A Course in Interacting Particle Systems

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Lecture 1: Introduction

Interacting Particle Systems

- Interacting particle systems are mathematical models for *collective behavior*.
- Applications in physics (atoms & molecules), biology (organisms) & sociology, financial mathematics (people).
- Simple rules lead to complicated behavior.
- *Markovian* dynamics.
- Easy to simulate, but not always easy to prove; open problems.
- Rigorous methods lead to better understanding.

Interacting Particle Systems

Interacting Particle Systems are continuous-time Markov processes $X = (X_t)_{t \geq 0}$ with state space of the form S^Λ, where:

- S is a finite set, called the *local state space*.
- Λ is a countable set, called the *lattice*.

We denote an element $x \in S^\Lambda$ as

$$x = \big(x(i)\big)_{i \in \Lambda} \quad \text{with} \quad x(i) \in S \ \forall \ i \in \Lambda.$$

We call $X_t(i)$ the *local state* of the process at time t and position $i \in \Lambda$.

Generator and local maps

The *generator* G of an interacting particle system can be written in the form

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \{ f(m(x)) - f(x) \},$$

where $(r_m)_{m \in \mathcal{G}}$ are nonnegative *rates* and $\mathcal{G}$ is a collection of *local maps* $m : S^\Lambda \to S^\Lambda$.

Poisson construction: The process can be constructed by applying each map $m \in \mathcal{G}$ at the times of a Poisson process with intensity r_m.

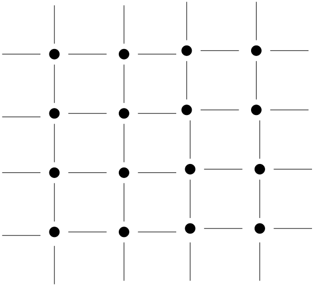
Generator construction: If Λ is finite, then the *transition probabilities* $P_t(x, y)$ are given by

$$P_t := e^{tG} := \sum_{n=0}^{\infty} \frac{1}{n!} (tG)^n.$$

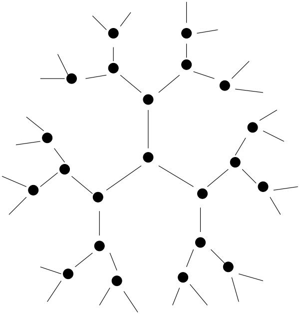
Examples of lattices

Often, the lattice is a *graph* (Λ, E) with (undirected) edge set E. We denote the corresponding set of *directed* edges by:

$$\mathcal{E} := \{(i,j) : \{i,j\} \in E\}$$



$\mathbb{Z}^2$



$\mathbb{T}_2$

Examples of lattices

In particular, we equip $\mathbb{Z}^d$ with the following edge sets:

$$E^d := \big\{\{i,j\} : \|i-j\|_1 = 1\big\},$$
$$E^d_R := \big\{\{i,j\} : 0 < \|i-j\|_\infty \leq R\big\}.$$

We let $\mathcal{E}^d$ and $\mathcal{E}^d_R$ denote the corresponding directed edges and let

$$\mathcal{N}_i \text{ resp. } \mathcal{N}_i^R := \{j \in \mathbb{Z}^d : \{i,j\} \in E^d \text{ resp. } E^d_R\}$$

denote the *neighborhood* of i.

The voter model

For each $i, j \in \Lambda$, the *voter model map* $\mathrm{vot}_{ij} : S^\Lambda \to S^\Lambda$ is defined as

$$\mathtt{vot}_{ij}(x)(k) := \begin{cases} x(i) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

In words, this copies the state of i onto j. The *nearest neighbor voter model* is defined by

$$G_{\mathrm{vot}} f(x) = \frac{1}{|\mathcal{N}_0|} \sum_{(i,j) \in \mathcal{E}^d} \big\{ f\big(\mathtt{vot}_{ij}(x)\big) - f\big(x\big) \big\} \qquad (x \in S^\Lambda).$$

Similarly, $\mathcal{N}_i^R$ gives the *range R voter model*.

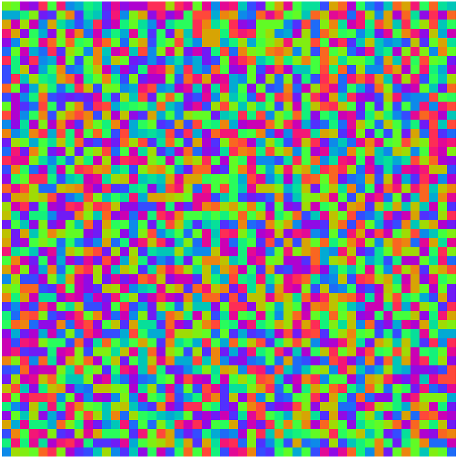
The voter model

With rate one, the *site* j adopts the *type* $x(i)$ of a randomly chosen neighbor.

Interpretation 1 Sites are people, types are political parties; at rate one, people ask their neighbor whom to vote for.

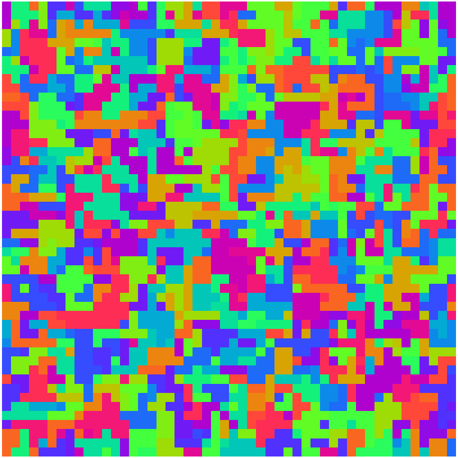
Interpretation 2 Sites are organisms, types are genetic types; at rate one, an organism dies and is replaced by a clone of a randomly chosen neighbor.

The voter model



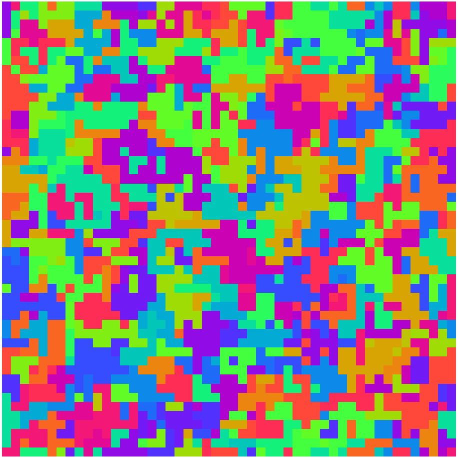
Time $t = 0$.

The voter model



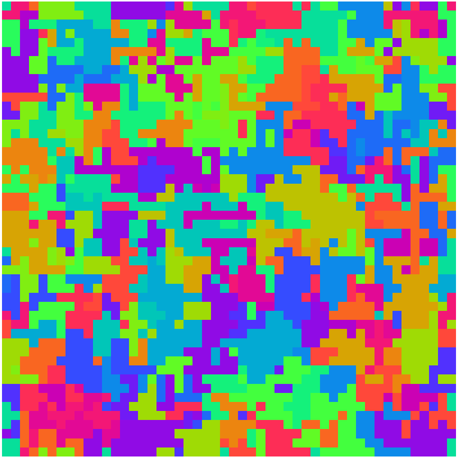
Time $t = 0.25$.

The voter model



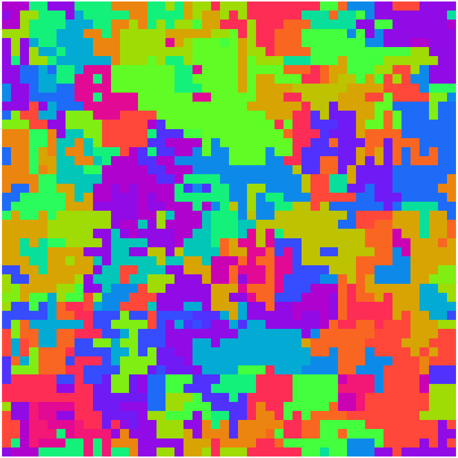
Time $t = 0.5$.

The voter model



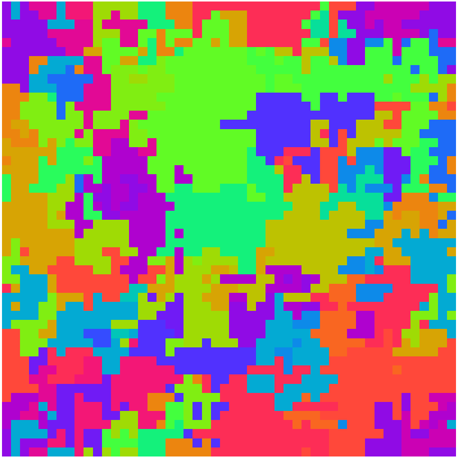
Time $t = 1$.

The voter model



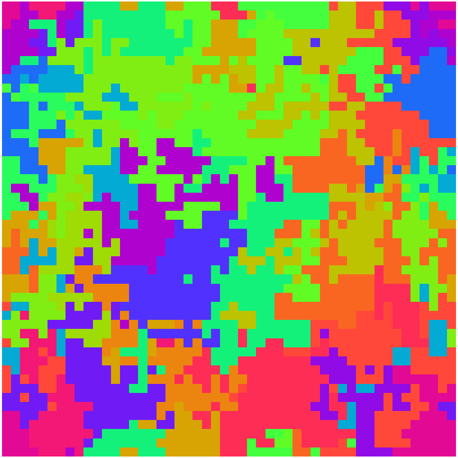
Time $t = 2$.

The voter model



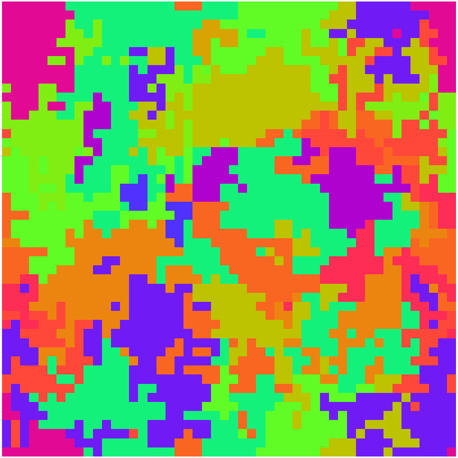
Time $t = 4$.

The voter model



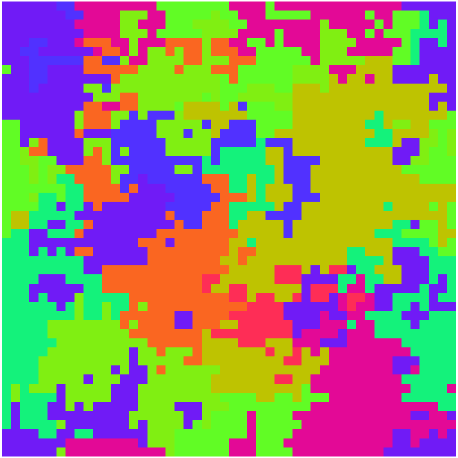
Time $t = 8$.

The voter model



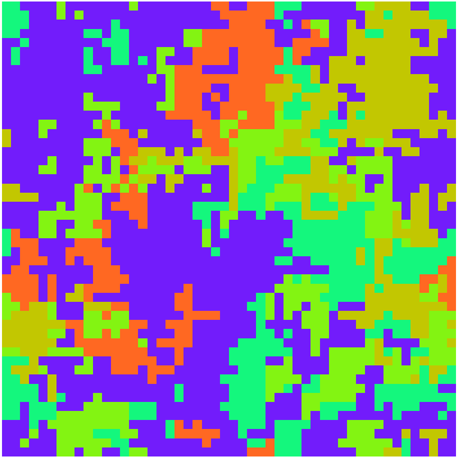
Time $t = 16$.

The voter model



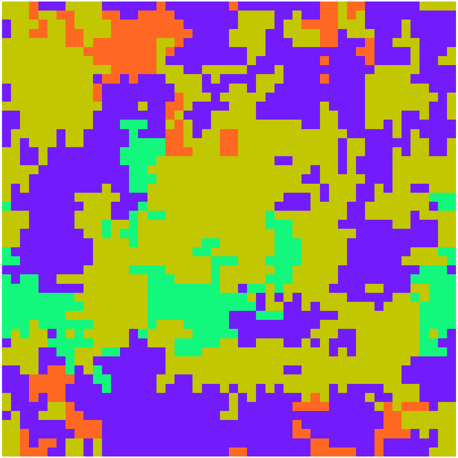
Time $t = 31.25$.

The voter model



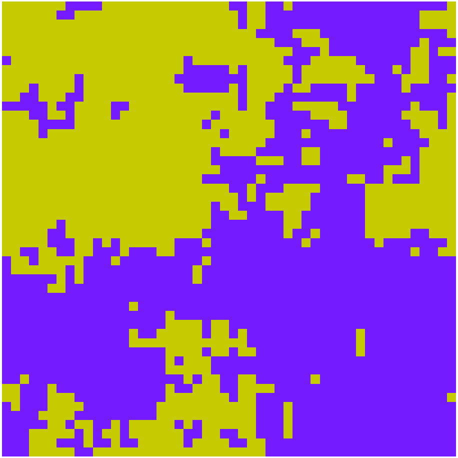
Time $t = 62.5$.

The voter model



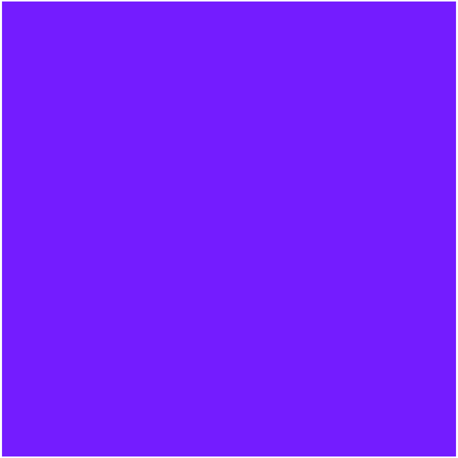
Time $t = 125$.

The voter model



Time $t = 250$.

The voter model



Time $t = 500$.

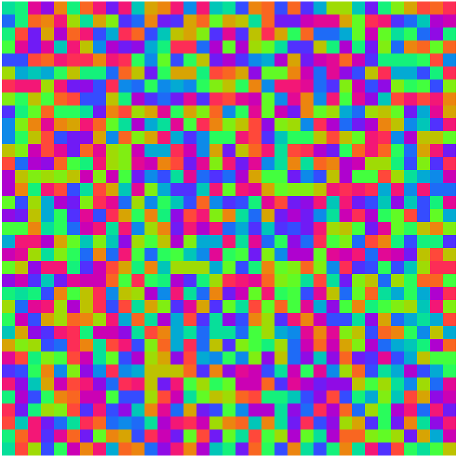
The voter model

The behavior of the voter model strongly depends on the dimension.

Clustering in dimensions $d = 1, 2$.

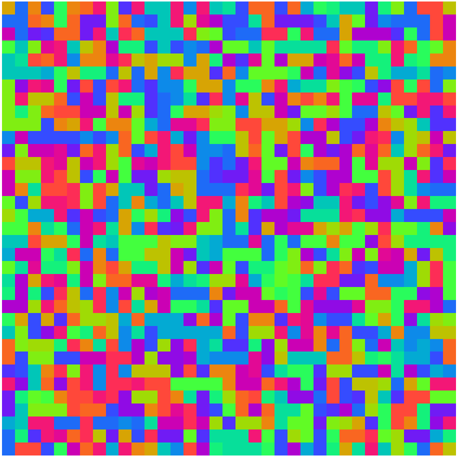
Stable behavior in dimensions $d \geq 3$.

The voter model



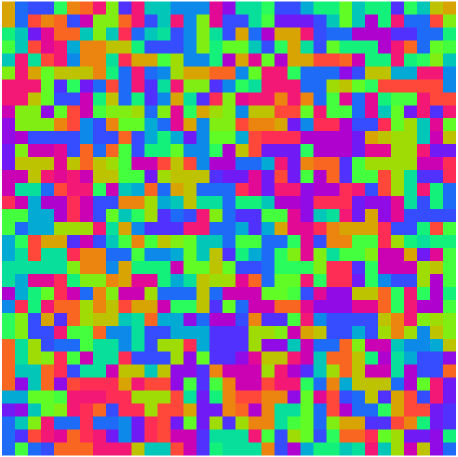
Cut of 3-dimensional model, time $t = 0$.

The voter model



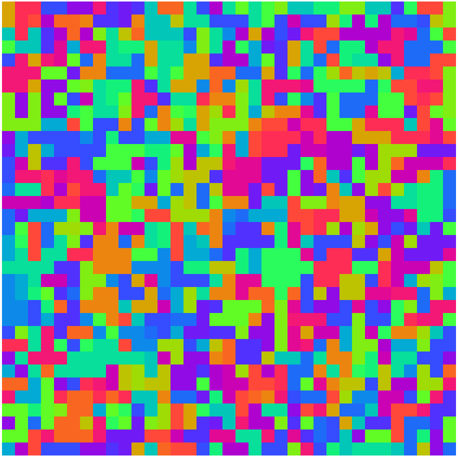
Cut of 3-dimensional model, time $t = 1$.

The voter model



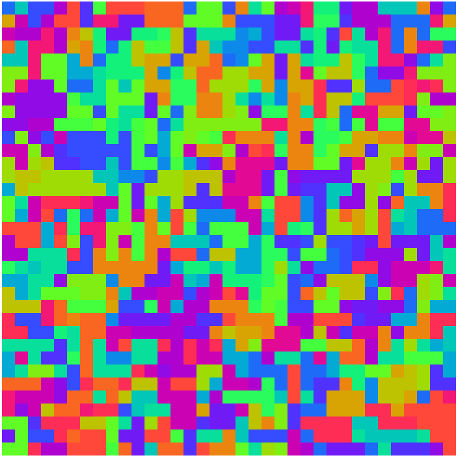
Cut of 3-dimensional model, time $t = 2$.

The voter model



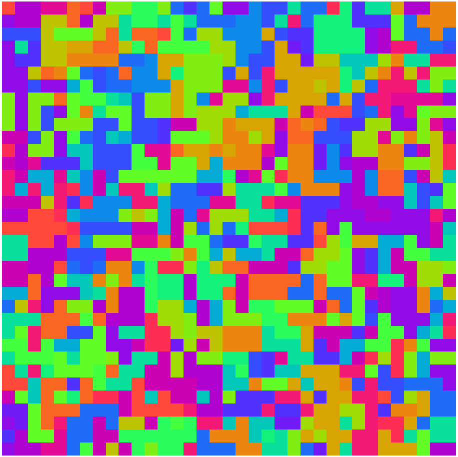
Cut of 3-dimensional model, time $t = 4$.

The voter model



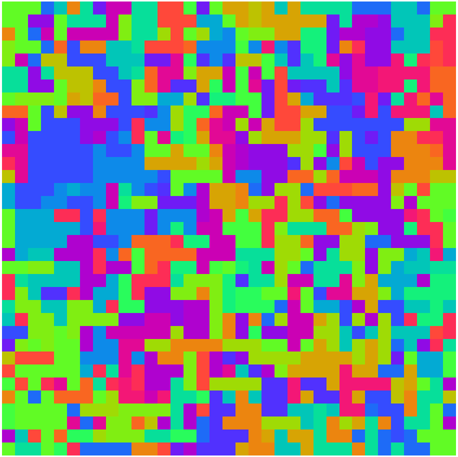
Cut of 3-dimensional model, time $t = 8$.

The voter model



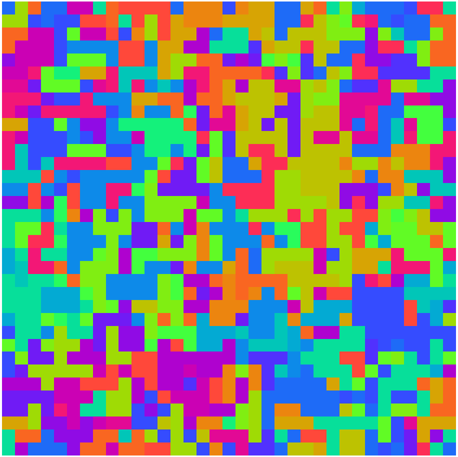
Cut of 3-dimensional model, time $t = 16$.

The voter model



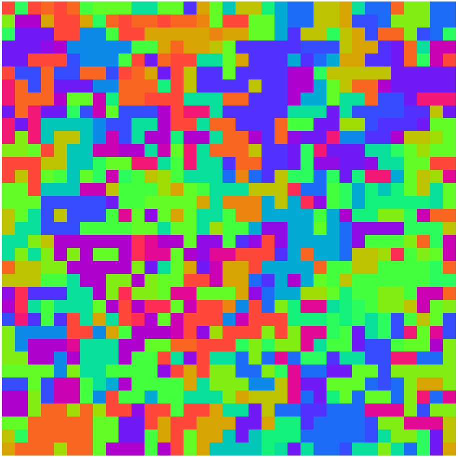
Cut of 3-dimensional model, time $t = 32$.

The voter model



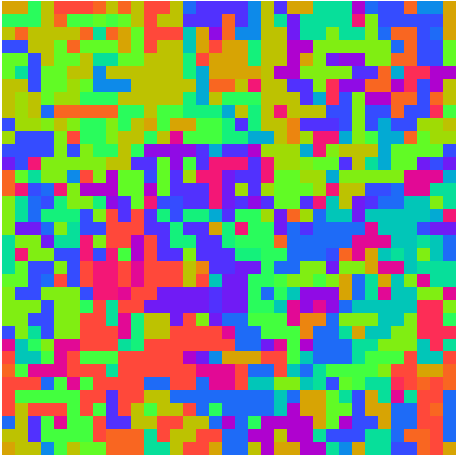
Cut of 3-dimensional model, time $t = 64$.

The voter model



Cut of 3-dimensional model, time $t = 125$.

The voter model



Cut of 3-dimensional model, time $t = 250$.

The Contact Process

Let $S = \{0, 1\}$ with $0 =$ empty and $1 =$ occupied.

For each $i, j \in \Lambda$, we define a *branching map* $\mathtt{bra}_{ij} : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ as

$$\mathtt{bra}_{ij}(x)(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

For each $i \in \Lambda$, we also define a *death map* $\mathtt{death}_i : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ as

$$\mathtt{death}_i(x)(k) := \begin{cases} 0 & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

The Contact Process

The nearest neighbor *contact process* with *infection rate* λ is defined by the generator

$$G_{\rm cont} f(x) := \lambda \sum_{(i,j)\in\mathcal{E}^d} \big\{ f\big(\big(\mathtt{bra}_{ij}(x)\big) - f\big(x\big)\big\} + \sum_{i\in\mathbb{Z}^d} \big\{ f\big(\big(\mathtt{death}_i(x)\big) - f\big(x\big)\big\} \qquad \big(x \in \{0,1\}^{\mathbb{Z}^d}\big).$$

Interpretation 1 $1 =$ infected, $0 =$ healthy, sites infect each neighbor with rate λ and recover with rate one.

Interpretation 2 $1 =$ occupied, $0 =$ empty, sites place offspring on each neighboring site with rate λ and die with rate one.

The contact process

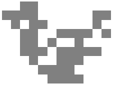
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 0$.

The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 1$.

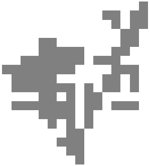
The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 2$.



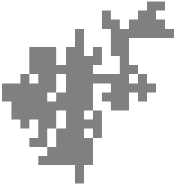
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 3$.

The contact process

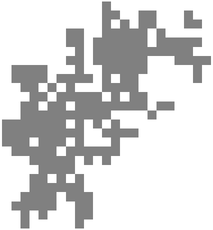


Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 4$.

The contact process

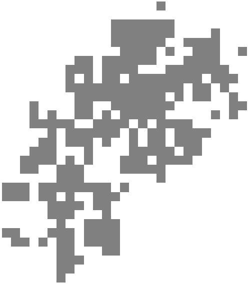


Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 5$.



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 6$.

The contact process

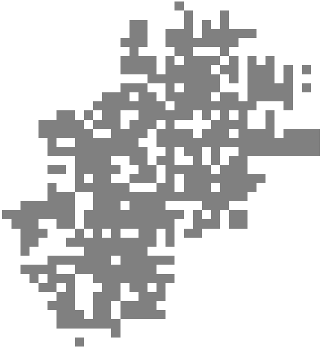


Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 7$.

The contact process

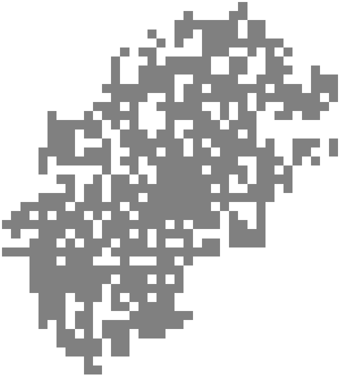
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 8$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 9$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 10$.

The contact process

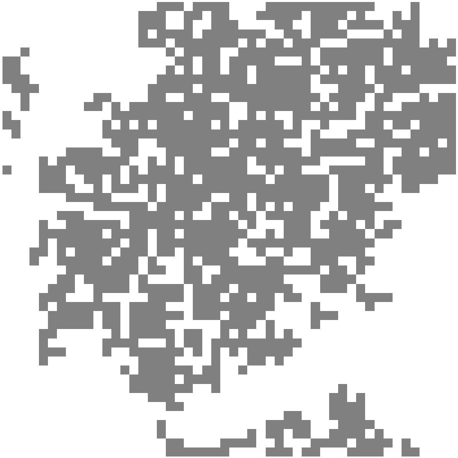
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 11$.

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 12$.

The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 13$.

The contact process

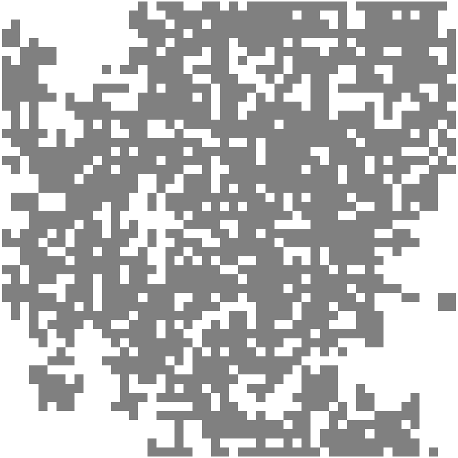


Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 14$.

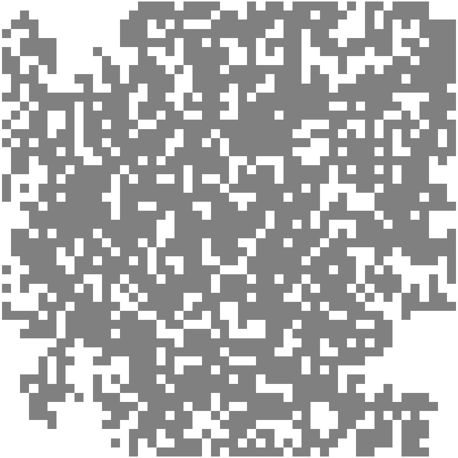
The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 15$.

The contact process

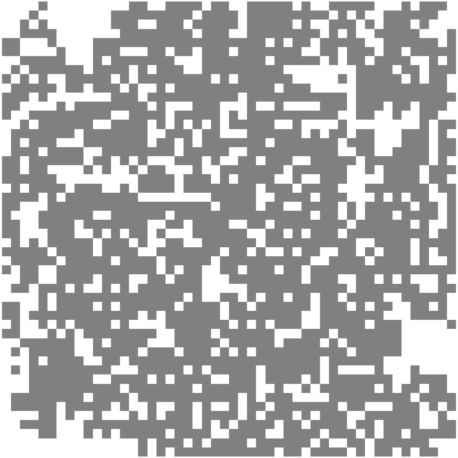


Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 16$.



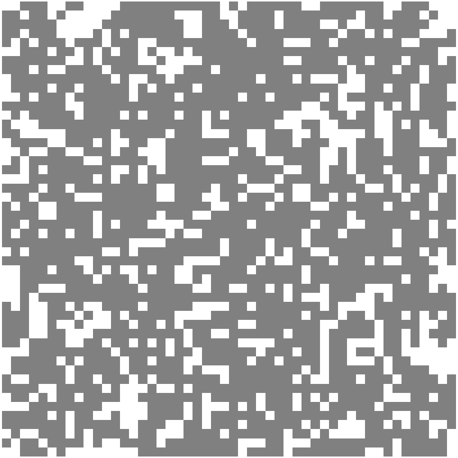
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 17$.

The contact process



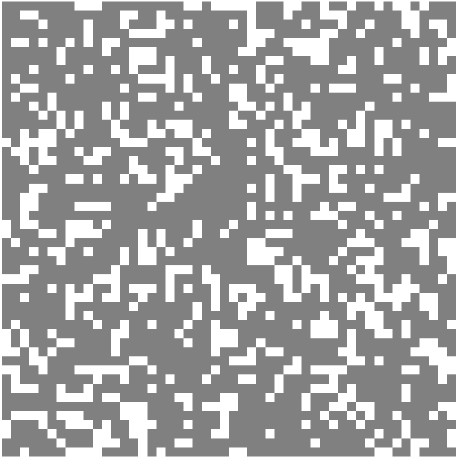
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 18$.

The contact process

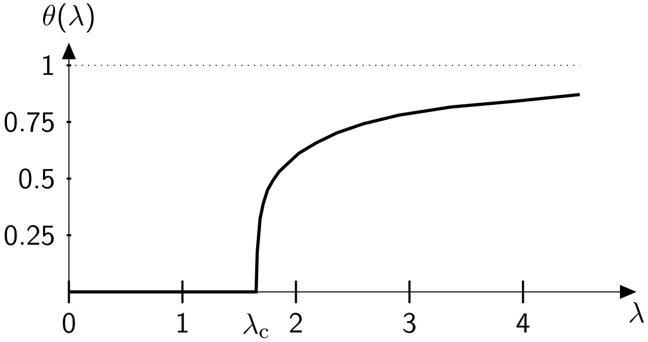


Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 19$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 20$.



The *survival probability*

$$\theta(\lambda) := \mathbb{P}^{1_{\{0\}}}\big[X_t \neq 0 \ \forall t \geq 0\big]$$

exhibits a phase transition. In one dimenson $\lambda_c \approx 1.649$.

Ising and Potts models

Let S be any finite set. For any $x \in S^\Lambda$ and $i \in \Lambda$, let

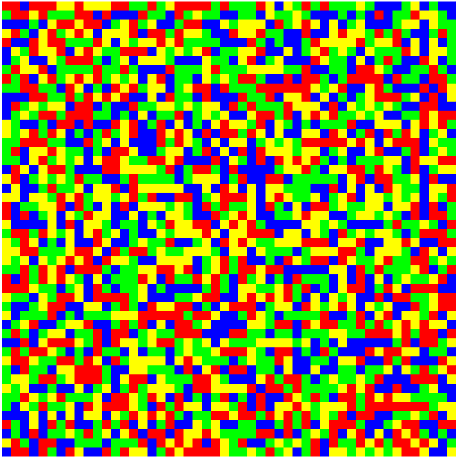
$$N_{x,i}(\sigma) := \sum_{j \in \mathcal{N}_i} 1_{\{x(j) = \sigma\}} \qquad (\sigma \in S)$$

denote the number of neighbors of site i that have the *spin value* $\sigma \in S$. In the *Potts model* with *Glauber dynamics*,

$$\text{site } i \text{ flips to the value } \sigma \text{ with rate } \quad \frac{e^{\beta N_{x,i}(\sigma)}}{\sum_{\tau \in S} e^{\beta N_{x,i}(\tau)}}.$$

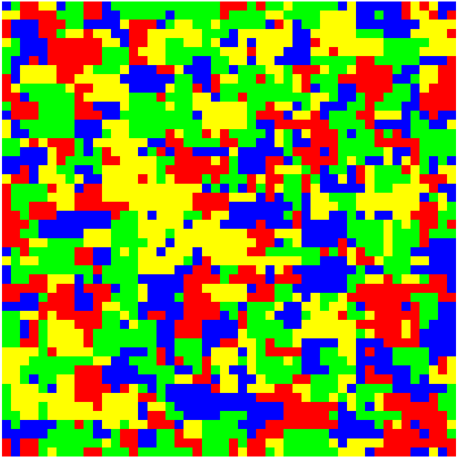
I.e., update with rate one, choose new value σ proportional to $e^{\beta N_{x,i}(\sigma)}$. For $\beta \geq 0$ *ferromagnetic*.

The Potts model



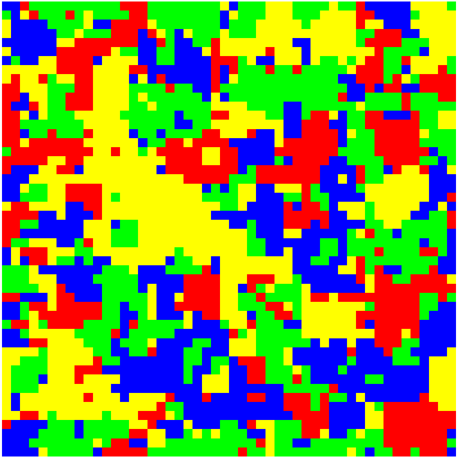
$\beta = 1.2$, time $t = 0$.

The Potts model



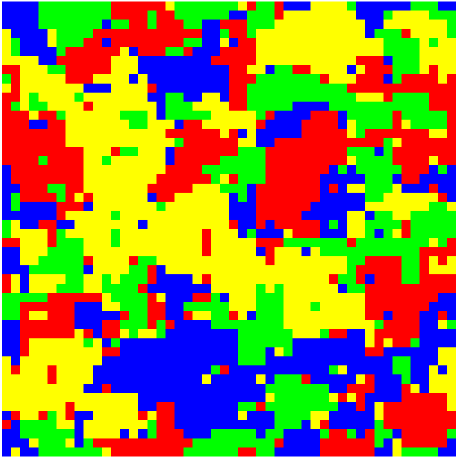
$\beta = 1.2$, time $t = 1$.

The Potts model



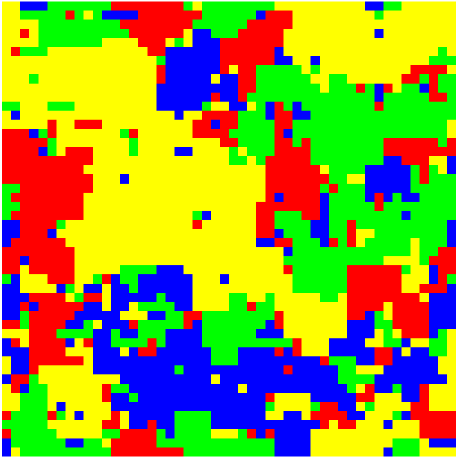
$\beta = 1.2$, time $t = 2$.

The Potts model



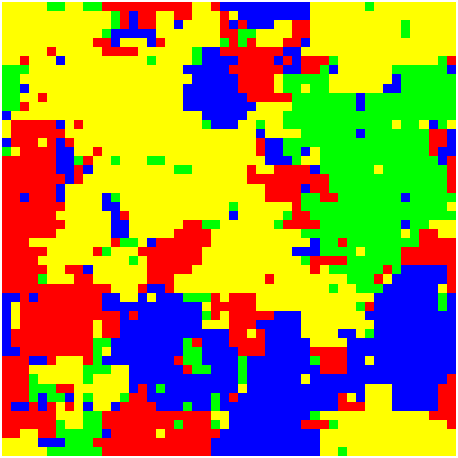
$\beta = 1.2$, time $t = 4$.

The Potts model



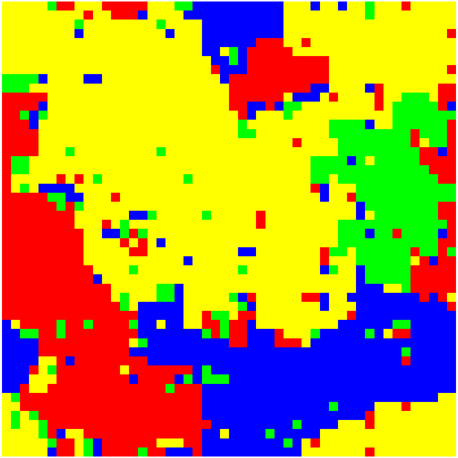
$\beta = 1.2$, time $t = 8$.

The Potts model



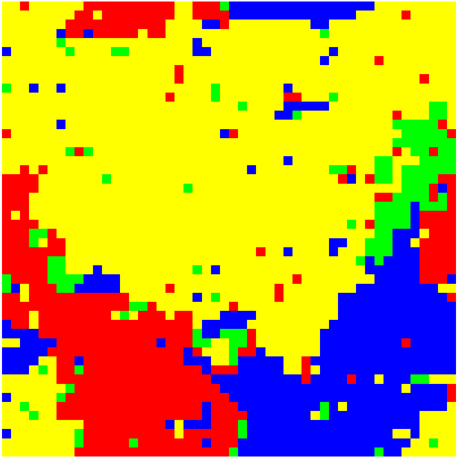
$\beta = 1.2$, time $t = 16$.

The Potts model



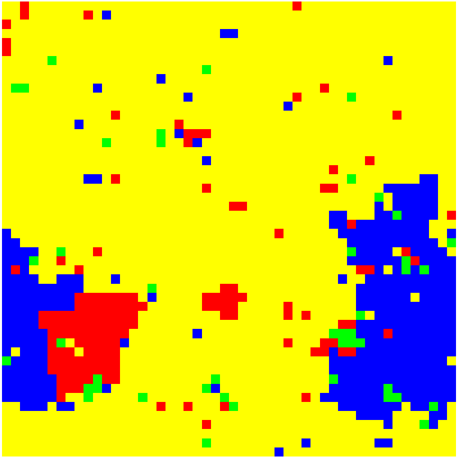
$\beta = 1.2$, time $t = 32$.

The Potts model



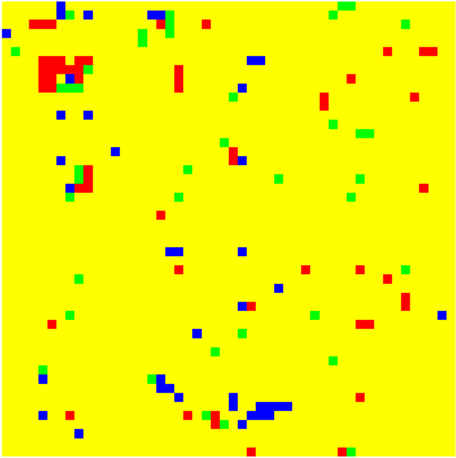
$\beta = 1.2$, time $t = 64$.

The Potts model



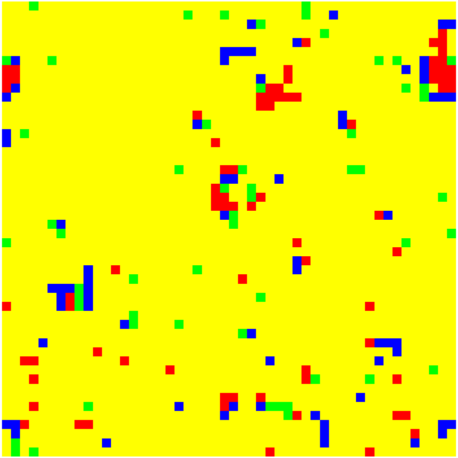
$\beta = 1.2$, time $t = 125$.

The Potts model



$\beta = 1.2$, time $t = 250$.

The Potts model



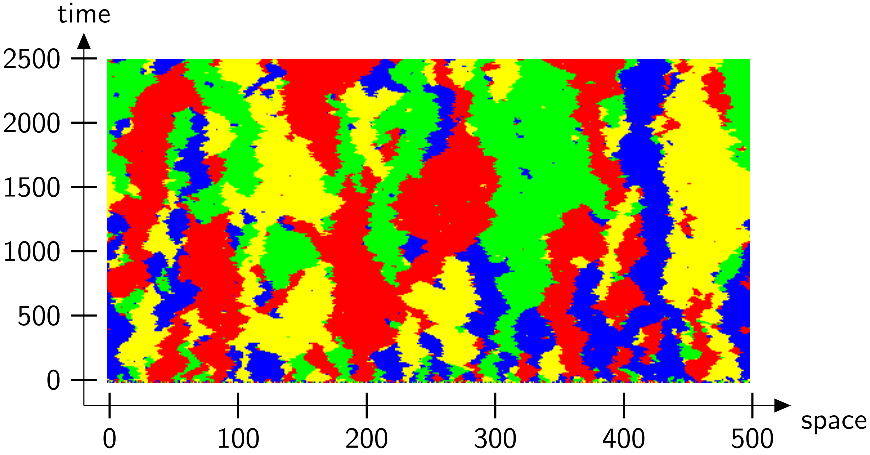
$\beta = 1.2$, time $t = 500$.

The Ising model

This looks superficially like the voter model, but:

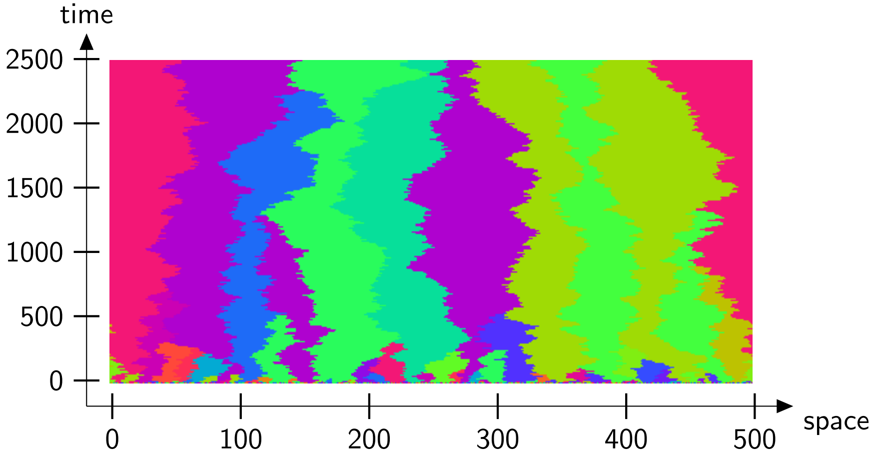
- Even in large clusters, single sites can still flip to other colors.
- Clustering happens only for β above a critical value $\beta_{\rm c}$.
- $0 < \beta_{\rm c} < \infty$ in dimensions $d \geq 2$ but $\beta_{\rm c} = \infty$ in dimension one.

A one-dimensional Potts model



In one-dimensional Potts models, the cluster size remains bounded in time even at very high β (= low temperature).

The voter model



A one-dimensional voter model.

The Ising model

The Ising model is the Potts model with $S = \{-1, +1\}$.

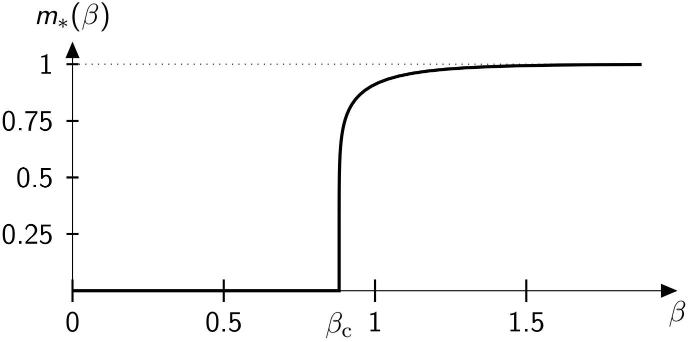
On a large, but finite lattice Λ, freeze the boundary spins to $+1$.

Since Λ is finite, there is a unique invariant law ν_+.
The *spontaneous magnetization* is the function

$$m_*(\beta) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \int \nu_+(\mathrm{d}x)\, x(0).$$

One has *clustering*, i.e., *long-range order*, iff $m_*(\beta) > 0$.

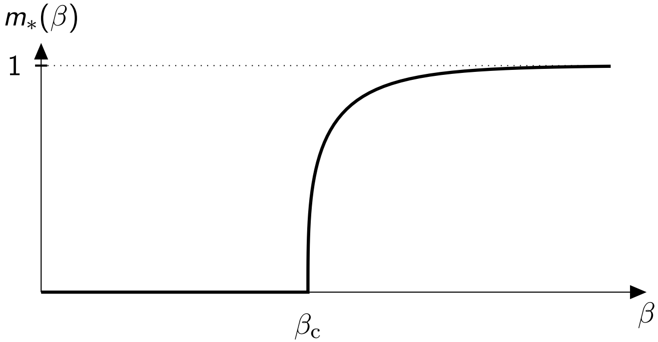
The magnetization of the Ising model



Onsager (1944) proved that for the model on $\mathbb{Z}^2$,

$$m_*(\beta) = \begin{cases} \big(1 - \sinh(\beta)^{-4}\big)^{1/8} & \text{for } \beta \geq \beta_c := \log(1+\sqrt{2}), \\ 0 & \text{for } \beta \leq \beta_c. \end{cases}$$

The magnetization of the Ising model



For the model on $\mathbb{Z}^3$, it is known that m_* is continuous, nondecreasing in β, and there exists a $0 < \beta_c < \infty$ such that $m_*(\beta) = 0$ for $\beta \leq \beta_c$ while $m_*(\beta) > 0$ for $\beta > \beta_c$. Continuity at β_c proved by Aizenman, Duminil-Copin & Sidoravicius (2014).

The contact process goes through a *phase transition* at $\lambda_{\rm c}$.
The Ising model goes through a *phase transition* at $\beta_{\rm c}$.

In both cases, below the *critical point* ($\lambda_{\rm c}$ resp. $\beta_{\rm c}$), the system is in a phase where there is a *unique invariant law* and *absence of long-range order*.

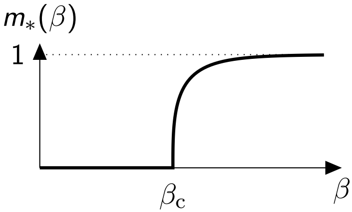
Above $\lambda_{\rm c}$ resp. $\beta_{\rm c}$, there are *multiple invariant laws* and *long-range order*.

Phase transitions

Since the order parameter $\theta(\lambda)$ resp. $m_*(\beta)$ is continuous at the critical point, the phase transitions of the contact process and Ising model are *second order* or *continuous* phase transitions.

By contrast, for Potts models with a high number of colors (in dimension two, for $q := |S| > 4$), the analogue of the magnetization m_* has a jump at $\beta_{\rm c}$. These systems have q ordered invariant laws for $\beta > \beta_{\rm c}$, one unordered invariant law for $\beta < \beta_{\rm c}$, while at $\beta = \beta_{\rm c}$, all $q+1$ invariant laws coexist (q ordered states and one disordered state). Such a phase transition is called *first order*.

Critical exponents



Nonrigorous renormalization group theory explains that

$$m_*(\beta) \propto (\beta - \beta_c)^c \quad \text{as } \beta \downarrow \beta_c,$$

where the *critical exponent* c is given by

$c = 1/8$ in dim 2, $\quad c \approx 0.326$ in dim 3, $\quad$ and $\quad c = 1/2$ in dim ≥ 4.

Critical exponents

For the contact process, it is believed that

$$\theta(\lambda) \propto (\lambda - \lambda_{\rm c})^c \quad \text{as } \beta \downarrow \beta_{\rm c},$$

where the *critical exponent* c is given by

$$c \approx 0.276 \text{ in dim } 1, \quad c \approx 0.583 \text{ in dim } 2,$$
$$c \approx 0.813 \text{ in dim } 3, \quad \text{and} \quad c = 1 \text{ in dim} \geq 4.$$

There are other critical exponents associated with other quantities (such as the correlation length) or with the power-law decay of correlations at the critical point.

Critical exponents

Critical exponents are believed to be *universal*. For example, for the range R model, the value of the critical point depends on R but the critical exponent does not. Critical exponents associated with the $d = 3$ Ising model have even been measured in real physical systems.

Critical exponents and more generally critical behavior are associated only with second order phase transitions, and for this reason physicists use the word "critical point" only for second order phase transitions.

Critical exponents

Critical exponents are explained by nonrigorous *renormalization group theory* but so far, there is no general mathematical theory. Powerlaw behavior with well-defined critical exponents has been proved in some special cases:

- In sufficiently high dimensions by means of the *lace expansion*.
- In a few exactly solvable models like the Ising model on $\mathbb{Z}^2$.
- In certain 2-dimensional models related to *conformal field theory* and the *Schramm Loewner Equation*.

Variations on the voter model

The *biased voter model* with *bias* $s \geq 0$ is the interacting particle system with state space $\{0,1\}^{\mathbb{Z}^d}$ and generator

$$G_{\text{bias}} f(x) := \frac{1}{|\mathcal{N}_i|} \sum_{(i,j)\in\mathcal{E}^d} \big\{ f\big(\texttt{vot}_{ij}(x)\big) - f\big(x\big) \big\} + \frac{s}{|\mathcal{N}_i|} \sum_{(i,j)\in\mathcal{E}^d} \big\{ f\big(\texttt{bra}_{ij}(x)\big) - f\big(x\big) \big\}.$$

The paremeter $s > 0$ gives type 1 a (small) advantage.

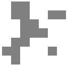
Contrary to the voter model, even if we start with just a single person of type 1, there is a positive probability that type 1 never dies out.

Models spread of *new idea* or *technology*, or *advantageous mutation* in biology.

The biased voter model

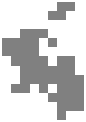
Biased voter model with $s = 0.2$. Time $t = 0$.

The biased voter model



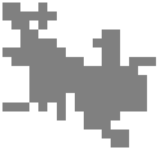
Biased voter model with $s = 0.2$. Time $t = 10$.

The biased voter model



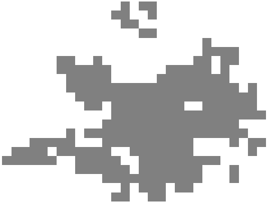
Biased voter model with $s = 0.2$. Time $t = 20$.

The biased voter model



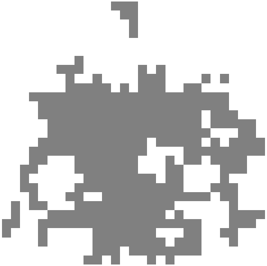
Biased voter model with $s = 0.2$. Time $t = 30$.

The biased voter model



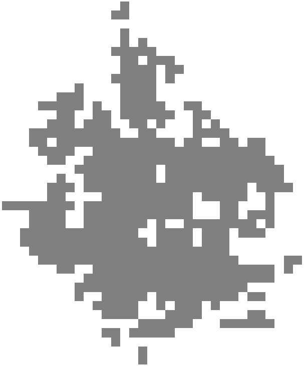
Biased voter model with $s = 0.2$. Time $t = 40$.

The biased voter model



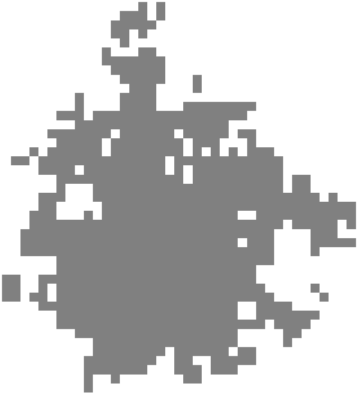
Biased voter model with $s = 0.2$. Time $t = 50$.

The biased voter model



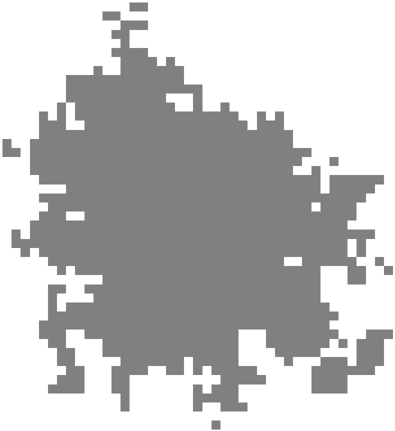
Biased voter model with $s = 0.2$. Time $t = 60$.

The biased voter model



Biased voter model with $s = 0.2$. Time $t = 70$.

The biased voter model



Biased voter model with $s = 0.2$. Time $t = 80$.

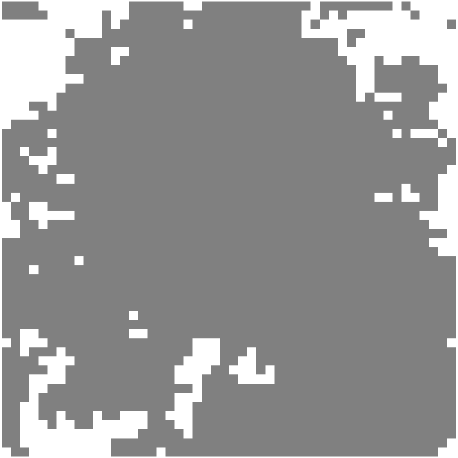
The biased voter model

Biased voter model with $s = 0.2$. Time $t = 90$.

The biased voter model

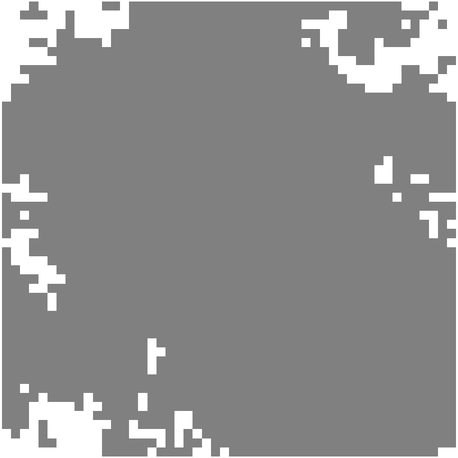
Biased voter model with $s = 0.2$. Time $t = 100$.

The biased voter model



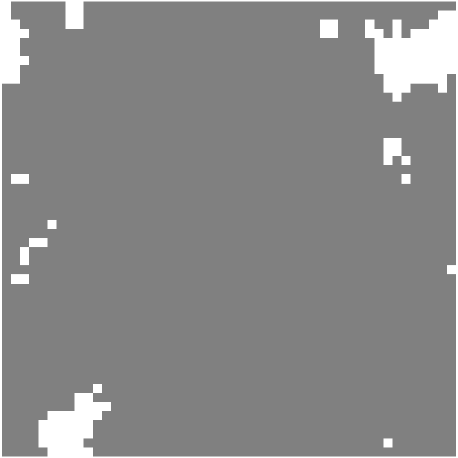
Biased voter model with $s = 0.2$. Time $t = 110$.

The biased voter model



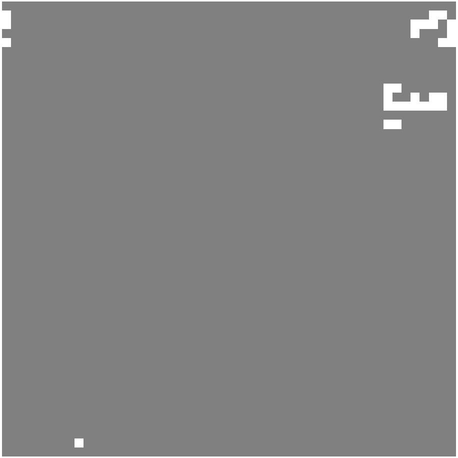
Biased voter model with $s = 0.2$. Time $t = 120$.

The biased voter model



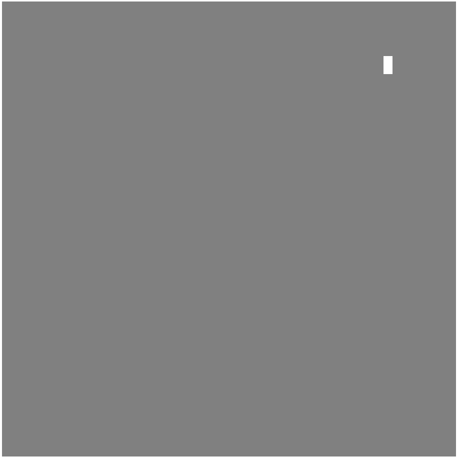
Biased voter model with $s = 0.2$. Time $t = 130$.

The biased voter model



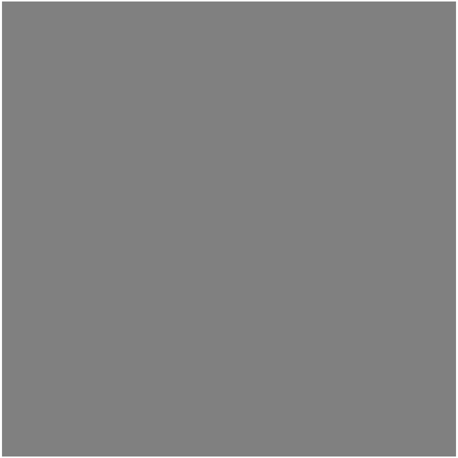
Biased voter model with $s = 0.2$. Time $t = 140$.

The biased voter model



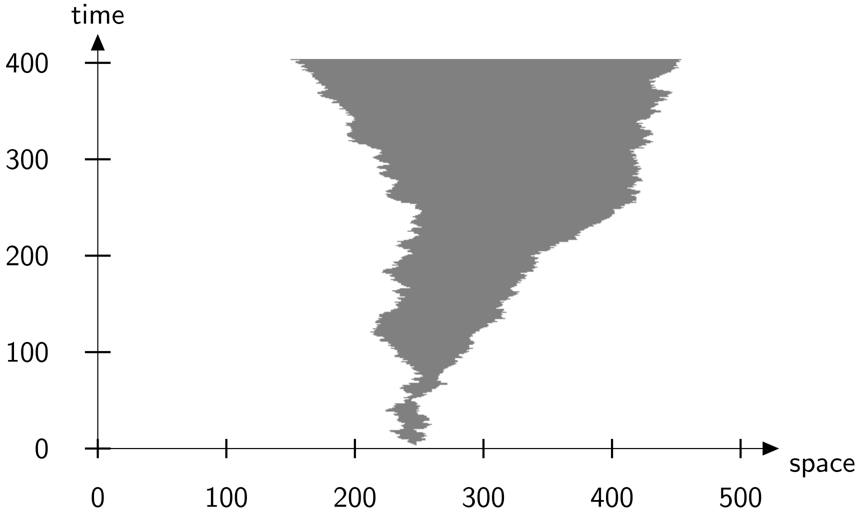
Biased voter model with $s = 0.2$. Time $t = 150$.

The biased voter model



Biased voter model with $s = 0.2$. Time $t = 160$.

The biased voter model



A one-dimensional biased voter model with bias $s = 0.2$.

The biased voter model

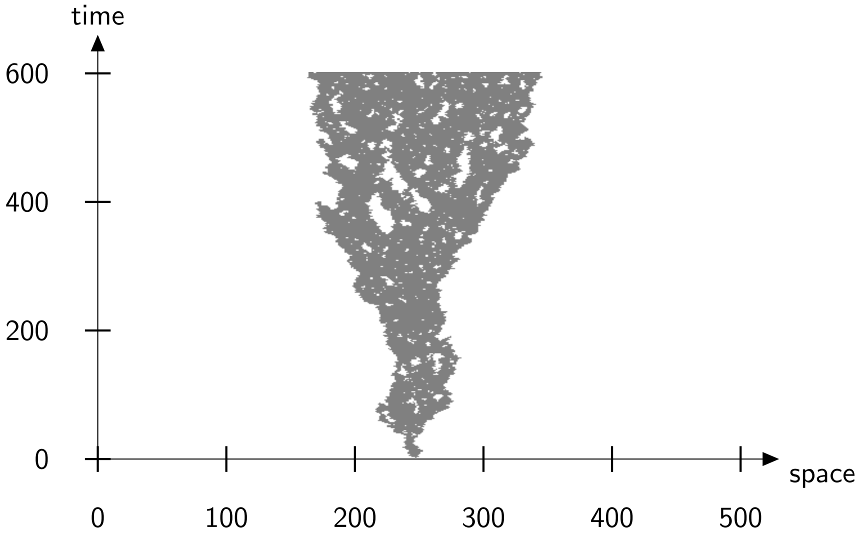
We can extend the biased voter model by also allowing deaths, i.e., spontaneous jumps from 1 to 0.

This models the fact that complicated new ideas may be forgotten or organisms may die.

Whether 1's have a positive probability to survive now depends in a nontrivial way on s and d.

In fact, the model appears to be in the same *universality class* as the contact process.

The biased voter model



Process with bias $s = 0.5$, death rate $d = 0.02$.

A rebellious voter model

Consider a model with two types $\{0, 1\}$ and let

$$f_\tau := \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} 1_{\{x(j) = \tau\}}$$

be the frequency of type τ in the neighborhood $\mathcal{N}_i$.

A person of type τ chooses a new type with rate

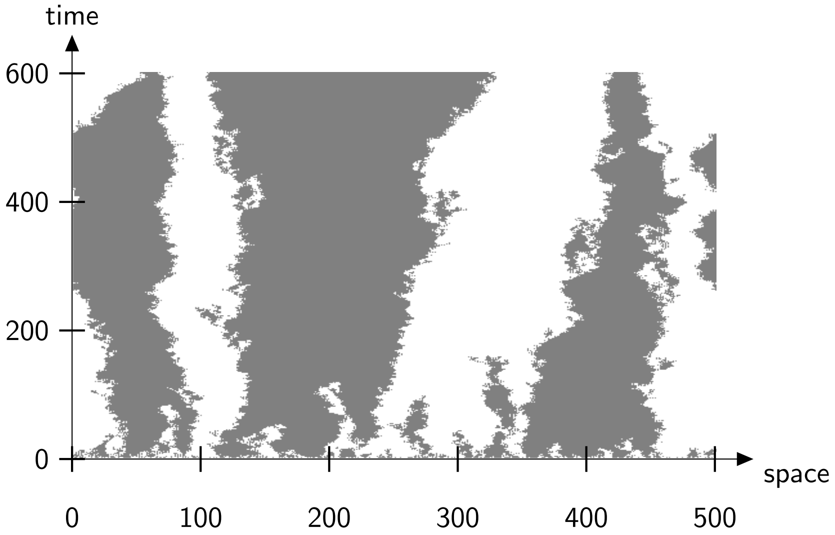
$$f_\tau + \alpha f_{1-\tau}.$$

For $\alpha < 1$, persons change their mind *less* often if they disagree with a lot of neighbors.

As in a normal voter model, the probability that the newly chosen type is τ' is $f_{\tau'}$.

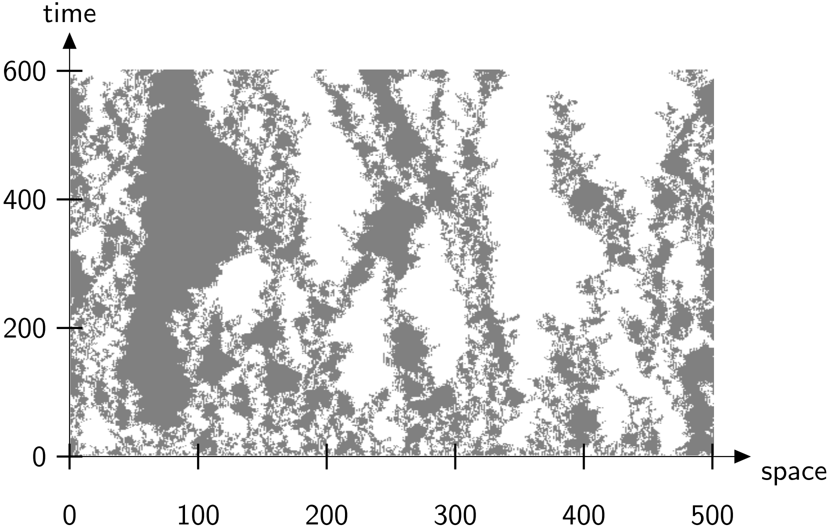
Used by Neuhauser & Pacala (1999) to model *balancing selection*.

A rebellious voter model



Process with $\alpha = 0.8$ behaves more or less as a voter model.

A rebellious voter model



In the process with $\alpha = 0.3$, cluster size remains bounded in time.

Coalescing random walks

Let $X_t(i) = 1$ (resp. 0) signify the presence (resp. absence) of a particle and consider the maps $\mathtt{rw}_{ij} : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$

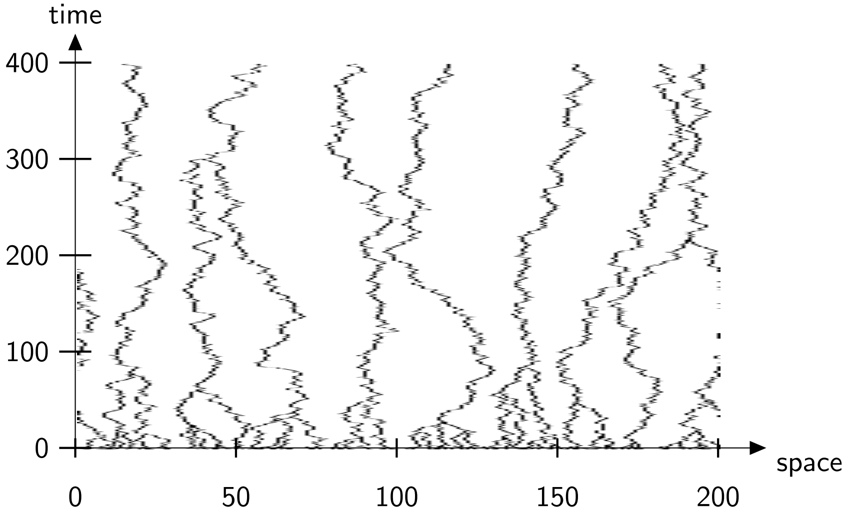
$$\mathtt{rw}_{ij}x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

The process with generator

$$G_{\mathrm{rw}} f(x) := \frac{1}{|\mathcal{N}_0|} \sum_{(i,j) \in \mathcal{E}^d} \big\{ f\big(\mathtt{rw}_{ij}x\big) - f\big(x\big) \big\}$$

describes *coalescing random walks*.

Coalescing random walks



Reaction diffusion models

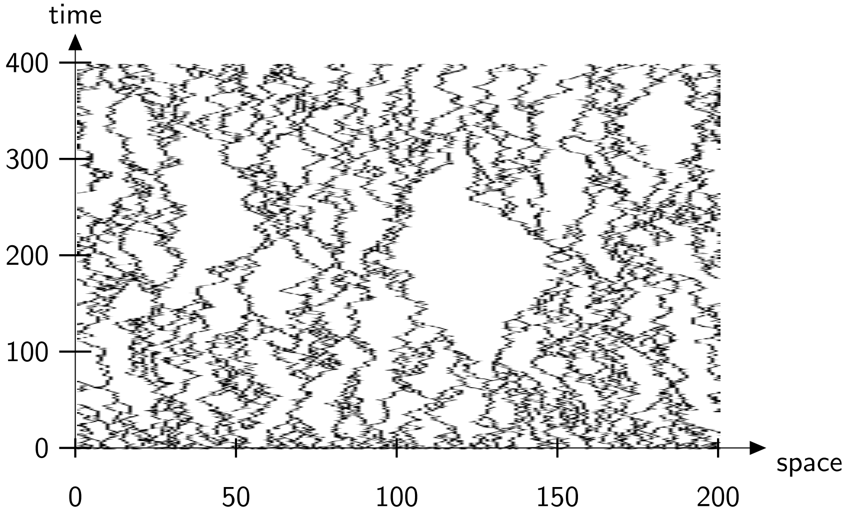
We can also add other maps to the dynamics, like the *branching map*

$$\mathtt{bra}_{ij}x(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

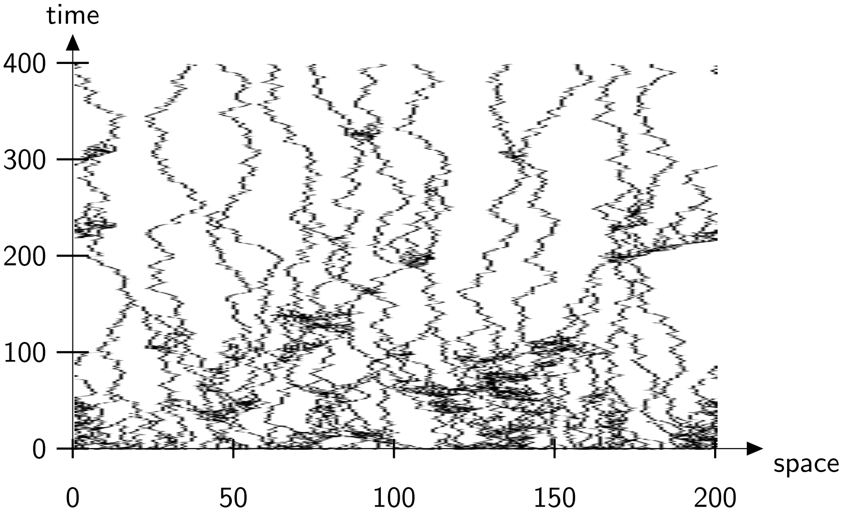
or even *cooperative branching*

$$\mathtt{coop}_{ii'j}x(k) := \begin{cases} \big(x(i) \wedge x(i')\big) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

Branching and coalescing random walks

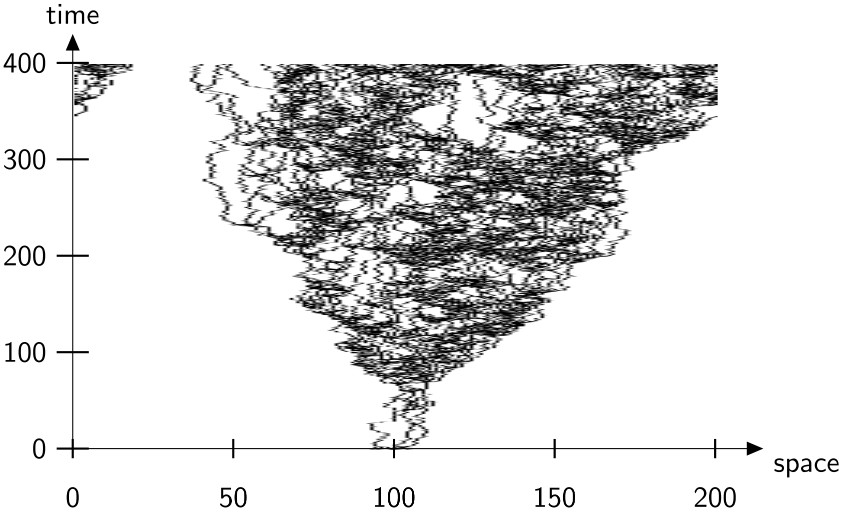


Cooperative branching and coalescence



Cooperative branching rate 2.2.

Cooperative branching



Cooperative branching rate 3.

A cancellative system

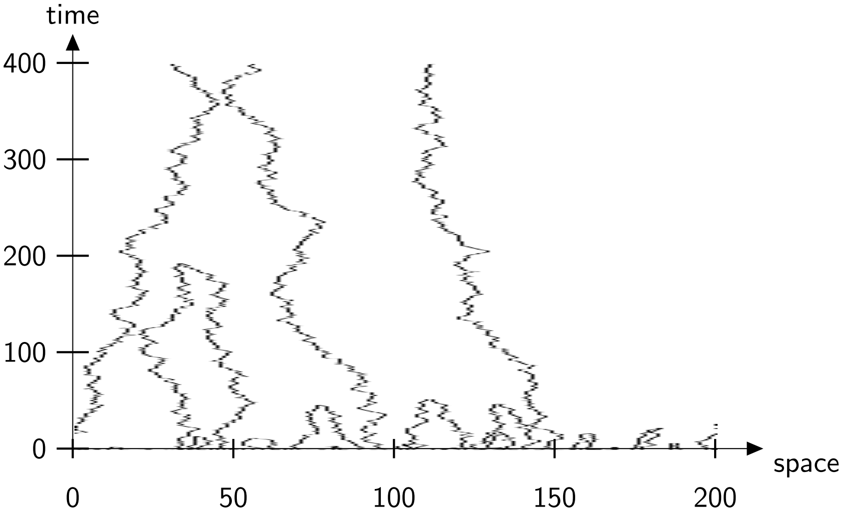
Two more maps of interest are the *annihilating random walk* map

$$\mathtt{ann}_{ij}x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(i) + x(j) \mod(2) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

and the *annihilating branching map*

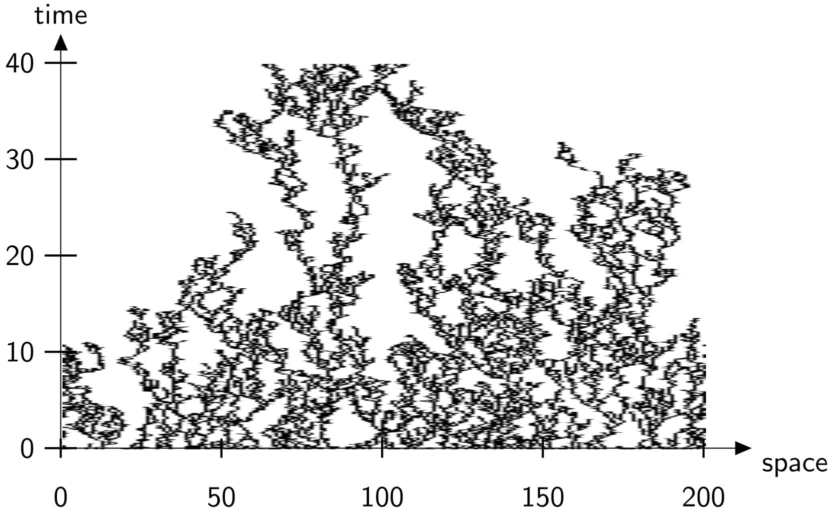
$$\mathtt{bran}_{ij}x(k) := \begin{cases} x(i) + x(j) \mod(2) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

A cancellative system



Annihilating random walks.

A cancellative system



A system of branching annihilating random walks.

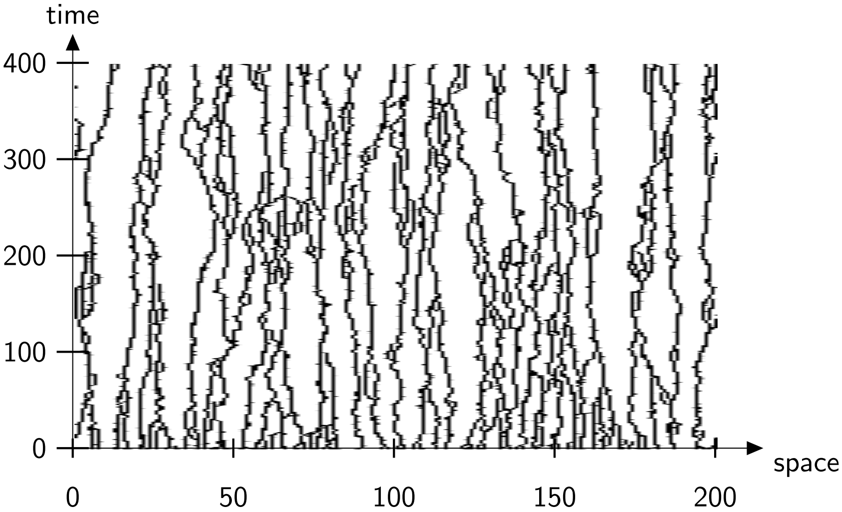
Killing

Define a killing map as

$$\mathtt{kill}_{ij}x(k) := \begin{cases} \big(1 - x(i)\big) \wedge x(j) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

which says that the particle at i, if present, kills any particle at j.

Branching and killing



A system with branching and killing.