrak{su}(2)$ by $\mathfrak{su}(1,1)$.

Let S be a finite set and let $r : S \times S \to [0, \infty)$ be a function that is symmetric in the sense that $r(i,j) = r(j,i)$. Consider the Markov process with state space $\Omega = \{0,1\}^S$ and generator

$$Lf(x) := \sum_{ij} r(i,j) 1_{\{(x_i,x_j)=(1,0)\}} \big\{ f(x - \delta_i + \delta_j) - f(x) \big\}, \tag{3.11}$$

where $\delta_i \in \Omega$ is defined as $\delta_i(j) := 1_{\{i=j\}}$. Then L is the generator of a *symmetric exclusion process* or *SEP*. We define operators $J_i^\pm$ and J_i^0 by

$$\begin{aligned} J_i^- f(x) &:= 1_{\{x_i=0\}} f(x + \delta_i), \quad J_i^+ f(x) := 1_{\{x_i=1\}} f(x - \delta_i), \\ \text{and} \quad J_i^0 f(x) &:= (x_i - \tfrac{1}{2}) f(x). \end{aligned} \tag{3.12}$$

It is straightforward to check that

$$[J_i^0, J_j^\pm] = \pm \delta_{ij} J_i^\pm \quad \text{and} \quad [J_i^-, J_j^+] = -2\delta_{ij} J_i^0. \tag{3.13}$$

It follows that the operators $J_i^\pm$ and J_i^0 define a representation of a Lie algebra that consists of a direct sum of copies of $\mathfrak{su}(2)$, with one copy for each site $i \in S$. We can write the generator L of the symmetric exclusion process in terms of the operators $J_i^\pm$ and J_i^0 as

$$L = \sum_{\{i,j\}} r(i,j) \big[J_i^- J_j^+ + J_j^- J_i^+ + 2J_i^0 J_j^0 - \tfrac{1}{2} I \big], \tag{3.14}$$

where we are summing over all unordered pairs $\{i,j\}$. We observe that the operators

$$K_i^\pm := J_i^\pm, \quad \text{and} \quad K_i^0 := -J_i^0 \tag{3.15}$$

satisfy the same commutation relations as $J_i^\pm$ and J_i^0, except that each commutation relation gets an extra minus sign. This shows that the operators $K_i^\pm$ and K_i^0 define a representation of the conjugate Lie algebra $\overline{\mathfrak{su}(2)}$. Moreover, we can alternatively write the generator in (3.14) as

$$L = \sum_{\{i,j\}} r(i,j) \big[K_j^+ K_i^- + K_i^+ K_j^- + 2K_j^0 K_i^0 - \tfrac{1}{2} I \big]. \tag{3.16}$$

We recall from Subsection 2.3 that two irreducible representations of $\mathfrak{su}(2)$ with the same dimension are necessarily equivalent. In view of this, we conjecture that there should exist an intertwiner D, unique up to a multiplicative constant, such that $J_i^{\pm} D = D(K_i^{\pm})^{\dagger}$ and $J_i^0 D = D(K_i^0)^{\dagger}$ for all i. By the general principle in Proposition 9, such an intertwiner is a self-duality function for the symmetric exclusion process.

We observe that all our operators act on the space of all complex functions on $\{0,1\}^S$, which in view of (2.26) is given by

$$\mathbb{C}^{\{0,1\}^S} \cong \bigotimes_{i \in S} \mathbb{C}^{\{0,1\}}. \tag{3.17}$$

For example, if $S = \{1,2,3\}$ consists of only three sites, then in line with (2.29),

$$J_1^0 = J^0 \otimes I \otimes I, \quad J_2^0 = I \otimes J^0 \otimes I, \quad \text{and} \quad J_3^0 = I \otimes I \otimes J^0,$$

and similarly for $J_1^{\pm}, J_2^{\pm}$, and $J_3^{\pm}$. Here

$$\begin{aligned} J^- f &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(1) \\ f(0) \end{pmatrix} = \begin{pmatrix} 0 \\ f(1) \end{pmatrix}, \\ J^+ f &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f(1) \\ f(0) \end{pmatrix} = \begin{pmatrix} f(0) \\ 0 \end{pmatrix}, \\ J^0 f &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} f(1) \\ f(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} f(1) \\ -\frac{1}{2} f(0) \end{pmatrix}. \end{aligned} \tag{3.18}$$

We equip $\mathbb{C}^{\{0,1\}}$ and the space in (3.17) with the standard inner product, which has the consequence that $A^* = A^{\dagger}$ and

$$(J_i^-)^* = J_i^+, \quad (J_i^+)^* = J_i^-, \quad \text{and} \quad (J_i^0)^* = J_i^0,$$

showing that the operators $J_i^{\pm}$ and J_i^0 define a unitary representation of our Lie algebra.

According to the general principle (2.30), to find an intertwiner D which acts on the product space (3.17), it suffices to find an intertwiner for the two-dimensional space corresponding to a single site, and then take the product over all sites. Setting

$$Q := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

it is straightforward to check that

$$J^{\pm} Q = Q J^{\mp} = Q (K^{\pm})^{\dagger} \quad \text{and} \quad J^0 Q = Q(-J^0) = Q(K^0)^{\dagger}.$$

Now, for example, if $S = \{1, 2, 3\}$ consists of only three sites, then in view of (2.30)

$$D := Q \otimes Q \otimes Q \quad \text{satisfies} \quad J_i^{\pm} D = D(K_i^{\pm})^{\dagger} \quad \text{and} \quad J_i^0 D = D(K_i^0)^{\dagger}$$

$(i = 1, 2, 3)$. In terms of matrix elements, we have $Q(x_i, y_j) = 1_{\{x_i \neq y_i\}}$ and hence the self-duality function of the symmetric exclusion process that we have found is

$$D(x, y) = \prod_{i \in S} 1\{x_i \neq y_i\} \qquad \big(x, y \in \{0, 1\}^S\big).$$

3.4 The symmetric inclusion process

Let S be a finite set and let $\alpha : S \to (0, \infty)$ and $q : S \times S \to [0, \infty)$ be functions such that $q(i, j) = q(j, i)$ and $q(i, i) = 0$ for each $i \in S$. By definition, the *Brownian energy process* or *BEP* with parameters α, q is the diffusion process $(Z_t)_{t \geq 0}$ with state space $[0, \infty)^S$ and generator

$$L := \tfrac{1}{2} \sum_{i,j \in S} q(i, j) \big[(\alpha_j z_i - \alpha_i z_j)(\tfrac{\partial}{\partial z_j} - \tfrac{\partial}{\partial z_i}) + z_i z_j (\tfrac{\partial}{\partial z_j} - \tfrac{\partial}{\partial z_i})^2 \big]. \tag{3.19}$$

This diffusion has the property that $\sum_i Z_t(i)$ is a preserved quantity. The drift part of the generator is zero if $z_i = \lambda \alpha_i$ for some $\lambda > 0$. If $z_i / \alpha_i > z_j / \alpha_j$, then the drift has the tendency to make z_i smaller and z_j larger.

In analogy with (2.13), we define operators acting on smooth functions $f : [0, \infty)^S \to \mathbb{R}$ by:

$$\begin{aligned} \mathcal{K}_i^- f(z) &= z_i \tfrac{\partial^2}{\partial z_i{}^2} f(z) + \alpha_i \tfrac{\partial}{\partial z_i} f(z), \\ \mathcal{K}_i^+ f(z) &= z_i f(z), \\ \mathcal{K}_i^0 f(z) &= z_i \tfrac{\partial}{\partial z_i} f(z) + \tfrac{1}{2} \alpha_i f(z). \end{aligned} \tag{3.20}$$

By (2.14), these operators satisfy the commutation relations

$$[\mathcal{K}_i^0, \mathcal{K}_j^{\pm}] = \pm \delta_{ij} \mathcal{K}_i^{\pm} \quad \text{and} \quad [\mathcal{K}_i^-, \mathcal{K}_j^+] = 2 \delta_{ij} \mathcal{K}_i^0.$$

It follows that these operators define a representation of the Lie algebra

$$\bigoplus_{i \in S} \mathfrak{g}_i,$$

where each $\mathfrak{g}_i$ is a copy of $\mathfrak{su}(1, 1)$, on the product space

$$\mathbb{C}^{[0,\infty)^S} \cong (\mathbb{C}^{[0,\infty)})^{\otimes S},$$

which is the tensor product of $|S|$ copies of $\mathbb{C}^{[0,\infty)}$.

We can express the generator (3.19) of the Brownian energy process in terms of the operators from (3.20) as

$$L = \tfrac{1}{2} \sum_{i,j \in S} q(i,j) \big[\mathcal{K}_i^+ \mathcal{K}_j^- + \mathcal{K}_i^- \mathcal{K}_j^+ - 2\mathcal{K}_i^0 \mathcal{K}_j^0 + \tfrac{1}{2}\alpha_i \alpha_j \big]. \tag{3.21}$$

Note that this is very similar to the expression for the symmetric exclusion process in (3.14).

We define operators acting on functions $f : \mathbb{N}^S \to \mathbb{R}$ by

$$\begin{aligned} K_i^- f(x) &= x_i f(x - \delta_i), \\ K_i^+ f(x) &= (\alpha_i + x_i) f(x + \delta_i), \\ K_i^0 f(x) &= (\tfrac{1}{2}\alpha_i + x_i) f(x). \end{aligned} \tag{3.22}$$

In view of (2.16), these operators define a representation of the conjugate of our Lie algebra. It turns out that the conjugate of this representation is equivalent to the representation defined by the operators in (3.20). This is a nontrivial statement that depends crucially on the fact that the parameters α_i are the same in both expressions. Indeed, we have seen in Subsection 2.4 that α is twice the Bargmann index and that representations with a different Bargmann index have a different Casimir operator and hence are not equivalent. Letting Φ denote the intertwiner of $\mathcal{K}_i^\pm$ and $(K_i^\pm)^\dagger$, we can write Φ in the form (3.9), where by Proposition 10 D is a duality function. Similar to what we did at the end of Subsection 3.3, we will choose a duality function of product form:

$$D(z,x) = \prod_{i \in S} Q(z_i, x_i) \qquad (z \in [0,\infty)^S,\ x \in \mathbb{N}^S), \tag{3.23}$$

where Q is a duality function for the single-site operators, i.e.,

$$\mathcal{K}^\pm Q(\,\cdot\,, x)(z) = K^\pm Q(z, \,\cdot\,)(x), \quad \mathcal{K}^0 Q(\,\cdot\,, x)(z) = K^0 Q(z, \,\cdot\,)(x) \tag{3.24}$$

$(z \in [0,\infty),\ x \in \mathbb{N})$. It turns out that

$$Q(z,x) := \frac{\Gamma(\alpha + x)}{\Gamma(\alpha)} z^x = z^x \prod_{k=0}^{x-1} (\alpha + k). \tag{3.25}$$

does the trick. This may look a bit complicated but the form of this duality function can in fact quite easily be guessed from the inductive relation

$$zQ(z,x) = \mathcal{K}^+ Q(\,\cdot\,, x)(z) = K^+ Q(z, \,\cdot\,)(x) = (\alpha + x) Q(z, x+1).$$

Our calculations so far imply that the generator in (3.21) is dual with respect to the duality function in (3.23)–(3.25) to the generator

$$\hat{L} = \tfrac{1}{2} \sum_{i,j \in S} q(i,j) \big[K_j^- K_i^+ + K_j^+ K_i^- - 2K_j^0 K_i^0 + \tfrac{1}{2}\alpha_j \alpha_i \big]. \tag{3.26}$$

It turns out that we are lucky in the sense that this is a Markov generator. In view of the similarity with (3.14) (with the role of $\mathfrak{su}(2)$ replaced by $\mathfrak{su}(1,1)$), in [GRV10], the corresponding process has been called the *symmetric inclusion process* or *SIP*. The fact that $\hat{L}$ is a Markov generator can be seen by rewriting it as

$$\begin{aligned} \hat{L} := \sum_{i,j \in S} q(i,j) \Big[& \alpha_j x_i \big\{ f\big(x - \delta_i + \delta_j\big) - f\big(x\big) \big\} \\ & + x_i x_j \big\{ f\big(x - \delta_i + \delta_j\big) - f\big(x\big) \big\} \Big]. \end{aligned} \tag{3.27}$$

The Markov process $(X_t)_{t \geq 0}$ with generator $\hat{L}$ has the property that $\sum_i X_t(i)$ is a preserved quantity. The terms in the generator involving the constants α_j describe a system of independent random walks, where each particle at i jumps with rate α_j to the site j. A reversible law for this part of the dynamics is a Poisson field with local intensity $\lambda\alpha_i$ for some $\lambda > 0$. The remaining terms in the generator describe a dynamics where particles at i jump to j with a rate that is proportional to the number $x(j)$ of particles at j. This part of the dynamics causes an attraction between particles.

3.5 Duality functions of product form

In the previous two subsections, we have seen that for a Markov process whose state space is a Carthesian product of other spaces, it is often natural to choose duality functions of *product form* as in (3.23). This idea does not depend on Lie algebras and is in fact older than the use of Lie algebras in duality theory.

In a series of papers [LS95, LS97, Sud00], Lloyd and Sudbury have systematically searched for dualities in a large class of interacting particle systems, which contains many well-known systems such as the voter model, contact process, and symmetric exclusion process. Let S be a finite set and let $q : S^2 \to [0, \infty)$ be a function such that $q(i,j) = q(j,i)$ and $q(i,i) = 0$ for all $i \in S$. Let $L = L(a,b,c,d,e)$ be the Markov generator, acting on functions

$f : \{0,1\}^S \to \mathbb{R}$, as

$$\begin{aligned}
Lf(x) = \sum_{i,j\in S} q(i,j)\Big[\; & \tfrac{1}{2}a1_{\{(x(i),x(j))=(1,1)\}}\big\{f(x-\delta_i-\delta_j)-f(x)\big\} \\
& b1_{\{(x(i),x(j))=(0,1)\}}\big\{f(x+\delta_i)-f(x)\big\} \\
& c1_{\{(x(i),x(j))=(1,1)\}}\big\{f(x-\delta_i)-f(x)\big\} \\
& d1_{\{(x(i),x(j))=(0,1)\}}\big\{f(x-\delta_j)-f(x)\big\} \\
& e1_{\{(x(i),x(j))=(0,1)\}}\big\{f(x+\delta_i-\delta_j)-f(x)\big\}\Big].
\end{aligned} \tag{3.28}$$

The dynamics of the Markov process with generator L can be described by saying that for each pair of sites i, j, the configuration of the process at these sites makes the following transitions with the following rates:

$$\begin{array}{lll}
11 \mapsto 00 & \text{with rate} \quad aq(i,j) & \text{(annihilation)}, \\
01 \mapsto 11 & \text{with rate} \quad bq(i,j) & \text{(branching)}, \\
11 \mapsto 01 & \text{with rate} \quad cq(i,j) & \text{(coalecence)}, \\
01 \mapsto 00 & \text{with rate} \quad dq(i,j) & \text{(death)}, \\
01 \mapsto 10 & \text{with rate} \quad eq(i,j) & \text{(exclusion dynamics)}.
\end{array}$$

Note that the factor $\frac{1}{2}$ in front of a disappears since the total rate of this transition is $\frac{1}{2}a(q(i,j)+q(j,i)) = aq(i,j)$. A lot of well-known interacting particle systems fall into this class. For example

$$\begin{array}{rl}
\text{voter model} & b = d = 1, \ \text{other parameters } 0, \\
\text{contact process} & b = \lambda, \ c = d = 1, \ \text{other parameters } 0, \\
\text{symmetric exclusion} & e = 1, \ \text{other parameters } 0.
\end{array}$$

As we have already seen in (3.17), the class of all functions $f : \{0,1\}^S \to \mathbb{R}$ can be written as the tensor product

$$\mathbb{R}^{\{0,1\}^S} \cong \bigotimes_{i\in S} \mathbb{R}^{\{0,1\}},$$

with one 'factor' $\mathbb{R}^{\{0,1\}}$ for each site $i \in S$. Moreover, duality functions D on the space $\{0,1\}^S \times \{0,1\}^S$ can be viewed as matrices corresponding to linear operators that act on $\mathbb{R}^{\{0,1\}^S}$. Based on various arguments that are not very important at this point, Lloyd and Sudbury decided to look for duality functions of *product form*

$$D(x,y) = \prod_{i\in S} Q(x_i, y_i), \tag{3.29}$$

where Q is a 2×2 matrix. After a more or less systematic search for suitable matrices Q, Lloyd and Sudbury find a rich class of dualities for matrices of the form

$$\begin{pmatrix} Q_q(0,0) & Q_q(0,1) \\ Q_q(1,0) & Q_q(1,1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix}, \tag{3.30}$$

where $q \in \mathbb{R}\backslash\{1\}$ is a constant. This choice of Q yields the duality function

$$D_q(x,y) := \prod_{i \in S} Q_q(x_i, y_i) = q^{\sum_{i \in S} x_i y_j} \qquad \big(x, y \in \{0,1\}^S\big). \tag{3.31}$$

In particular, setting $q = 0$ yields

$$D_0(x,y) = 1_{\{\sum_{i \in S} x_i y_j = 0\}},$$

which corresponds to the well-known *additive systems duality*, while $q = -1$ is known as *cancellative systems duality*. For these special values of q, the duality can in fact be upgraded to a pathwise duality as in Subsection 1.4, using a construction in terms of open paths in a graphical representation. Interestingly, for other values of q, there seems to be no pathwise interpretation of the duality with duality function D_q.

We cite the following theorem from [LS95, Sud00]. A somewhat more general version of this theorem which drops the symmetry assumption $q(i,j) = q(j,i)$ at the cost of replacing (3.32) by a somewhat more complicated set of conditions can be found in [Swa06, Appendix A in the version on the ArXiv].

Theorem 11 (q-duality) *The generators $L(a,b,c,d,e)$ and $L(a',b',c',d',e')$ from (3.28) are dual with respect to the duality function D_q from (3.31) if and only if*

$$a' = a + 2q\gamma, \quad b' = b + \gamma, \quad c' = c - (1+q)\gamma, \quad d' = d + \gamma, \quad e' = e - \gamma, \tag{3.32}$$

where $\gamma := (a + c - d + qb)/(1 - q)$.

3.6 Intertwining and thinning

In Subsection 1.2, when we introduced Markov process duality, we also defined the similar concept of intertwining of Markov processes. So far, we have not discussed this second concept very much, but it turns out that the two are closely related. In particular, as Lloyd and Sudbury already observed [LS95, Sud00], there is a close connection between q-duality and thinning relations. To explain this, we start with a general principle, that says that if two Markov processes are both dual to a third Markov process, then we can expect an intertwining relation between the first two processes.

Lemma 12 (Duality and intertwining) *Let Ω and $\hat{\Omega}$ be finite sets, and let $L_i : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$, $\hat{L} : \mathbb{R}^{\hat{\Omega}} \to \mathbb{R}^{\hat{\Omega}}$, and $D_i : \mathbb{R}^{\hat{\Omega}} \to \mathbb{R}^\Omega$ be linear operators such that*

$$L_i D_i = D_i \hat{L}^\dagger \qquad (i = 1, 2). \tag{3.33}$$

Assume that D_1 and D_2 are invertible. Then

$$L_1(D_1 D_2^{-1}) = (D_1 D_2^{-1}) L_2. \tag{3.34}$$

Proof This follows by writing $D_1^{-1} L_1 D_1 = \hat{L}^\dagger = D_2^{-1} L_2 D_2$. ∎

We have seen that for interacting particle systems, there are good reasons to look for duality functions of product form as in (3.29). Likewise, it is natural to look for intertwining probability kernels of product form. If the state space is of the form $\{0,1\}^S$, this means that we are looking for kernels of the form

$$K(x, y) = \prod_{i \in S} M(x_i, y_i) \qquad \big(x, y \in \{0,1\}^S\big),$$

where M is a probability kernel on $\{0,1\}$. If we moreover require that $M(0,0) = 1$ (which is natural for interacting particle systems for which the all zero state is a trap), then there is only a one-parameter family of such kernels. For $p \in [0,1]$, let M_p be the probability kernel on $\{0,1\}$ given by

$$M_p = \begin{pmatrix} M_p(0,0) & M_p(0,1) \\ M_p(1,0) & M_p(1,1) \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 1-p & p \end{pmatrix}, \tag{3.35}$$

and let

$$K_p(x, y) := \prod_{i \in S} M_p(x_i, y_i) \qquad \big(x, y \in \{0,1\}^S\big) \tag{3.36}$$

the corresponding kernel on $\{0,1\}^S$ of product form. We can interpret a configuration of particles, where $x_i = 1$ if the site i is occupied by a particle, and $x_i = 0$ otherwise. Then K_p is a *thinning* kernel that independently for each site throws away particles with probability $1 - p$ or keeps them with probability p. It is easy to see that

$$K_p K_{p'} = K_{pp'},$$

i.e., first thinning with p and then with p' is the same as thinning with pp'. There is a close relation between Lloyd and Sudbury's duality function D_q from (3.31) and thinning kernels of the form (3.36). We claim that

$$D_q D_{q'}^{-1} = K_p \quad \text{with} \quad p = \frac{1-q}{1-q'} \qquad (q, q' \in \mathbb{R},\ q' \neq 1). \tag{3.37}$$

Since both D_q and K_p are of product form, i.e.,

$$D_q = \bigotimes_{i\in S} Q_q \quad \text{and} \quad K_p = \bigotimes_{i\in S} M_p$$

with Q_q and M_p as in (3.30) and (3.35), it suffices to check that

$$Q_q Q_{q'}^{-1} = M_p \quad \text{with} \quad p = \frac{1-q}{1-q'}.$$

Indeed, one can check that

$$Q_q^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix}^{-1} = (1-q)^{-1} \begin{pmatrix} -q & 1 \\ 1 & -1 \end{pmatrix} \qquad (q \neq 1),$$

and that

$$Q_q Q_{q'}^{-1} = (1-q')^{-1} \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} \begin{pmatrix} -q' & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{q-q'}{1-q'} & \frac{1-q}{1-q'} \end{pmatrix} = M_p,$$

as claimed.

Proposition 13 (Thinning and q-duality) *Let L_1 and L_2 be generators of Markov processes with state space $\{0,1\}^S$. Assume that there exists an operator $\hat{L}$ such that*

$$L_i D_{q_i} = D_{q_i} \hat{L}^\dagger \qquad (i = 1, 2) \tag{3.38}$$

for some $q_1, q_2 \in \mathbb{R}$ such that $q_2 \neq 1$ and $p := (1-q_1)/(1-q_2) \in [0,1]$. Then

$$L_1 K_p = K_p L_2. \tag{3.39}$$

Proof This follows from (3.37) and Lemma 12. Note that in general, there is no guarantee that the operator $D_1 D_2^{-1}$ from Lemma 12 is a probability kernel. In a way, Proposition 13 explains why the q-duality function D_q is natural, because it is closely linked to the natural concept of thinning. ∎

3.7 The biased voter model

In this section, we demonstrate Lloyd-Sudbury theory on the example of the *biased voter model* with *selection parameter* $s > 0$, which is the interacting particle system with generator

$$L(a, b, c, d, e) = L(0, 1+s, 0, 1, 0) =: L_{\text{bias}}.$$

We apply Theorem 11 to find q-duals of the biased voter model. For simplicity, we restrict ourselves here to dual generators of the form $L(a',b',c',d',e')$ with $a'=0$, which means that we must choose the parameter q as

$$q=0 \quad \text{or} \quad q=(1+s)^{-1}.$$

For $q=0$ we find the dual generator

$$L(a',b',c',d',e')=L(0,s,1,0,1)=:L_{\rm braco},$$

which describes a system of branching and coalescing random walks with branching parameter s. For $q=(1+s)^{-1}$, we find a self-duality, i.e., in this case $L(a',b',c',d',e')=L(a,b,c,d,e)=L_{\rm bias}$.

Since $L_{\rm bias}$ and $L_{\rm braco}$ are both q-dual to $\hat{L}=L_{\rm bias}$, Proposition 13 tells us that there is a thinning relation between biased voter models and systems of branching and coalescing random walks of the form

$$L_{\rm bias}K_p=K_pL_{\rm braco} \quad \text{with} \quad p=\frac{1-(1+s)^{-1}}{1-0}=\frac{s}{1+s}.$$

As explained in Subsection 3.6, this implies that if we start a biased voter model $(X_t)_{t\geq 0}$ and a system of branching and coalescing random walks $(Y_t)_{t\geq 0}$ in initial states $\mu^{\rm bias}_t$ and $\mu^{\rm braco}_t$ denote the laws of X_t and Y_t, then

$$\mu^{\rm bias}_0K_p=\mu^{\rm braco}_0 \quad \text{implies} \quad \mu^{\rm bias}_tK_p=\mu^{\rm braco}_t \quad (t\geq 0).$$

In other words, the following two procedures are equivalent:

(i) Evolve a particle configuration for time t according to biased voter model dynamics, then thin with p.

(ii) Thin a particle configuration with p, then evolve for time t according to branching coalescing random walk dynamics.

In particular, if we start X in the initial state $X_0(i)=1$ for all $i\in S$, then because of the nature of the voter model, we will have $X_t(i)=1$ for all $i\in S$ and $t\geq 0$. Applying the thinning relation now shows that product measure with intensity p is an invariant law for branching coalescing random walk dynamics. Thus, there is a close connection between:

I. q-duality,

II. thinning relations,

III. invariant laws of product form.

Although Lloyd-Sudbury theory is restricted to Markov processes with state space of the form $\{0,1\}^S$, many other dualities, including the self-duality of the Wright-Fisher diffusion from Section 3.1, can be derived from Lloyd-Sudbury duals by taking a suitable limit [Swa06].

3.8 Time-reversal and symmetry

In this subsection we present an idea from [GKRV09], which says that nontrivial dualities can sometimes be found by starting from a "trivial" duality which is based on time reversal, and then using a symmetry of the model to transform such a duality into a nontrivial one. Although Lie algebras are not strictly needed in this approach, writing generators in terms of the basis elements of a representation of a Lie algebra can help finding suitable symmetries.

Each irreducible Markov process with finite state space Ω has a unique invariant measure, i.e., a probability measure μ such that

$$\mu L = 0 \quad \text{or equivalently} \quad \mu P_t = \mu \quad (t \geq 0),$$

where L denotes the generator and $(P_t)_{t\geq 0}$ the semigroup of the Markov process. Irreducibility implies that $\mu(x) > 0$ for all $x \in \Omega$. Letting $(X_t)_{t\in\mathbb{R}}$ denote the stationary process, we see that the semigroup $(\tilde{P}_t)_{t\geq 0}$ of the time-reversed process is given by

$$\begin{aligned}\tilde{P}_t(x,y) &= \frac{\mathbb{P}[X_0 = y,\ X_t = x]}{\mathbb{P}[X_t = x]} \\ &= \frac{\mu(y)P_t(y,x)}{\mu(x)} = \mu(y)P_t(y,x)\mu(x)^{-1} \quad (t \geq 0).\end{aligned}$$

Differentiating shows that the generator $\tilde{L}$ of the time-reversed process is given by[10]

$$\tilde{L}(x,y) = \mu(y)L(y,x)\mu(x)^{-1}.$$

Let R denote the diagonal matrix

$$R(x,y) := \delta_{x,y}\mu(x)^{-1}.$$

Then $L(y,x)\mu(x)^{-1} = \tilde{L}(x,y)\mu(y)^{-1} = \mu(y)^{-1}\tilde{L}^\dagger(y,x)$ can be rewritten as

$$LR = R\tilde{L}^\dagger,$$

which shows that $\tilde{L}$ is dual to L with duality function R. In particular, reversible processes (for which $\tilde{L} = L$) are always *self-dual* with duality function $R(x,y)$. Note that since R is diagonal, it is invertible with

$$R^{-1}(x,y) := \delta_{x,y}\mu(x) \qquad (x,y \in \Omega).$$

[10] This formula is wrong in [GKRV09, below (12)].

Let V be a finite dimensional complex linear space and let $L : V \to V$ be any linear operator (not necessarily a Markov generator). Then it is known that there exists an invertible matrix $Q \in \mathcal{L}(V)$ such that

$$LQ = QL^\dagger \quad \text{or equivalently} \quad L^\dagger Q^{-1} = Q^{-1}L \tag{3.40}$$

Thus, *every* finite dimensional linear operator is self-dual and the self-duality function Q can be chosen such that it is invertible, viewed as a matrix. Let

$$\mathcal{C}_L := \{A \in \mathcal{L}(V) : AL = LA\}$$

be the algebra of all elements of $\mathcal{L}(V)$ that commute with L. We call this the space of *symmetries* of L. In [GKRV09, Thm 2.6], the following simple observation is made.

Lemma 14 (Self-duality functions) *Let L be a linear operator on some finite dimensional linear space V. Fix some Q as in (3.40). Then the set of all self-duality functions of L is given by*

$$\{SQ : S \in \mathcal{C}_L\}.$$

Proof Clearly, if $S \in \mathcal{C}_L$, then

$$LSQ = SLQ = SQL^\dagger,$$

showing that SQ is a self-duality function. Conversely, if D is a self-duality function, then we can write $D = SQ$ with $S = DQ^{-1}$. Now, since D is a self-duality function,

$$SL = DQ^{-1}L = DL^\dagger Q^{-1} = LDQ^{-1} = LS,$$

which shows that $S \in \mathcal{C}_L$. ∎

For dualities, we can play a similar game. Once we have two operators $L, \hat{L}$ that are dual with duality function D, i.e.,

$$LD = D\hat{L}^\dagger,$$

we have that for any $S \in \mathcal{C}_L$, the operators $L, \hat{L}$ are also dual with duality function SD, as follows by writing

$$LSD = SLD = SD\hat{L}^\dagger.$$

If D is invertible, then every duality function of L and $\hat{L}$ is of this form. Indeed, if $\tilde{D}$ is any duality function, then we can write $\tilde{D} = SD$ with $S = \tilde{D}D^{-1}$. Now

$$SL = \tilde{D}D^{-1}L = \tilde{D}L^\dagger D^{-1} = L\tilde{D}D^{-1} = LS,$$

proving that $S \in \mathcal{C}_L$. See also [GKRV09, Thm 2.10].

3.9 The symmetric exclusion process revisited

Following [GKRV09, Sect. 3.1], we demonstrate the principles explained in the previous subsections to derive a self-duality of the symmetric exclusion process. Our starting point is formula (3.14), which expresses the generator L in terms of operators $J_i^\pm, J_i^0$ that define a representation (V,π) of a Lie algebra $\mathfrak{g}$ that is the direct sum of finitely many copies of the Lie algebra $\mathfrak{su}(2)$, with one copy for each site $i \in S$. Since $r(i,j) = r(j,i)$, we can rewrite this formula as

$$L = \tfrac{1}{2}\sum_{i,j} r(i,j)\big[J_i^- J_j^+ + J_j^- J_i^+ + 2J_i^0 J_j^0 - \tfrac{1}{2}I\big]. \tag{3.41}$$

A straightforward calculation shows that

$$\sum_k [J_k^\pm, L] = 0 \quad \text{and} \quad \sum_k [J_k^0, L] = 0 \qquad (k \in S). \tag{3.42}$$

We need a bit of general theory. If U, V, W are representations of the same Lie algebra $\mathfrak{g}$, then we can equip their tensor product $U \otimes V \otimes W$ with the structure of a representation of $\mathfrak{g}$ by putting

$$A(u \otimes v \otimes w) := Au \otimes v \otimes w + u \otimes Av \otimes w + u \otimes v \otimes Aw \qquad (A \in \mathfrak{g}), \tag{3.43}$$

and similar for the tensor product of any finite number of representations, see formula (A.13) in the appendix. This definition also naturally equips $U \otimes V \otimes W$ with the structure of a representation of the Lie group G associated with $\mathfrak{g}$, in such a way that

$$e^{tA}(u \otimes v \otimes w) = e^{tA}u \otimes e^{tA}v \otimes e^{tA}w \qquad (A \in \mathfrak{g},\ t \geq 0),$$

where for each $A \in \mathfrak{g}$, the operator e^{tA} is an element of the Lie group G associated with $\mathfrak{g}$. Thus, the representation (3.43) corresponds to letting the Lie group act in the same way on each space in the tensor product.

In our specific set-up, this means that the operators K^-, K^+, K^0 defined by

$$K^- := \sum_k J_k^-, \quad K^+ := \sum_k J_k^+, \quad K^0 := \sum_k J_k^0 \tag{3.44}$$

define a representation of $\mathfrak{su}(2)$ on the product space

$$\mathbb{C}^{\{0,1\}^S} \cong \bigotimes_{i \in S} \mathbb{C}^{\{0,1\}}.$$

(Indeed, one can check that K^-, K^+, K^0 satisfy the commutation relations of $\mathfrak{su}(2)$.) Let $c_-K^- + c_+K^+ + c_0K^0$ be an operator in the linear space spanned by K^-, K^+, K^0. Then

$$e^{t(c_-K^- + c_+K^+ + c_0K^0)} = \bigotimes_{i\in S} e^{t(c_-J^- + c_+J^+ + c_0J^0)} \qquad (t \geq 0), \tag{3.45}$$

i.e., a natural group of symmetries of the generator L is formed by all operators of the form (3.45) and their products, and this actually corresponds to a representation of the Lie group SU(2).

We take this as our motivation to look at one specific operator of the form (3.45), which is e^{K^+}. One can check that the uniform distribution is an invariant law for the exclusion process, so by the principle of Subsection 3.8, the function

$$D(x,y) = 1_{\{x=y\}} = \prod_{i\in S} 1_{\{x_i=y_i\}}$$

is a trivial self-duality function. Applying Lemma 14 to the symmetry $S = e^{K^+}$, we see that $SD = SI = S$ is also a self-duality function. Since S factorizes over the sites, it suffices to calculate S for a single site, and then take the product. We recall from (3.18) that

$$J^+ f \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f(1) \\ f(0) \end{pmatrix} = \begin{pmatrix} f(0) \\ 0 \end{pmatrix},$$

which gives

$$e^{J^+} = \sum_{n=0}^{\infty} \frac{1}{n!}(J^+)^n = I + J^+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and finally yields the duality function

$$S(x,y) = \prod_{i\in S} 1_{\{x_i \geq y_i\}} \qquad \big(x, y \in \{0,1\}^S\big).$$

A A crash course in Lie algebras

A.1 Lie groups

In the present appendix, we give a bit more background on Lie algebras. In particular, we explain how Lie algebras are closely linked to Lie groups, and how every Lie algebra can naturally be embedded in an algebra, called the universal enveloping algebra. We also explain how properties of the Lie group (in particular, compactness) are related to representations of its associated Lie algebra.

A *group* is a set G which contains a special element I, called the *identity*, and on which a *group product* $(A,B) \mapsto AB$ and *inverse operation* $A \mapsto A^{-1}$ are defined such that

(i) $IA = AI = A$

(ii) $(AB)C = A(BC)$

(iii) $A^{-1}A = AA^{-1} = I$.

A group is *abelian* (also called *commutative*) if $AB = BA$ for all $A, B \in G$. A *group homomorphism* is a map Φ from one group G into another group H that preserves the group structure, i.e.,

(i) $\Phi(I) = I$,

(ii) $\Phi(AB) = \Phi(A)\Phi(B)$,

(iii) $\Phi(A^{-1}) = \Phi(A)^{-1}$.

If Φ is a bijection, then Φ^{-1} is also a group homomorphism. In this case, we call Φ a *group isomorphism.* A *subgroup* of a group G is a subset $H \subset G$ such that $I \in H$ and H is closed under the product and inverse, i.e., $A, B \in H$ imply $AB \in H$ and $A \in H$ implies $A^{-1} \in H$. A subgroup is in a natural way itself a group.

A *Lie group* is a smooth manifold G which is also a group such that the group product and inverse functions

$$G \times G \ni (A,B) \mapsto AB \in G \quad \text{and} \quad G \ni A \mapsto A^{-1} \in G$$

are smooth. A *finite-dimensional representation* of G is a finite-dimensional linear space V over $\mathbb{R}$ or $\mathbb{C}$ together with a map

$$G \times V \ni (A, v) \mapsto Av \in V$$

such that

(i) $v \mapsto Av$ is linear,

(ii) $Iv = v$,

(iii) $A(Bv) = (AB)v$.

Letting $\mathcal{L}(V)$ denote the space of all linear operators $A : V \to V$, these conditions are equivalent to saying that the map $\Pi : G \to \mathcal{L}(V)$ defined by

$$\Pi(A)v := Av$$

is a group homomorphism from G into the *general linear group* $\mathrm{GL}(V)$ of all invertible linear maps $A : V \to V$. A representation is *faithful* if Π is one-to-one, i.e., if $A \mapsto \Pi(A)$ is a group isomorphism between G and the subgroup $\Pi(G) := \{\Pi(A) : A \in G\}$ of $\mathrm{GL}(V)$.

One can prove that if G is a Lie group and V is a faithful finite-dimensional representation, then $\Pi(G)$ is a closed subset of $\mathrm{GL}(V)$ and $\Pi : G \to \Pi(G)$ is a homeomorhism. Conversely, each closed subgroup of $\mathrm{GL}(V)$ is a Lie group. Such Lie groups are called *matrix Lie groups.* Not every Lie group has a finite dimensional faithful representation, so not every Lie group is a matrix Lie group, but many important Lie groups are matrix Lie groups and following [Hal03] we will mostly focus on them from now on.

A.2 Lie algebras

An *algebra* is a finite-dimensional linear space $\mathfrak{a}$ over $\mathbb{R}$ or $\mathbb{C}$ with a special element I called *unit element* and on which there is defined a product

$$\mathfrak{a} \times \mathfrak{a} \ni (A, B) \mapsto AB \in \mathfrak{a}$$

such that

(i) $(A, B) \mapsto AB$ is bilinear,

(ii) $IA = AI = A$,

(iii) $(AB)C = A(BC)$.

In some textbooks, algebras are not required to contain a unit element. We speak of a *real* resp. *complex* algebra depending on whether $\mathfrak{a}$ is a linear space over $\mathbb{R}$ or $\mathbb{C}$. An algebra is *abelian* if $AB = BA$ for all $A, B \in G$. In any algebra, the *commutator* of two elements A, B is defined as $[A, B] = AB - BA$. If V is a linear space, then $\mathcal{L}(V)$ is an algebra.

An *algebra homomorphism* is a map $\phi : \mathfrak{a} \to \mathfrak{b}$ from one algebra into another that preserves the structure, i.e.,

(i) ϕ is linear,

(ii) $\phi(I) = I$,

(iii) $\phi(AB) = \phi(A)\phi(B)$.

Algebra homomorphisms that are bijections have the property that ϕ^{-1} is also a homomorphism; these are called algebra isomorphisms. A *subalgebra* of an algebra $\mathfrak{a}$ is a linear subspace $\mathfrak{b} \subset \mathfrak{a}$ that contains I and is closed under the product.

Lie algebras, *Lie algebra homomorphisms*, and *isomorphisms* have already been defined in Section 2.1. A *sub-Lie-algebra* is a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ such that

$$A, B \in \mathfrak{h} \quad \text{implies} \quad [A, B] \in \mathfrak{h}.$$

If $\mathfrak{g}$ is an algebra, then $\mathfrak{g}$, equipped with the commutator map $[\,\cdot\,,\,\cdot\,]$, is a Lie algebra. As the example in Section 2.1 shows. Lie algebras need not be an algebras.

A *representation* of an algebra $\mathfrak{a}$ is a linear space V together with a map $\mathfrak{a} \times V \to V$ that satisfies

(i) $(A, v) \mapsto Av$ is bilinear,

(ii) $Iv = v$,

(iii) $A(Bv) = (AB)v$.

If $\mathfrak{a}$ is a complex algebra, then we require V to be a linear space over $\mathbb{C}$, but even when $\mathfrak{a}$ is a real algebra, it is often useful to allow for the case that V is a linear space over $\mathbb{C}$. In this case, bilinearity means real linearity in the first argument and complex linearity in the second argument. We speak of *real* or *complex* representations depending on whether V is a linear space over $\mathbb{R}$ or $\mathbb{C}$.

A representation V of an algebra $\mathfrak{a}$ gives in a natural way rise to an algebra homomorphism $\pi : \mathfrak{a} \to \mathcal{L}(V)$ defined as

$$\pi(A)v := Av \qquad (A \in \mathfrak{a},\ v \in V).$$

Conversely, given an algebra homomorphism $\pi : \mathfrak{a} \to \mathcal{L}(V)$ we can equip V with the structure of a representation by defining $Av := \pi(A)v$. Thus, a representation V of an algebra $\mathfrak{a}$ is equivalent to a pair (V, π) where V is a linear space and $\pi : \mathfrak{a} \to \mathcal{L}(V)$ is an algebra homomorphism. A representation (V, π) is *faithful* if π is an isomorphism between $\mathfrak{a}$ and the subalgebra $\pi(\mathfrak{a}) = \{\pi(A) : A \in \mathfrak{a}\}$ of $\mathcal{L}(V)$.

Representations of Lie algebras have already been defined in Section 2.2. If V is a complex representation of a real algebra or Lie algebra $\mathfrak{a}$, then the image of $\mathfrak{a}$ under π is only a real subspace of $\mathcal{L}(V)$. We can define a complex algebra or Lie algebra $\mathfrak{a}_{\mathbb{C}}$ whose elements can formally be written as $A+iB$ with $A, B \in \mathfrak{a}$; this is called the *complexification* of $\mathfrak{a}$. Then π extends uniquely to a homomorphism from $\mathfrak{a}_{\mathbb{C}}$ to $\mathcal{L}(V)$, see [Hal03, Prop. 3.39], so V is also a representation of $\mathfrak{a}_{\mathbb{C}}$.

Every algebra has a faithful representation. Indeed, $\mathfrak{a}$ together with the map $(A, B) \mapsto AB$ is a representation of itself, and it is not hard to see (using our assumption that $I \in \mathfrak{a}$) that this representation is faithful. Lie algebras can be represented on themselves in a construction that is very similar to the one for algebras.

Lemma 15 (Lie algebra represented on itself) *A Lie algebra $\mathfrak{g}$, equipped with the map $(A, B) \mapsto [A, B]$, is a representation of itself.*

Proof It will be convenient to use somewhat different notation for the Lie bracket. If $\mathfrak{g}$ is a Lie algebra and $X \in \mathfrak{g}$, then we define $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}_X(A) := [X, A].$$

We need to show that $\mathfrak{g} \ni X \mapsto \mathrm{ad}_X \in \mathcal{L}(\mathfrak{g})$ is a Lie algebra homomorphism. Bilinearity follows immediately from the bilinear property (i) of the Lie bracket, so it remains to show that

$$\mathrm{ad}_{[X,Y]}(Z) = \mathrm{ad}_X(\mathrm{ad}_Y(Z)) - \mathrm{ad}_Y(\mathrm{ad}_X(Z)).$$

This can be rewritten as

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]].$$

Using also the skew symmetric property (ii) of the Lie bracket, this can be rewritten as

$$0 = [Z, [X, Y]] + [X, [Y, Z]] + [Y, [Z, X]],$$

which is the Jacobi identity. ∎

In general, representing a Lie algebra on itself as in Lemma 15 need not yield a faithful representation. (For example, any abelian algebra is also a Lie algebra and for such Lie algebras $\mathrm{ad}_X = 0$ for each X.) By definition, the *center* of a Lie algebra $\mathfrak{g}$ is the set

$$\{X \in \mathfrak{g} : [X, A] = 0 \ \forall A \in \mathfrak{g}\}. \tag{A.1}$$

We say that the center is *trivial* if it contains only the zero element. If $\mathfrak{g}$ has a trivial center, then the representation $X \mapsto \mathrm{ad}_X$ of $\mathfrak{g}$ on itself is faithful. Indeed, $\mathrm{ad}_X = \mathrm{ad}_Y$ implies $[X, A] = [Y, A]$ for all $A \in \mathfrak{g}$ and hence $X - Y$ is an element of the center of $\mathfrak{g}$. If the center is trivial, this implies $X = Y$.

A.3 Relation between Lie groups and Lie algebras

Let V be a linear space and let $G \subset \mathrm{GL}(V)$ be a matrix Lie group. By definition, the *Lie algebra* $\mathfrak{g}$ of G is the space of all matrices A such that there exists a smooth curve γ in G with

$$\gamma(0) = I \quad \text{and} \quad \tfrac{\partial}{\partial t}\gamma(t)\big|_{t=0} = A.$$

In manifold terminology, this says that $\mathfrak{g}$ is the *tangent space* to G at I. For any matrix A, we define

$$e^A := \sum_{k=0}^{\infty} \frac{1}{n!} A^n. \tag{A.2}$$

The following lemma follows from [Hal03, Cor. 3.46]. The main idea behind this lemma is that the elements of the Lie algebra act as "infinitesimal generators" of the Lie group.

Lemma 16 (Exponential formula) *Let $\mathfrak{g}$ be the Lie algebra of a Lie group $G \subset \mathrm{GL}(V)$. Then the following conditions are equivalent.*

(i) $A \in \mathfrak{g}$

(ii) $e^{tA} \in G$ *for all* $t \in \mathbb{R}$.

The following lemma (a precise proof of which can be found in [Hal03, Thm 3.20]) says that our terminology is justified.

Lemma 17 (Lie algebra property) *The Lie algebra of any matrix Lie group is a real Lie algebra.*

Proof (sketch) Let $\lambda \in \mathbb{R}$ and $A \in \mathfrak{g}$. By assumption, there exists a smooth curve γ such that $\gamma(0) = I$ and $\frac{\partial}{\partial t}\gamma(t)\big|_{t=0} = A$. But now $t \mapsto \gamma(\lambda t)$ is also smooth and $\frac{\partial}{\partial t}\gamma(\lambda t)\big|_{t=0} = \lambda A$, showing that $\mathfrak{g}$ is closed under multiplication with real scalars.

Also, if $A, B \in \mathfrak{g}$, then in the limit as $t \to 0$,

$$e^{tA}e^{tB} = \big((I + tA + O(t^2)\big)\big((I + tB + O(t^2)\big) = I + (A+B)t + O(t^2),$$

which suggests that $A + B$ lies in the tangent space to G at I; making this idea precise proves that indeed $A + B \in \mathfrak{g}$, so $\mathfrak{g}$ is a real linear space.

To complete the proof, we must show that $[A, B] \in \mathfrak{g}$ for all $A, B \in \mathfrak{g}$. It is easy to see that for any $A, B \in \mathfrak{g}$, as $t \to 0$

$$[e^{tA}, e^{tB}] = t^2[A, B] + O(t^3),$$

and hence

$$e^{tA}e^{tB}e^{-tA}e^{-tB} = e^{tA}\{e^{-tA}e^{tB} + [e^{tB}, e^{-tA}]\}e^{-tB} = I + t^2[A, B] + O(t^3).$$

Since $e^{tA}e^{tB}e^{-tA}e^{-tB} \in G$, this suggests that $[A, B]$ lies in the tangent space to G at I. ∎

By [Hal03, Cor. 3.47], if $\mathfrak{g}$ is the Lie algebra of a Lie group G, then there exist open neighbourhoods $0 \in O \subset \mathfrak{g}$ and $I \in U \subset G$ such that the map

$$O \ni A \mapsto e^{A} \in U$$

is a homeomorphism (a continuous bijection whose inverse is also continuous). The *identity component* G_0 of a Lie group G is the connected component that contains the identity. By [Hal03, Prop. 1.10], G_0 is a subgroup[11] of G. If U is an open neighbourhood of I, then each element of G_0 can be written as the product of finitely many elements of U. In particular, if G is connected, then U generates G. Therefore (see [Hal03, Cor. 3.47]), if G is a connected Lie group, then each element $X \in G$ can be written as

$$X = e^{A_1} \cdots e^{A_n} \tag{A.3}$$

for some $A_1, \ldots, A_n \in \mathfrak{g}$. As [Hal03, Example 3.41] shows, even if G is connected, it is in general not true that for each $A, B \in \mathfrak{g}$ there exists a $C \in \mathfrak{g}$ such that $e^A e^B = e^C$ and hence in general $\{e^A : A \in \mathfrak{g}\}$ need not be a group; in particular, this is not always G.

Anyway, the Lie algebra uniquely characterizes the local structure of a Lie group, so it should be true that if two Lie groups G and H are isomorphic, then their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are also isomorphic. Indeed, by [Hal03, Thm. 3.28], each Lie group homomorphism $\Phi : G \to H$ gives rise to a unique homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras such that

$$\Phi(e^A) = e^{\phi(A)} \qquad (A \in \mathfrak{g}). \tag{A.4}$$

In general, the converse conclusion cannot be drawn, i.e., two different Lie groups may have the same Lie algebra. By definition, a Lie group G is *simply connected* if it is connected and "has no holes", i.e., every continuous loop can be continuously shrunk to a point. (E.g., the surface of a ball is simply connected but a torus is not.) We cite the following theorem from [Hal03, Thm. 5.6].

[11] In fact, G_0 is a normal subgroup -see formula (A.9) below for the definition of a normal subgroup.

Theorem 18 (Simply connected Lie groups) *Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of Lie algebras. If G is simply connected, then there exists a unique Lie group homomorphism $\Phi : G \to H$ such that (A.4) holds.*

In particular ([Hal03, Cor. 5.7]), this implies that two simply connected Lie groups are isomorphic if and only if their Lie algebras are isomorphic. Every connected Lie group G has a *universal cover* (H, Φ) (this is stated without proof in [Hal03, Sect. 5.8]), which is a simply connected Lie group H together with a Lie group homomorphism $\Phi : H \to G$ such that the associated Lie algebra homomorphism as in (A.4) is a Lie algebra isomorphism. The following lemma says that such a universal cover is unique up to natural isomorphisms.

Lemma 19 (Uniqueness of the universal cover) *Let G be a connected Lie group and let (H_i, Φ_i) $(i = 1, 2)$ be universal covers of G. Then there exists a unique Lie group isomorphism $\Psi : H_1 \to H_2$ such that $\Psi(\Phi_1(A)) = \Phi_2(A)$ $(A \in G)$.*

Proof Let $\phi_i : \mathfrak{g} \to \mathfrak{h}_i$ denote the Lie algebra homomorphism associated with Φ_i as in (A.4). If a Lie group isomorphism Ψ as in the lemma exists, then the associated Lie algebra isomorphism ψ must satisfy $\psi \circ \phi_1 = \phi_2$. By assumption, ϕ_i $(i = 1, 2)$ are isomorphisms, so setting $\psi := \phi_2 \circ \phi_1^{-1}$ defines a Lie algebra isomorphism from $\mathfrak{h}_1$ to $\mathfrak{h}_2$. By assumption, H_1 is simply connected, so by Theorem 18, there exists a unique Lie group homomorphism $\Psi : H_1 \to H_2$ such that $\Psi(e^A) = e^{\psi(A)}$ $(A \in \mathfrak{h}_1)$. Similarly, there exists a unique Lie group homomorphism $\tilde{\Psi} : H_2 \to H_1$ such that $\tilde{\Psi}(e^A) = e^{\psi^{-1}(A)}$ $(A \in \mathfrak{h}_2)$. Now

$$\tilde{\Psi}(\Psi(e^A)) = \tilde{\Psi}(e^{\psi(A)}) = e^{\psi^{-1} \circ \psi(A)} = e^A \qquad (A \in \mathfrak{h}_1)$$

and similarly $\Psi(\tilde{\Psi}(e^A))$ $(A \in \mathfrak{h}_2)$, which (using the fact that elements of the form e^A with $A \in \mathfrak{h}_i$ generate H_i) proves that Ψ is invertible and $\tilde{\Psi} = \Psi^{-1}$. ∎

Informally, the universal cover H of G is the unique simply connected Lie group that has the same Lie algebra as G. The universal cover of a matrix Lie group need in general not be a matrix Lie group. Lie's third theorem [Hal03, Thm 5.25] says:

Theorem 20 (Lie's third theorem) *Every real Lie algebra $\mathfrak{g}$ is the Lie algebra of some connected Lie group G.*

By [Hal03, Conclusion 5.26], we can even take G to be a matrix Lie group, and by restricting to the identity component we can take G to be connected. By going to the universal cover, we can also take G to be simply connected, but in this case we may loose the property that G is a *matrix* Lie group. Anyway, we can conclude:

> There is a one-to-one correspondence between Lie algebras and simply connected Lie groups. Every Lie group has a unique universal cover, which is a simply connected Lie group with the same Lie algebra.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let (V, Π) be a representation of G. Then, by (A.4), there exists a unique Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathcal{L}(V)$ such that

$$\Pi(e^A) = e^{\pi(A)} \qquad (A \in \mathfrak{g}). \tag{A.5}$$

More concretely, one has (see [Hal03, Prop. 4.4])

$$\pi(A)v = \frac{\partial}{\partial t}\Pi(e^{tA})v\Big|_{t=0} \qquad (A \in \mathfrak{g},\ v \in V). \tag{A.6}$$

We say that (V, π) is the representation of $\mathfrak{g}$ *associated* with the representation (V, Π) of G. Conversely, if G is simply connected, then by grace of Theorem 18, through (A.5), each representation (V, π) of $\mathfrak{g}$ gives rise to a unique associated representation (V, Π) of G.

A.4 Relation between algebras and Lie algebras

If $\mathfrak{a}$ is an algebra and $\mathfrak{c} \subset \mathfrak{a}$ is any subset of $\mathfrak{a}$, then there exists a smallest subalgebra $\mathfrak{b} \subset \mathfrak{a}$ such that $\mathfrak{b}$ contains $\mathfrak{c}$. This algebra consists of the linear span of the unit element I and all finite products of elements of $\mathfrak{c}$. We call $\mathfrak{b}$ the algebra *generated* by $\mathfrak{c}$. If $\mathfrak{b} = \mathfrak{a}$, then we say that $\mathfrak{c}$ *generates* $\mathfrak{a}$.

Let $\mathfrak{g}$ be a Lie algebra. By definition, an *enveloping algebra* for $\mathfrak{g}$ is a pair $(\mathfrak{a}, \mathfrak{i})$ such that

(i) $\mathfrak{a}$ is an algebra and $\mathfrak{i} : \mathfrak{g} \to \mathfrak{a}$ is a Lie algebra homomorphism.

(ii) The image $\mathfrak{i}(\mathfrak{g})$ of $\mathfrak{g}$ under $\mathfrak{i}$ generates $\mathfrak{a}$.

We cite the following theorem from [Hal03, Thms 9.7 and 9.9].

Theorem 21 (Universal enveloping algebra) *For every Lie algebra $\mathfrak{g}$, there exists an enveloping algebra $(\mathfrak{a}, \mathfrak{i})$ with the following properties.*

(i) *If* $(\mathfrak{b}, \mathfrak{j})$ *is an enveloping algebra of* $\mathfrak{g}$*, then there exists a unique algebra homomorphism* $\phi : \mathfrak{a} \to \mathfrak{b}$ *such that* $\phi(\mathfrak{i}(A)) = \mathfrak{j}(A)$ *for all* $A \in \mathfrak{g}$*.*

(ii) *If* $\{X_1, \ldots, X_n\}$ *is a basis for* $\mathfrak{g}$*, then a basis for* $\mathfrak{a}$ *is formed by all elements of the form*

$$\mathfrak{i}(X_1)^{k_1} \cdots \mathfrak{i}(X_n)^{k_n},$$

where $k_1, \ldots, k_n \geq 0$ *are integers. In particular, these elements are linearly independent.*

An argument similar to the proof of Lemma 19 shows that the pair $(\mathfrak{a}, \mathfrak{i})$ from Theorem 21 is unique up to natural isomorphisms. We call $(\mathfrak{a}, \mathfrak{i})$ the *universal enveloping algebra* of $\mathfrak{g}$ and use the notation $U(\mathfrak{g}) := \mathfrak{a}$. By property (ii), the map $\mathfrak{i}$ is one-to-one, so we often identify $\mathfrak{g}$ with its image under $\mathfrak{i}$ and pretend $\mathfrak{g}$ is a sub-Lie-algebra of $U(\mathfrak{g})$.

As an immediate consequence of property (i) of Theorem 21, we see that if V is a representation of a Lie algebra $\mathfrak{g}$ and $\pi : \mathfrak{g} \to \mathcal{L}(V)$ is the associated Lie algebra homomorphism, then there exists a unique algebra homomorphism $\overline{\pi} : U(\mathfrak{g}) \to \mathcal{L}(V)$ such that $\overline{\pi}(A) = \pi(A)$ $(A \in \mathfrak{g})$. (Here we view $\mathfrak{g}$ as a sub-Lie-algebra of $U(\mathfrak{g})$.) Conversely, of course, every representation of $U(\mathfrak{g})$ is also a representation of $\mathfrak{g}$.

If (V, π) is a representation of a Lie algebra $\mathfrak{g}$, then we usually denote the associated representation of $U(\mathfrak{g})$ also by (V, π), i.e., we identify the map π with its extension $\overline{\pi}$. Note, however, that a representation (V, π) of a Lie algebra $\mathfrak{g}$ can be faithful even when the associated representation (V, π) of $U(\mathfrak{g})$ is not. Indeed, by property (ii) of Theorem 21, $U(\mathfrak{g})$ is always infinite dimensional, even though $\mathfrak{g}$ is finite dimensional, so finite-dimensional faithful representations of $\mathfrak{g}$ are not faithful when viewed as a representation of $U(\mathfrak{g})$.

A.5 Adjoints and unitary representations

Let V be a finite dimensional linear space equipped with an inner product $\langle \cdot | \cdot \rangle$, which for linear spaces over $\mathbb{C}$ is conjugate linear in its first argument and linear in its second argument. Each $A \in \mathcal{L}(V)$ has a unique *adjoint* $A^* \in \mathcal{L}(V)$ such that

$$\langle A^* v | w \rangle = \langle v | A w \rangle \qquad (v, w \in V). \tag{A.7}$$

An operator A is *self-adjoint* (also called *hermitian*) if $A^* = A$ and *skew symmetric* if $A^* = -A$. A *positive operator* is an operator such that $\langle v | A v \rangle \geq 0$ for all v. If V, W are linear spaces equipped with inner products, then an operator $U \in \mathcal{L}(V, W)$ is called *unitary* if it preserves the inner product, i.e.,

$$\langle U v | U w \rangle = \langle v | w \rangle \qquad (v, w \in V). \tag{A.8}$$

In particular, an operator $U \in \mathcal{L}(V)$ is unitary if and only if it is invertible and $U^{-1} = U$. If V is a finite dimensional linear space over $\mathbb{C}$, then for $v \in V$ we define operators $\langle v| \in \mathcal{L}(V, \mathbb{C})$ and $|v\rangle \in \mathcal{L}(\mathbb{C}, V)$ by

$$\langle v|w := \langle v|w\rangle \quad \text{and} \quad |v\rangle c := cv.$$

Then $\langle v||w\rangle$ is an operator in $\mathcal{L}(\mathbb{C}, \mathbb{C})$ which we can identify with the complex number $\langle v|w\rangle$. Moreover, $|v\rangle\langle w|$ is an operator in $\mathcal{L}(V)$. An *orthonormal* basis $\{e(1), \ldots, e(n)\}$ of V is a basis such that $\langle e(i)|e(j)\rangle = \delta_{ij}$. Then

$$A = \sum_{ij} A_{ij} |e(i)\rangle\langle e(j)|,$$

where A_{ij} denotes the matrix of A with respect to the orthonormal basis $\{e(1), \ldots, e(n)\}$. An operator $A \in \mathcal{L}(V)$ is *normal* if $[A, A^*] = 0$. An operator is normal if and only if it is diagonal w.r.t. some orthonormal basis, i.e., if it can be written as

$$A = \sum_{i} \lambda_i |e(i)\rangle\langle e(i)|,$$

where the λ_i are the eigenvalues of A. For operators, the following properties are equivalent.

$$\begin{aligned}
A \text{ is hermitian} \quad &\Leftrightarrow \quad A \text{ is normal with real eigenvalues,} \\
A \text{ is skew symmetric} \quad &\Leftrightarrow \quad A \text{ is normal with imaginary eigenvalues,} \\
A \text{ is positive} \quad &\Leftrightarrow \quad A \text{ is normal with nonnegative eigenvalues,} \\
A \text{ is unitary} \quad &\Leftrightarrow \quad A \text{ is normal with eigenvalues of norm 1.}
\end{aligned}$$

By definition, a *unitary* representation of a Lie group G is a complex representation (V, Π) where V is equipped with an inner product such that $\Pi(A)$ is a unitary operator for each $A \in G$. A *unitary* representation of a real Lie algebra $\mathfrak{g}$ is a complex representation V that is equipped with an inner product such that

$$\pi(A) \quad \text{is skew symmetric for all } A \in \mathfrak{g}.$$

Since $e^{\pi(A)}$ is unitary if and only if $\pi(A)$ is skew symmetric, our definitions imply that a representation (V, Π) of a Lie group G is unitary if and only if the associated representation (V, π) of the real Lie algebra $\mathfrak{g}$ of G is unitary.

Theorem 22 (Compact Lie groups) *Let K be a compact Lie group and let V be a representation of K. Then it is possible to equip V with an inner product so that V becomes a unitary representation of K.*

Proof (sketch) Choose an arbitrary inner product $\langle \cdot | \cdot \rangle$ on V and define

$$\langle v|w\rangle_K := \int \langle \Pi(A)v|\Pi(A)w\rangle \mathrm{d}A,$$

where $\mathrm{d}A$ denotes the *Haar measure* on K, which is finite by the assumption that K is compact. It is easy to check that $\langle \cdot | \cdot \rangle_K$ is an inner product. In particular, since $\Pi(A)$ is invertible for each $A \in K$, we have $\Pi(A)v \neq 0$ and hence $\langle \Pi(A)v|\Pi(A)v\rangle > 0$ for all $v \in V$ and $A \in K$. Now by the fact that the Haar measure is invariant under the action of the group

$$\begin{aligned}\langle \Pi(B)v|\Pi(B)w\rangle_K &= \int \langle \Pi(A)\Pi(B)v|\Pi(A)\Pi(B)w\rangle \mathrm{d}A \\ &= \int \langle \Pi(AB)v|\Pi(AB)w\rangle \mathrm{d}A = \int \langle \Pi(C)v|\Pi(C)w\rangle \mathrm{d}C = \langle v|w\rangle_K,\end{aligned}$$

which proves that V, equipped with the inner product $\langle \cdot | \cdot \rangle_K$, is a unitary representation of K. ∎

The following lemma is a sort of converse to Theorem 22 since it says that noncompact Lie groups do not have faithful unitary representations, at least when we restrict ourselves to finite-dimensional representations, as we do here.

Lemma 23 (Noncompact Lie groups) *Let K be a noncompact Lie group and let V be a faithful (finite dimensional) representation of K. Then it is not possible to equip V with an inner product so that V becomes a unitary representation of K.*

Proof Equip V with an inner product and let $\mathrm{U}(V)$ denote the group of all unitary maps $A : V \to V$. If (V, Π) is a faithful representation of K, then the image $\Pi(K)$ of K under Π is a closed subset of $\mathrm{GL}(V)$ and $\Pi : K \to \Pi(K)$ is a homeomorphism. If (V, Π) is a unitary representation, then $\Pi(K) \subset \mathrm{U}(V)$ and hence by the compactness of the latter, $\Pi(K)$ is compact. Since $\Pi : K \to \Pi(K)$ is a homeomorphism, it follows that K is compact. ∎

A $*$-*algebra* is a complex algebra on which there is defined an *adjoint* operation $A \mapsto A^*$ such that

(i) $A \mapsto A^*$ is conjugate linear,

(ii) $(A^*)^* = A$,

(iii) $(AB)^* = B^*A^*$.

If V is a complex finite dimensional linear space equipped with an inner product, then $\mathcal{L}(V)$, equipped with the adjoint operation (A.7), is a $*$-algebra.

A $*$-*algebra homomorphism* is an algebra homomorphism that satisfies

$$\phi(A^*) = \phi(A)^*.$$

A *sub-$*$-algebra* of a $*$-algebra is a subalgebra that is closed under the adjoint operation. By definition, a $*$-*representation* of a $*$-algebra $\mathfrak{a}$ is a representation (V, π) such that V is equipped with an inner product and π is a $*$-algebra homomorphism.

In general, a $*$-algebra may fail to have a faithful $*$-representation. For finite dimensional $*$-algebras, a necessary and sufficient condition for the existence of a faithful representation is that

$$A^*A = 0 \quad \text{implies} \quad A = 0,$$

but it is rather difficult to prove this; see [Swa17] and references therein. In infinite dimensions, one needs the theory of C$*$-algebras, which are $*$-algebras equipped with a norm that in faithful representations corresponds to the operator norm $\|A\| = \sup_{\|v\| \leq 1} \|Av\|$.

Recall the definition of an *adjoint* operation on a complex Lie algebra $\mathfrak{g}$ from Section 2.1. Recall also that we called a Lie algebra homomorphism *unitary* if $\phi(A^*) = \phi(A)^*$, and that a *unitary* representation is a representation (V, π) such that V is equipped with an inner product and π is a unitary Lie algebra homomorphism.

Lemma 24 (Universal enveloping $*$-algebra) *Let $\mathfrak{g}$ be a Lie-$*$-algebra. Then there exists a unique adjoint operation on its universal enveloping algebra $U(\mathfrak{g})$ that coincides with the adjoint operation on $\mathfrak{g}$.*

Proof Recall from Sections 2.2 that every complex linear space V has a *conjugate* space which is a linear space $\overline{V}$ together with a conjugate linear bijection $V \ni v \mapsto \overline{v} \in \overline{V}$. If $\mathfrak{a}$ is a complex algebra, then we can equip $\overline{\mathfrak{a}}$ with the structure of an algebra by putting

$$\overline{A}\,\overline{B} := \overline{BA}.$$

It is not hard to see that a map $A \mapsto A^*$ defined on some algebra $\mathfrak{a}$ is an adjoint operation if and only if the map $A \mapsto \overline{A^*}$ from $\mathfrak{a}$ into $\overline{\mathfrak{a}}$ is an algebra homomorphism. By the definition of an adjoint operation on a Lie algebra, $[A^*, B^*] = -[A, B]^*$ for all $A, B \in \mathfrak{g}$. It follows that the map

$$\mathfrak{g} \ni X \mapsto \overline{X^*} \in \overline{U(\mathfrak{g})}$$

is a Lie algebra homomorphism, which by the defining property of the universal enveloping algebra (Theorem 21 (i)) extends to a unique algebra homomorphism from $U(\mathfrak{g})$ to $\overline{U(\mathfrak{g})}$. ∎

A.6 Dual, quotient, sum, and product spaces

Dual spaces

The *dual* V' of a finite dimensional linear space V over $\mathbb{K} = \mathbb{R}$ or $= \mathbb{C}$ is the space of all linear forms $l : V \to \mathbb{K}$. Each element $v \in V$ naturally defines a linear form L_v on V' by $L_v(l) := l(v)$ and each linear form on V arises in this way, so we can identify $V'' \cong V$. If $\{e(1), \ldots, e(n)\}$ is a basis for V, then setting $f(i)(e(j)) := 1_{\{i=j\}}$ defines a basis $\{f(1), \ldots, f(n)\}$ for V' called the *dual basis*. If V is equipped with an inner product, then setting

$$\langle v|w := \langle v|w\rangle$$

defines a linear form on V and $V' := \{\langle v| : v \in V\}$. Through this identification, we also equip V' with an inner product. Then if $\{e(1), \ldots, e(n)\}$ is an orthonormal basis for V, the dual basis is an orthonormal basis for V'. Each linear map $A : V \to W$ gives naturally rise to a *dual map* $A' : W' \to V'$ defined by

$$A'(l) := l \circ A,$$

and indeed every linear map from W' to V' arises in this way, i.e., $\mathcal{L}(W', V') = \{A' : A \in \mathcal{L}(V, W)\}$. If V, W are equipped with inner products and $A \in \mathcal{L}(V, W)$, then

$$A'(\langle\phi|) = \langle A^*\phi|,$$

where A^* denotes the adjoint of A, i.e., this is the linear map $A^* \in \mathcal{L}(W, V)$ defined by

$$\langle\phi|A\psi\rangle = \langle A^*\phi|\psi\rangle \qquad (\phi \in W,\ \psi \in V).$$

If (V, Π) is a representation of a Lie group G, then we can define group homomorphism $\Pi' : G \to \mathcal{L}(V')$ by

$$\Pi'(A)l := \Pi(A^{-1})'l = l \circ \Pi(A^{-1}).$$

In this way, the dual space V' naturally obtains the structure of a representation of G. Note that

$$\Pi'(AB)l = l \circ \Pi((AB)^{-1}) = l \circ \Pi(A^{-1})\Pi(B^{-1}) = \Pi'(A)(\Pi'(B)l),$$

proving that Π' is indeed a group homomorphism. Similarly, if (V, π) is a representation of a Lie algebra $\mathfrak{g}$, then we can equip the dual space V' with the structure of a representation of $\mathfrak{g}$ by putting

$$\pi'(A)l := -\pi(A)'(l) = -l \circ \pi(A),$$

where in this case the minus sign guarantees that

$$\begin{aligned}\pi'([A,B])l &= -l\circ\pi([A,B]) = -l\circ\big(\pi(A)\pi(B)-\pi(B)\pi(A)\big)\\ &= -\big(\pi'(B)(\pi'(A)l)-\pi'(A)(\pi'(B)l\big) = \pi'(A)(\pi'(B)l)-\pi'(B)(\pi'(A)l).\end{aligned}$$

This is called the *dual representation* or *contragredient representation* of G or $\mathfrak{g}$, respectively, associated with V, see [Hal03, Def. 4.21]. If two representations of G and $\mathfrak{g}$ are associated as in (A.6), then their dual representations are also associated.

Quotient spaces

By definition, a *normal subgroup* of a group $\mathcal{G}$ is a subgroup $\mathcal{H}$ such that

$$A\mathcal{H} := \{AB : B\in\mathcal{H}\} = \{BA : B\in\mathcal{H}\} =: \mathcal{H}A \qquad \forall A\in\mathcal{G}, \tag{A.9}$$

or equivalently, if $B\in\mathcal{H}$ implies $ABA^{-1}\in\mathcal{H}$ for all $A\in\mathcal{G}$. Sets of the form $A\mathcal{H}$ and $\mathcal{H}A$ are called *left* and *right cosets*, respectively. If $\mathcal{H}$ is a normal subgroup, then left cosets are right cosets and vice versa, and we can equip the set

$$\mathcal{G}/\mathcal{H} := \big\{A\mathcal{H} : A\in\mathcal{G}\big\} = \big\{\mathcal{H}A : A\in\mathcal{G}\big\}$$

of all cosets with a group structure such that

$$(A\mathcal{H})(B\mathcal{H}) = (AB)\mathcal{H}.$$

We call $\mathcal{G}/\mathcal{H}$ the *quotient group* of $\mathcal{G}$ and $\mathcal{H}$. Note that as a set this is obtained from $\mathcal{G}$ by dividing out the equivalence relation

$$A\sim B \quad\Leftrightarrow\quad A = BC \quad\text{for some}\quad C\in\mathcal{H}.$$

If V is a linear space and $W\subset V$ is a linear subspace, then we can define an equivalence relation on V by setting

$$v_1\sim v_2 \quad\Leftrightarrow\quad v_1 = v_2 + w \quad\text{for some}\quad w\in W.$$

The equivalence classes with respect to this equivalence relation are the sets of the form

$$v+W := \{v+w : w\in W\}$$

and we can equip the space

$$V/W := \{v+W : v\in V\}$$

with the structure of a linear space by setting

$$a_1(v_1 + W) + a_2(v_2 + W) := \big(a_1v_1 + a_2v_2\big) + W.$$

An *invariant subspace* of a representation V of a Lie group G, Lie algebra $\mathfrak{g}$, or algebra $\mathfrak{a}$ is a linear space $W \subset V$ such that $Aw \in W$ for all $w \in W$ and A from G, $\mathfrak{g}$, or $\mathfrak{a}$, respectively. If W is an invariant subspace, then we can equip the quotient space V/W with the structure of a representation by setting

$$A(v + W) := (Av) + W.$$

Note that this is a good definition since $v_1 = v_2 + w$ for some $w \in W$ implies $Av_1 = Av_2 + Aw$ where $Aw \in W$ by the assumption that W is invariant.

A *left ideal* (resp. *right ideal*) of an algebra $\mathfrak{a}$ is a linear subspace $\mathfrak{i} \subset \mathfrak{a}$ such that $AB \in \mathfrak{i}$ (resp. $BA \in \mathfrak{i}$) for all $A \in \mathfrak{a}$ and $B \in \mathfrak{i}$. An *ideal* is a linear subspace that is both a left and right ideal. If $\mathfrak{i}$ is an ideal of $\mathfrak{a}$, then we can equip the quotient space $\mathfrak{a}/\mathfrak{i}$ with the structure of an algebra by putting

$$(A + \mathfrak{i})(B + \mathfrak{i}) := (AB) + \mathfrak{i}.$$

To see that this is a good definition, write $A_1 \sim A_2$ if $A_1 = A_2 + B$ for some $B \in \mathfrak{i}$. Then $A_1 \sim A_2$ and $B_1 \sim B_2$ imply that $A_1 = A_2 + C$ and $B_1 = B_2 + D$ for some $C, D \in \mathfrak{i}$ and hence

$$A_1B_1 = (A_2 + C)(B_2 + D) = A_2B_2 + \big(CB_2 + A_2D + CD\big)$$

with $CB_2 + A_2D + CD \in \mathfrak{i}$, so $A_1B_1 \sim A_2B_2$. If $\mathfrak{a}$ is a $*$-algebra, then a $*$-*ideal* of $\mathfrak{a}$ is an ideal $\mathfrak{i}$ such that $A \in \mathfrak{i}$ implies $A^* \in \mathfrak{i}$. If $\mathfrak{i}$ is a $*$-ideal, then we can equip the quotient algebra $\mathfrak{a}/\mathfrak{i}$ with an adjoint operation by putting

$$(A + \mathfrak{i})^* := A^* + \mathfrak{i}.$$

A linear subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is said to be an *ideal* if $[A, B] \in \mathfrak{h}$ for all $A \in \mathfrak{g}$ and $B \in \mathfrak{h}$. Note that this automatically implies that also $[B, A] = -[A, B] \in \mathfrak{h}$. If $\mathfrak{h}$ is an ideal of a Lie algebra, then we can equip the quotient space $\mathfrak{g}/\mathfrak{h}$ with the structure of a Lie algebra by putting

$$[A + \mathfrak{h}, B + \mathfrak{h}] := [A, B] + \mathfrak{h}.$$

The proof that this is a good definition is the same as for algebras.

The direct sum

The direct sum $V_1 \oplus \cdots \oplus V_n$ of linear spaces $V_1, \ldots, V_n$ has already been defined in Section 2.6. There is a natural isomorphism between $V_1 \oplus \cdots \oplus V_n$ and the Carthesian product

$$V_1 \times \cdots \times V_n = \big\{ \big(\phi(1), \ldots, \phi(n)\big) : \phi(i) \in V_i \ \forall i \big\},$$

which we equip with a linear structure by defining

$$a\big(\phi(1), \ldots, \phi(n)\big) + b\big(\psi(1), \ldots, \psi(n)\big) := \big(a\phi(1) + b\phi(1), \ldots, a\phi(n) + b\phi(n)\big).$$

If $V_1, \ldots, V_n$ are equipped with inner products, then we require that the inner product on $V_1 \oplus \cdots \oplus V_n$ is given by

$$\langle \phi(1) + \cdots + \phi(n) | \psi(1) + \cdots + \psi(n) \rangle := \sum_{k=1}^{n} \langle \phi(k) | \psi(k) \rangle, \tag{A.10}$$

which has the effect that $V_1, \ldots, V_n$ are (mutually) orthogonal. One has the natural isomorphism

$$(V_1 \oplus V_2)/V_2 \cong V_1.$$

In general, given a subspace V_1 of some larger linear space W, there are many possible ways to choose another subspace V_2 such that $W = V_1 \oplus V_2$ and hence $W \cong (W/V_1) \oplus V_1$.

If V is a linear subspace of some larger linear space W, and W is equipped with an inner product, then we define the *orthogonal complement* of V as

$$V^\perp := \{ w \in W : \langle v | w \rangle = 0 \ \forall v \in V \}.$$

Then one has the natural isomorphisms

$$W/V \cong V^\perp \quad \text{and} \quad W \cong V \oplus V^\perp,$$

where the inner product $V \oplus V^\perp$ is given in terms of the inner products on V and $V^\perp$ as in (A.10). Thus, given a linear subspace V_1 of a linear space W that is equipped with an inner product, there is a canonical way to choose another subspace V_2 such that $W = V_1 \oplus V_2$.

If $V_1, \ldots, V_n$ are representations of the same Lie group, Lie algebra, or algebra, then we equip $V_1 \oplus \cdots \oplus V_n$ with the structure of a representation by putting

$$A\big(\phi(1) + \cdots + \phi(n)\big) := A\phi(1) + \cdots + A\phi(n).$$

If V, W are representations, then W is an invariant subspace of $V \oplus W$ and one has the natural isomorphism of representations $(V \oplus W)/W \cong V$.

If $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are algebras, then we equip their direct sum $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$ with the structure of an algebra by putting

$$\big(A(1)+\cdots+A(n)\big)\big(B(1)+\cdots+B(n)\big) := A(1)B(1)+\cdots+A(n)B(n). \quad \text{(A.11)}$$

If $\mathfrak{a}, \mathfrak{b}$ are algebras, then $\mathfrak{b}$ is an ideal of $\mathfrak{a} \oplus \mathfrak{b}$ and one has the natural isomorphism $(\mathfrak{a} \oplus \mathfrak{b})/\mathfrak{b} \cong \mathfrak{a}$. Note that $\mathfrak{b}$ is not a subalgebra of $\mathfrak{a} \oplus \mathfrak{b}$ since $I \notin \mathfrak{b}$ (unless $\mathfrak{a} = \{0\}$). For $*$-algebras, we also put

$$\big(A(1) + \cdots + A(n)\big)^* := \big(A(1)^* + \cdots + A(n)^*\big).$$

The direct sum of Lie algebras has already been defined in Section 2.6. It is easy to see that this is consistent with the definition of the direct sum of algebras.

The tensor product

The *tensor product* of two (or more) linear spaces has already been defined in Section 2.6. A proof similar to the proof of Lemma 19 shows that the tensor product is unique up to natural isomorphisms, i.e., if $V \tilde{\otimes} W$ and $(\phi, \psi) \mapsto \phi \tilde{\otimes} \psi$ are another linear space and bilinear map which satisfy the defining property of the tensor product, then there exists a unique linear bijection $\Psi : V \otimes W \to V \tilde{\otimes} W$ such that $\Psi(V \otimes W) = V \tilde{\otimes} W$.

If V, W are representations of the same Lie group, then we equip $V \otimes W$ with the structure of a representation by putting

$$A(\phi \otimes \psi) := A\phi \otimes A\psi. \quad \text{(A.12)}$$

If V, W are representations of the same Lie algebra or algebra, then we equip $V \otimes W$ with the structure of a representation by putting

$$A(\phi \otimes \psi) := A\phi \otimes \psi + \phi \otimes A\psi. \quad \text{(A.13)}$$

The reason why we define things in this way is that in view of (A.6), if $\mathfrak{g}$ is the Lie algebra of G, then the representation of $\mathfrak{g}$ defined in (A.13) is the representation of $\mathfrak{g}$ that is associated with the representation of G defined in (A.12). Note that (A.13) is bilinear in ϕ and ψ and hence by the defining property of the tensor product uniquely defines a linear operator on $V \otimes W$.

If $\mathfrak{a}, \mathfrak{b}$ are algebras, then we equip their tensor product $\mathfrak{a} \otimes \mathfrak{b}$ with the structure of an algebra by putting

$$\big(A(1) \otimes B(1)\big)\big(A(2) \otimes B(2)\big) := \big(A(1)A(2) \otimes B(1)B(2)\big).$$

Using the defining property of the tensor product, one can show that this unambiguously defines a linear map

$$(\mathfrak{a} \otimes \mathfrak{b})^2 \ni (A, B) \mapsto AB \in \mathfrak{a} \otimes \mathfrak{b}.$$

We can identify $\mathfrak{a}$ and $\mathfrak{b}$ with the subalgebras of $\mathfrak{a} \otimes \mathfrak{b}$ given by

$$\mathfrak{a} \cong \{A \otimes I : A \in \mathfrak{a}\} \quad \text{and} \quad \mathfrak{b} \cong \{I \otimes B : B \in \mathfrak{b}\}.$$

Note that if we identify $\mathfrak{a}$ and $\mathfrak{b}$ with subalgebras of $\mathfrak{a} \otimes \mathfrak{b}$, then every element of $\mathfrak{a}$ commutes with every element of $\mathfrak{b}$. If $\mathfrak{a}, \mathfrak{b}$ are $*$-algebras, then we equip the algebra $\mathfrak{a} \otimes \mathfrak{b}$ with an adjoint operation by setting

$$(A \otimes B)^* := (A^* \otimes B^*).$$

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then the universal enveloping algebra of their direct sum is naturally isomorphic to the tensor product of their universal enveloping algebras:

$$U(\mathfrak{g} \oplus \mathfrak{h}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{h}). \tag{A.14}$$

Indeed, if $\{X_1, \ldots, X_n\}$ is a basis for $\mathfrak{g}$ and $\{Y_1, \ldots, Y_m\}$ is a basis for $\mathfrak{h}$, then we can define a bilinear map $(A, B) \mapsto A \otimes B$ from $U(\mathfrak{g}) \times U(\mathfrak{h})$ into $U(\mathfrak{g} \oplus \mathfrak{h})$ by

$$\begin{aligned} &\big(X_1^{k_1} \cdots X_n^{k_n}, Y_1^{l_1} \cdots Y_m^{l_m}\big) \\ &\quad \mapsto X_1^{k_1} \cdots X_n^{k_n} \otimes Y_1^{l_1} \cdots Y_m^{l_m} := X_1^{k_1} \cdots X_n^{k_n} Y_1^{l_1} \cdots Y_m^{l_m}. \end{aligned}$$

where we view $\mathfrak{g}$ and $\mathfrak{h}$ as sub-Lie-algebras of $\mathfrak{g} \oplus \mathfrak{h}$ such that $[X, Y] = 0$ for each $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. In view of Theorem 21, the space $U(\mathfrak{g} \oplus \mathfrak{h})$ together with this bilinear map is a realization of the tensor product $U(\mathfrak{g}) \otimes U(\mathfrak{h})$.

On a philosophical note, recall that elements of a Lie algebra are related to elements of a matrix Lie group via an exponential map. We can view (A.14) as a reflection of the property of the exponential map that converts sums into products.

If V and W are representations of algebras $\mathfrak{a}$ and $\mathfrak{b}$, respectively, then we can make $V \otimes W$ into a representation of $\mathfrak{a} \otimes \mathfrak{b}$ by setting

$$(A \otimes B)(\phi \otimes \psi) := (A\phi) \otimes (B\psi). \tag{A.15}$$

Again, by bilinearity and the defining property of the tensor product, this is a good definition. Note that this is consistent with (A.14) and our definition in (2.29) where we showed that if V and W are representations of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, then $V \otimes W$ is naturally a representation of $\mathfrak{g} \oplus \mathfrak{h}$. On the other hand, one should observe that in the special case that $\mathfrak{a} = \mathfrak{b}$, our present construction differs from our earlier construction in (A.13).

A.7 Irreducible representations

Let $\mathfrak{g}$ be a Lie algebra on which an adjoint operation is defined, and let $\mathfrak{h} := \{\mathbf{a} \in \mathfrak{g} : \mathbf{a}^* = -\mathbf{a}\}$ denote the real sub-Lie-algebra[12] consisting of all skew-symmetric elements of $\mathfrak{g}$. It is not hard to see that $\mathfrak{g}$ is the complexification of $\mathfrak{h}$, i.e., each $\mathbf{a} \in \mathfrak{g}$ can uniquely be written as $\mathbf{a} = \mathbf{a}_1 + i\mathbf{a}_2$ with $\mathbf{a}_1, \mathbf{a}_2 \in \mathfrak{h}$.[13] Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis for $\mathfrak{g}$. The Lie bracket on $\mathfrak{g}$ is uniquely characterized by the commutation relations

$$[\mathbf{x}_i, \mathbf{x}_j] = \sum_{j=1}^{n} c_{ijk}\mathbf{x}_k, \tag{A.16}$$

where c_{ijk} are the structure constants (see (A.16)). Likewise, the adjoint operation on $\mathfrak{g}$ is uniquely characterized by its action on basis elements

$$\mathbf{x}_i^* = \sum_j d_{ij}\mathbf{x}_j, \tag{A.17}$$

where d_{ij} is another set of constants.

By Theorem 20, the real Lie algebra $\mathfrak{h}$ is the Lie algebra of some Lie group G. By going to the universal cover, we can take G to be simply connected, in which case it is uniquely determined by $\mathfrak{h}$. Conversely, if G is a simply connected Lie group, $\mathfrak{h}$ is its real Lie algebra, and $\mathfrak{g} := \mathfrak{h}_\mathbb{C}$ is the complexification of $\mathfrak{h}$, then we can equip $\mathfrak{g}$ with an adjoint operation such that the set of skew symmetric elements is exactly $\mathfrak{h}$, by putting $(\mathbf{a}_1 + i\mathbf{a}_2)^* := -\mathbf{a}_1 + i\mathbf{a}_2$ for each $\mathbf{a}_1, \mathbf{a}_2 \in \mathfrak{h}$.

If V is a linear space and $X_1, \ldots, X_n \in \mathcal{L}(V)$ satisfy (A.16), then there exists a unique Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathcal{L}(V)$ such that $\pi(\mathbf{x}_i) = X_i$ ($i = 1, \ldots, n$). If V is equipped with an inner product and the operators $X_1, \ldots, X_n$ moreover satisfy (A.17), then π is a unitary representation. By Theorem 21 (i) and Lemma 24, π can in a unique way be extended to a $*$-algebra homomorphism $\overline{\pi} : U(\mathfrak{g}) \to \mathcal{L}(V)$. Moreover, if G is the simply connected Lie group associated with $\mathfrak{h}$, then by Theorem 18, there exists a unique Lie group homomorphism $\Pi : G \to \mathcal{L}(V)$ such that (A.5) holds, so (V, Π) is a representation of G. Since every element of $\mathfrak{h}$ is skew symmetric, (V, π) and hence also (V, Π) are unitary representations of $\mathfrak{h}$ and G, respectively.

[12] To see that this is a sub-Lie-algebra, note that $\mathbf{a}, \mathbf{b} \in \mathfrak{h}$ imply $[\mathbf{a}, \mathbf{b}]^* = -[\mathbf{a}^*, \mathbf{b}^*]$ and hence $[\mathbf{a}, \mathbf{b}] \in \mathfrak{h}$.

[13] Equivalently, we may show that each $\mathbf{a} \in \mathfrak{g}$ can uniquely be written as $\mathbf{a} = \mathrm{Re}(\mathbf{a}) + i\mathrm{Im}(\mathbf{a})$ with $\mathrm{Re}(\mathbf{a}), \mathrm{Im}(\mathbf{a})$ self-adjoint. This follows easily by putting $\mathrm{Re}(\mathbf{a}) := \frac{1}{2}(\mathbf{a} + \mathbf{a}^*)$ and $\mathrm{Im}(\mathbf{a}) := \frac{1}{2}i(\mathbf{a}^* - \mathbf{a})$.

Let $W \subset V$ be a linear subspace. It is not hard to see that

$$\begin{aligned} & W \text{ is an invariant subspace of } (V, \Pi) \\ \Leftrightarrow \quad & W \text{ is an invariant subspace of } (V, \pi) \\ \Leftrightarrow \quad & W \text{ is an invariant subspace of } (V, \overline{\pi}). \end{aligned}$$

We say that V is *irreducible* if its only invariant subspaces are $\{0\}$ and V.

Let V, W be two representations of the same Lie group G, Lie algebra $\mathfrak{g}$, or algebra $\mathfrak{a}$. Generalizing our earlier definition for ie algebras, a *homomorphism* of representations (of any kind) is a linear map $\phi : V \to W$ such that

$$\phi(\mathbf{a}v) = \mathbf{a}\phi(v) \tag{A.18}$$

for all $\mathbf{a} \in G$, $\mathbf{a} \in \mathfrak{g}$, or $\mathbf{a} \in \mathfrak{a}$, respectively. Homomorphisms of representations are called *intertwiners* of representations. If ϕ is a bijection, then its inverse is also an intertwining map. In this case we call ϕ an *isomorphism* and say that the representations are *equivalent* (or *isomorphic*). If G is a simply connected Lie group, $\mathfrak{g}$ its associated complexified Lie algebra, and $U(\mathfrak{g})$ its universal enveloping algebra, then it is not hard to see that

$$\text{(A.18) holds } \forall \mathbf{a} \in G \;\Leftrightarrow\; \text{(A.18) holds } \forall \mathbf{a} \in \mathfrak{g} \;\Leftrightarrow\; \text{(A.18) holds } \forall \mathbf{a} \in U(\mathfrak{g}).$$

The following result can be found in, e.g., [Hal03, Thm 4.29]. In the special case of complex Lie algebras, we have already stated this in Proposition 4.

Proposition 25 (Schur's lemma)

- **(a)** *Let V and W be irreducible representations of a Lie group, Lie algebra, or algebra, and let $\phi : V \to W$ be an intertwiner. Then either $\phi = 0$ or ϕ is an isomorphism.*

- **(b)** *Let V be an irreducible complex representation of a Lie group, Lie algebra, or algebra, and let $\phi : V \to V$ be an intertwiner. Then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$.*

By definition, the *center* of an algebra is the subalgebra $\mathcal{C}(\mathfrak{a}) := \{C \in \mathfrak{a} : [A, C] = 0 \ \forall A \in \mathfrak{a}\}$. The center is *trivial* if $\mathcal{C}(\mathfrak{a}) = \{\lambda I : \lambda \in \mathbb{K}\}$. The following is adapted from [Hal03, Cor. 4.30].

Corollary 26 (Center) *Let (V, π) be an irreducible complex representation of an algebra $\mathfrak{a}$ and let $C \in \mathcal{C}(\mathfrak{a})$. Then $\pi(C) = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Proof Define $\phi : V \to V$ by $\phi v := \pi(C)v$. Then $\phi(Av) = \pi(C)\pi(A)v = \pi(CA)v = \pi(AC)v = \pi(A)\pi(C)v = A(\phi v)$ for all $A \in \mathfrak{a}$, so $\phi : V \to V$ is an intertwiner. By part (b) of Schur's lemma, $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$. ∎

A.8 Semisimple Lie algebras

A Lie algebra $\mathfrak{g}$ is called *irreducible* (see [Hal03, Def. 3.11]) if its only ideals are $\{0\}$ and $\mathfrak{g}$, and *simple* if it is irreducible and has dimension $\dim(\mathfrak{g}) \geq 2$. A Lie algebra is called *semisimple* if it can be written as the direct sum of simple Lie algebras. Recall the definition of the center of a Lie algebra in (A.1).

Lemma 27 (Trivial center) *The center of a semisimple Lie algebra is trivial.*

Proof If $\mathfrak{g}$ is simple and A is an element of its center, then the linear space spanned by A is an ideal. Since $\dim(\mathfrak{g}) \geq 2$ and its only ideals are $\{0\}$ and $\mathfrak{g}$, this implies that $A = 0$. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ is the direct sum of simple Lie algebras, then we can write any element A of the center of $\mathfrak{g}$ as $A = A(1) + \cdots + A(n)$ with $A(k) \in \mathfrak{g}$. By the definition of the Lie bracket on $\mathfrak{g}$ (see (2.24)), $A(k)$ lies in the center of $\mathfrak{g}$ for each k, and hence $A = 0$ by what we have already proved. ∎

The following proposition is similar to [Hal03, Prop. 7.4].

Proposition 28 (Inner product on Lie algebra) *Let $\mathfrak{g}$ be a Lie algebra on which an adjoint operation is defined, let $\mathfrak{h} := \{\mathbf{a} \in \mathfrak{g} : \mathbf{a}^* = -\mathbf{a}\}$, and let G be the simply connected Lie group with Lie algebra $\mathfrak{h}$. Assume that G is compact. Then the Lie algebra $\mathfrak{g}$, equipped with the map*

$$\mathfrak{g} \ni \mathbf{x} \mapsto \mathrm{ad}_{\mathbf{x}} \in \mathcal{L}(\mathfrak{g}),$$

is a faithful representation of itself. It is possible to equip $\mathfrak{g}$ with an inner product such that this is a unitary representation, i.e., $\mathrm{ad}_{\mathbf{x}^*} = (\mathrm{ad}_{\mathbf{x}})^*$ $(\mathbf{x} \in \mathfrak{g})$.

Proof By [Hal03, Prop. 7.7], the center of $\mathfrak{g}$ is trivial. By Lemma 15 and the remarks below it, it follows that $\mathfrak{g}$, equipped with the map $\mathfrak{g} \ni \mathrm{ad}_X \in \mathcal{L}(\mathfrak{g})$, is a faithful representation of itself. This representation naturally gives rise to a representation of G. By assumption, G is compact, so by Theorem 22, we can equip $\mathfrak{g}$ with an inner product so that this representation is unitary. It follows that the representation of $\mathfrak{h}$ on $\mathfrak{g}$ is also unitary and hence the representation of $\mathfrak{g}$ on itself is a unitary representation. ∎

The following theorem follows from [Hal03, Thm 7.8].

Theorem 29 (Semisimple algebras) *Let G be a compact simply connected Lie group and let $\mathfrak{g}$ be the complexification of its Lie algebra. Then $\mathfrak{g}$ is semisimple.*

Proof (main idea) If $\mathfrak{g}$ is not simple, then it has some ideal $\mathfrak{i}$ that is neither $\{0\}$ nor $\mathfrak{g}$. Let $\mathfrak{i}^\perp$ denote the orthogonal complement of $\mathfrak{i}$ with respect to the inner product on $\mathfrak{g}$ defined in Proposition 28. It is shown in [Hal03, Prop. 7.5] that $\mathfrak{i}^\perp$ is an ideal of $\mathfrak{g}$ and one has $\mathfrak{g} \cong \mathfrak{i} \oplus \mathfrak{i}^\perp$, where $\oplus$ denotes the direct sum of Lie algebras. Continuing this process, one arrives at a decomposition of $\mathfrak{g}$ as a direct sum of simple Lie algebras. ∎

In fact, the converse statement to Theorem 29 also holds: if $\mathfrak{g}$ is a semisimple complex Lie algebra, then it is the complexification of the Lie algebra of a compact simply connected Lie group. This is stated (with references for a proof) in [Hal03, Sect. 10.3].

Let G be a compact simply connected Lie group, let $\mathfrak{h}$ be its real Lie algebra, let $\mathfrak{g} := \mathfrak{h}_{\mathbb{C}}$ be the complexification of $\mathfrak{h}$, and let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. The *Casimir element* is the element $C \in U(\mathfrak{g})$ defined as

$$\mathbf{c} := -\sum_j \mathbf{x}_j^2,$$

where $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis for $\mathfrak{h}$ that is orthonormal with respect to the inner product from Proposition 28.[14] We cite the following result from [Hal03, Prop. 10.5].

Proposition 30 (Casimir element) *The definition of the Casimir element does not depend on the choice of the orthonormal basis* $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ *of* $\mathfrak{h}$. *Moreover* $\mathbf{c}$ *lies in the center of* $U(\mathfrak{g})$.

In irreducible representations, the Casimir element has a simple form.

Lemma 31 (Representations of Casimir element) *For each irreducible representation* (V, π) *of* $\mathfrak{g}$, *there exists a constant* $\lambda_V \geq 0$ *such that* $\pi(\mathbf{c}) = \lambda_V I$.

Proof Proposition 30 and Corollary 26 imply that for each irreducible representation (V, π) of $U(\mathfrak{g})$, there exists a constant $\lambda \in \mathbb{C}$ such that $\pi(\mathbf{c}) = \lambda I$. By Theorem 22, we can equip V with an inner product such that it is a unitary representation of $\mathfrak{h}$. This means that $\mathbf{x}_j$ is skew symmetric and hence $i\mathbf{x}_j$ is hermitian, so $\mathbf{c} = \sum_i (i\mathbf{x}_j)^2$ is a positive operator. In particular, its eigenvalues are ≥ 0. ∎

[14] The inner product from Proposition 28 is not completely unique; at best it is only determined up to a multiplicative constant. So the Casimir operator depends on the choice of the inner product, but once this is fixed, it does not depend on the choice of the orthonormal basis.

A.9 Some basic matrix Lie groups

For any finite-dimensional linear space V over $V = \mathbb{R}$ or $= \mathbb{C}$, we let $\mathrm{GL}(V)$ denote the *general linear group* of all invertible linear maps $A : V \to V$. In particular, we write $\mathrm{GL}(n;\mathbb{R}) = \mathrm{GL}(\mathbb{R}^n)$ and $\mathrm{GL}(n;\mathbb{C}) = \mathrm{GL}(\mathbb{C}^n)$.

The *special linear group* $\mathrm{SL}(V)$ is defined as

$$\mathrm{SL}(V) := \big\{A \in \mathrm{GL}(V) : \det(A) = 1\big\}.$$

Again, we write $\mathrm{SL}(n;\mathbb{R}) = \mathrm{SL}(\mathbb{R}^n)$ and $\mathrm{SL}(n;\mathbb{C}) = \mathrm{SL}(\mathbb{C}^n)$. If V is a finite-dimensional linear space over $\mathbb{C}$ and V is equipped with an inner product $\langle\,\cdot|\cdot\,\rangle$, then we call

$$\mathrm{U}(V) := \{A \in \mathcal{L}(V) : A \text{ is unitary}\}$$

the *unitary group* and

$$\mathrm{SU}(V) := \{A \in \mathrm{U}(V) : \det(A) = 1\}$$

the *special unitary group*, and write $\mathrm{U}(n) := \mathrm{U}(\mathbb{C}^n)$ and $\mathrm{SU}(n) := \mathrm{SU}(\mathbb{C}^n)$.

If V is a finite-dimensional linear space over $\mathbb{R}$ and V is equipped with an inner product $\langle\,\cdot|\cdot\,\rangle$, then an operator $O \in \mathcal{L}(V)$ that preserves the inner product as in (A.8) is called *orthogonal*. (This is the equivalent of unitarity in the real setting.) We call

$$\mathrm{O}(V) := \{A \in \mathcal{L}(V) : A \text{ is orthogonal}\}$$

denote the *orthogonal group* and

$$\mathrm{SO}(V) := \{A \in \mathrm{O}(V) : \det(A) = 1\}$$

the *special orthogonal group*, and write $\mathrm{O}(n) := \mathrm{O}(\mathbb{R}^n)$ and $\mathrm{SO}(n) := \mathrm{SO}(\mathbb{R}^n)$. There also exists a group $\mathrm{O}(n;\mathbb{C})$, which consists of all complex matrices that preserve the bilinear form $(v, w) := \sum_i v_i w_i$. Not that this is *not* the inner product on $\mathbb{C}^n$; as a result $\mathrm{O}(n;\mathbb{C})$ is not the same as $\mathrm{U}(n)$.

Unitary operators satisfy $|\det(A)| = 1$ and orthogonal operators satisfy $\det(A) = \pm 1$. The group $\mathrm{O}(3)$ consists of rotations and reflections (and combinations thereof) while $\mathrm{SO}(3)$ consists only of rotations.

By [Hal03, Prop. 3.23], for $\mathbb{K} = \mathbb{R}$ or $= \mathbb{C}$, the Lie algebra of $\mathrm{GL}(n, \mathbb{K})$ is the space $M_n(\mathbb{K})$ of all $\mathbb{K}$-valued $n \times n$ matrices, and the Lie algebra of $\mathrm{SL}(n, \mathbb{K})$ is given by

$$\mathfrak{sl}(n, \mathbb{K}) = \{A \in M_n(\mathbb{K}) : \mathrm{tr}(A) = 0\}.$$

By [Hal03, Prop. 3.24], the Lie algebras of $\mathrm{U}(n)$ and $\mathrm{O}(n)$ are given by

$$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) : A^* = -A\} \quad \text{and} \quad \mathfrak{o}(n) = \{A \in M_n(\mathbb{R}) : A^* = -A\}.$$

Moreover, again by [Hal03, Prop. 3.24], the Lie algebras of $\mathrm{SU}(n)$ and $\mathrm{SO}(n)$ are given by

$$\mathfrak{su}(n) = \{A \in M_n(\mathbb{C}) : A^* = -A,\ \mathrm{tr}(A) = 0\} \quad \text{and} \quad \mathfrak{so}(n) = \mathfrak{o}(n).$$

By [Hal03, formula (3.17)], the complexifications of the real Lie algebras introduced above are given by

$$\begin{aligned}
\mathfrak{gl}(n,\mathbb{R})_{\mathbb{C}} &\cong \mathfrak{gl}(n,\mathbb{C}),\\
\mathfrak{u}(n)_{\mathbb{C}} &\cong \mathfrak{gl}(n,\mathbb{C}),\\
\mathfrak{su}(n)_{\mathbb{C}} &\cong \mathfrak{sl}(n,\mathbb{C}),\\
\mathfrak{sl}(n,\mathbb{R})_{\mathbb{C}} &\cong \mathfrak{sl}(n,\mathbb{C}),\\
\mathfrak{so}(n,\mathbb{R})_{\mathbb{C}} &\cong \mathfrak{so}(n,\mathbb{C}).
\end{aligned}$$

As mentioned in [Hal03, Sect. 1.3.1], the following Lie groups are compact:

$$\mathrm{O}(n), \quad \mathrm{SO}(n), \quad \mathrm{U}(n), \quad \text{and} \quad \mathrm{SU}(n).$$

By [Hal03, Prop 1.11, 1.12, and 1.13] and [Hal03, Exercise 1.13], the following Lie groups are connected:

$$\mathrm{GL}(n;\mathbb{C}) \quad \mathrm{SL}(n;\mathbb{C}) \quad \mathrm{U}(n) \quad \mathrm{SU}(n), \quad \text{and} \quad \mathrm{SO}(n).$$

By [Hal03, Prop. 13.11], the group $\mathrm{SU}(n)$ is simply connected. By [Hal03, Example 5.15], $\mathrm{SU}(2)$ is the universal cover of $\mathrm{SO}(3)$.

Of further interest are the real and complex *symplectic groups* $\mathrm{SP}(n,\mathbb{R})$ and $\mathrm{SP}(n,\mathbb{C})$, and the *compact symplectic group* $\mathrm{SP}(n)$; for their definitions we refer to [Hal03, Sect. 1.2.4].

A.10 The Lie group SU(1,1)

Let us define a Minkowski form $\{\,\cdot\,,\,\cdot\,\} : \mathbb{C}^2 \to \mathbb{C}$ by

$$\{v, w\} := v_1^* w_1 - v_2^* w_2.$$

Note that this is almost identical to the usual definition of the inner product on $\mathbb{C}^2$ (in particular, it is conjugate linear in its first argument and linear in its second argument), except for the minus sign in front of the second term. Letting

$$M := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we can write

$$\{v, w\} = \langle v|M|w\rangle,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the usual inner product. The Lie group $\mathrm{SU}(1,1)$ is the matrix Lie group consisting of all matrices $Y \in \mathcal{L}(\mathbb{C}^2)$ with determinant 1 that preserve this Minkowski form, i.e.,

$$\det(Y) = 1 \quad \text{and} \quad \{Yv, Yw\} = \{v, w\} \quad (v, w \in \mathbb{C}^2).$$

The second condition can be rewritten as $\langle Yv|M|Yw\rangle = \langle v|M|w\rangle$ which holds for all v, w if and only if

$$Y^* M Y = M, \tag{A.19}$$

where Y^* denotes the usual adjoint of a matrix. Since

$$(e^{tA})^* M e^{tA} = M + t(A^* M + MA) + O(t^2),$$

it is not hard to see that a matrix of the form $Y = e^{tA}$ satisfies (A.19) if and only if

$$A^* M + MA = 0 \quad \Leftrightarrow \quad MA^* M = -A,$$

and the Lie algebra $\mathfrak{su}(1,1)$ associated with $\mathrm{SU}(1,1)$ is given by

$$\mathfrak{su}(1,1) = \big\{A \in M_2(\mathbb{C}) : MA^* M = -A,\ \mathrm{tr}(A) = 0\big\}.$$

It is easy to see that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \Rightarrow \quad MA^* M = \begin{pmatrix} A_{11} & -(A_{21})^* \\ -(A_{12})^* & A_{22} \end{pmatrix}$$

and in fact the map $A \mapsto MA^* M$ satisfies the axioms of an adjoint operation. Let $\mathfrak{su}(1,1)_\mathbb{C}$ denote the Lie algebra

$$\mathfrak{su}(1,1)_\mathbb{C} := \big\{A \in M_2(\mathbb{C}) : \mathrm{tr}(A) = 0\big\},$$

equipped with the adjoint operation $A \mapsto MA^* M$. Then $\mathfrak{su}(1,1)$ is the real sub-Lie algebra of $\mathfrak{su}(1,1)_\mathbb{C}$ consisting of all elements that are skew symmetric with respect to the adjoint operation $A \mapsto MA^* M$.

A basis for $\mathfrak{su}(1,1)_\mathbb{C}$ is formed by the matrices in (2.8), which satisfy the commutation relations (2.7). The adjoint operation $A \mapsto MA^* M$ leads to the adjoint relations (2.9). Some elementary facts about the Lie algebra $\mathfrak{su}(1,1)_\mathbb{C}$ are already stated in Section 2.4. Note that the definition of the "Casimir operator" in (2.10) does not follow the general definition for compact Lie groups in Proposition 30, but is instead defined in an analogous way, replacing the inner product by a Minkowski form.

A.11 The Heisenberg group

Consider the matrices

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We observe that

$$\begin{array}{lll} XX = 0, & XY = Z, & XZ = 0, \\ YX = 0, & YY = 0, & YZ = 0, \\ ZX = 0, & ZY = 0, & ZZ = 0. \end{array}$$

The *Heisenberg group* H [Hal03, Sect. 1.2.6] is the matrix Lie group consisting of all 3×3 real matrices of the form

$$B = I + xX + yY + zZ \qquad (x, y, z \in \mathbb{R}).$$

To see that this is really a group, we note that if B is as above, then its inverse B^{-1} is given by

$$B^{-1} = -xX - yY + (xy - z)Z.$$

It is easy to see that $\{X, Y, Z\}$ is a basis for the Lie algebra $\mathfrak{h}$ of H. In fact, the expansion formula for $e^{t(xX+yY+zZ)}$ terminates and

$$e^{t(xX+yY+zZ)} = I + t(xX + yY + zZ) + \tfrac{1}{2}t^2 xyZ \qquad (t \geq 0).$$

The basis elements X, Y, Z satisfy the commutation relations

$$[X, Y] = Z, \quad [X, Z] = 0, \quad [Y, Z] = 0.$$

Thus, we can abstractly define the *Heisenberg Lie algebra* as the real Lie algebra $\mathfrak{h}$ with basis elements $\mathbf{x}, \mathbf{y}, \mathbf{z}$ that satisfy the commutation relations

$$[\mathbf{x}, \mathbf{y}] = \mathbf{z}, \quad [\mathbf{x}, \mathbf{z}] = 0, \quad [\mathbf{y}, \mathbf{z}] = 0. \tag{A.20}$$

Representations of the Heisenberg algebra have already been discussed in Subsection 2.5.

References

[AH07] R. Alkemper and M. Hutzenthaler. Graphical representation of some duality relations in stochastic population models. *Electron. Commun. Probab.* 12 (2007), 206–220.

[Bar47] V. Bargmann. Irreducible unitary representations of the Lorentz group. *Ann. Math.* 48 (1947), 568–640.

[Bar61] V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform. *Commun. Pure Appl. Math.* 14 (1961), 187–214.

[CGGR15] G. Carinci, C. Giardinà, C. Giberti, and F. Redig. Dualities in population genetics: a fresh look with new dualities. *Stochastic Processes Appl.* 125(3) (2015), 941–969.

[Fil92] J.A. Fill. Strong stationary duality for continuous-time Markov chains. I. Theory. *J. Theor. Probab.* 5(1) (1992), 45–70.

[GKRV09] C. Giardinà, J. Kurchan, F. Redig, and K. Vafayi. Duality and hidden symmetries in interacting particle systems. *J. Stat. Phys.* 135(1) (2009), 25–55.

[GRV10] C. Giardinà, F. Redig, and K. Vafayi. Correlation Inequalities for Interacting Particle Systems with Duality. *J. Stat. Phys.* 141(2) (2010), 242–263.

[Hal03] B.C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.* Graduate Texts in Mathematics, vol. 222, Springer, 2003.

[JK14] S. Jansen and N. Kurt. On the notion(s) of duality for Markov processes. *Prob. Surveys* 11 (2014), 59–120.

[LS95] A. Sudbury and P. Lloyd. Quantum operators in classical probability theory. II: The concept of duality in interacting particle systems. *Ann. Probab.* 23(4) (1995), 1816–1830.

[LS97] A. Sudbury and P. Lloyd. Quantum operators in classical probability theory. IV: Quasi-duality and thinnings of interacting particle systems. *Ann. Probab.* 25(1) (1997), 96–114.

[Nov04] M. Novaes. Some basics of $\mathfrak{su}(1,1)$. *Revista Brasileira de Ensino de Física* 26(4) (2014), 351–357.

[Ros04] J. Rosenberg. A selective history of the Stone-von Neumann theorem. *Contemp. Math.* 365 (2004), 331–353.

[SS16] A. Sturm and J.M. Swart. Pathwise duals of monotone and additive Markov processes. *J. Theor. Probab.* (2016). 52 pages. doi:10.1007/s10959-016-0721-5.

[Sud00] A. Sudbury. Dual families of interacting particle systems on graphs. *J. Theor. Probab.* 13(3) (2000), 695–716.

[Swa06] J.M. Swart. Duals and thinnings of some relatives of the contact process. Pages 203-214 in: *Prague Stochastics 2006*, M. Hušková and M. Janžura (eds.), Matfyzpress, Prague, 2006. ArXiv:math.PR/0604335.

[Swa13] J.M. Swart. *Duality and Intertwining of Markov Chains.* Lecture notes for the ALEA in Europe School October 21-25 2013, Luminy (Marseille). Available at:
`http://staff.utia.cas.cz/swart/tea_index.html`.

[Swa17] J.M. Swart. *Introduction to Quantum Probability.* Lecture notes (2017) available at:
`http://staff.utia.cas.cz/swart/tea_index.html`.

[VK91] N.Ya. Vilenkin and A.U. Klimyk. *Representation of Lie groups and special functions. Volume 1: Simplest Lie groups, special functions and integral transforms.* Kluwer, Dordrecht, 1991.