Interacting Particle Systems: Almost sure uniqueness, pathwise duality, and the mean-field limit

Jan M. Swart

Lecture 1: Poisson Construction of Interacting Particle Systems

Probability kernels

E	a compact metrizable space,
$\mathcal{C}(E)$	the space of all continuous functions $f : E \to \mathbb{R}$, equipped with the supremumnorm $\|\cdot\|_\infty$,
$\mathcal{P}(E)$	space of probability measures on E, equipped with the topology of *weak convergence*, and the associated Borel-σ-algebra.

Def *probability kernel* K on E is a measurable map

$$E \ni x \mapsto K(x,\cdot\,) \in \mathcal{P}(E).$$

Def K is *continuous* if this map is continuous.

For any probability kernel $K(x,\mathrm{d}y)$ on E and measurable function $f : E \to \mathbb{R}$, we define

$$Kf(x) := \int_E K(x,\mathrm{d}y)f(y) \qquad (x \in E).$$

K continuous $\Leftrightarrow$ K maps $\mathcal{C}(E)$ into itself.

Random mapping representations

A probability kernel K on E is *deterministic* if it is of the form

$$K(x,\,\cdot\,) = \delta_{m(x)} \qquad (x \in E)$$

for some measurable map $m : E \to E$.

Note 1 Deterministic kernels are the extremal elements of the convex set of all probability kernels on E.
Note 2 K is continuous if and only if m is.

A *random mapping representation* of a probability kernel K is a random map M such that

$$K(x,\,\cdot\,) = \mathbb{P}\big[M(x) \in \,\cdot\,\big] \qquad (x \in E).$$

Feller semigroups

By definition, a *Feller semigroup* is a collection of probability kernels $(P_t)_{t\geq 0}$ on E such that:

(i) $P_0 = 1$ and $P_s P_t = P_{s+t}$ $(s,t\geq 0)$,

(ii) $E\times[0,\infty)\ni(x,s)\mapsto P_s(x,\cdot\,)\in\mathcal{P}(E)$ is continuous.

The *generator* of a Feller semigroup is the operator G defined as

$$Gf := \lim_{t\downarrow 0} t^{-1}(P_t f - f) \qquad\qquad (*),$$

with domain

$$\mathcal{D}(G) := \big\{f\in\mathcal{C}(E):\ \text{the limit in } (*) \text{ exists in the norm } \|\cdot\|_\infty\big\}.$$

G is *closed* and *densely defined*.

The *Hille-Yosida theorem* gives necessary and sufficient conditions for (the closure of) an operator to generate a Feller semigroup.

Feller processes

Let $(P_t)_{t\geq 0}$ be a Feller semigroup and let $\mu \in \mathcal{P}(E)$.

Then there exists a process $(X_t)_{t\geq 0}$, unique in law, such that

- The sample paths $t \mapsto X_t$ are a.s. *cadlag*, i.e., right-continuous and the left limit $X_{t-} := \lim_{s\uparrow t} X_s$ exists $\forall t > 0$.
- $\mathbb{P}[X_0 \in \cdot\,] = \mu$.
- $\mathbb{P}\big[X_u \in \cdot\,\big|\,(X_s)_{0\leq s\leq t}\big] = P_{u-t}(X_t,\,\cdot\,)$ a.s. $(t \leq u)$.

Such a process is called a *Feller process*.

Stochastic flows

Def A *stochastic flow* on E is a collection $(\mathbb{X}_{s,u})_{s\leq u}$ of random measurable maps $\mathbb{X}_{s,u}: E \to E$ such that

$$\mathbb{X}_{s,s} = 1 \quad \text{and} \quad \mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u} \qquad (s \leq t \leq u).$$

We say that $(\mathbb{X}_{s,u})_{s\leq u}$ has *independent increments* if

$$\mathbb{X}_{t_1,t_2}, \ldots, \mathbb{X}_{t_{n-1},t_n} \text{ are independent for all } t_1 < \cdots < t_n.$$

Let $(\mathbb{X}_{s,u})_{s\leq u}$ have independent increments, let $s \in \mathbb{R}$ and let X_0 be an E-valued random variable, independent of $(\mathbb{X}_{s,u})_{s\leq u}$. Then

$$X_t := \mathbb{X}_{s,s+t}(X_0) \qquad (t \geq 0)$$

defines a Markov process $(X_t)_{t\geq 0}$. Many Feller processes can be constructed from a stochastic flow. In this case,

$$P_{u-s}(x, \cdot\,) = \mathbb{P}\big[\mathbb{X}_{s,u}(x) \in \cdot\,\big] \qquad (x \in E,\ s \leq u),$$

so $\mathbb{X}_{s,u}$ is a random mapping representation of P_{u-s}.

Product spaces

S	a finite set called *local state space*,
Λ	a countable set called *lattice*,
S^Λ	the set of all $(x(i))_{i\in\Lambda}$ with $x(i) \in S\ \forall i \in \Lambda$, equipped with the *product topology*.

Note Tychonoff $\Rightarrow$ S^Λ compact.

Def A probability kernel K on S^Λ is *local* if

(i) K is continuous,

(ii) there exists a finite $\Delta \subset \Lambda$ such that $K(x, \cdot\,)$ is concentrated on $\{y \in S^\Lambda : y(i) = x(i)\ \forall i \in \Lambda\backslash\Delta\}$.

An *interacting particle system* is a Feller process with state space of the form S^Λ and generator of the form

$$Gf = \sum_{K \in \mathcal{K}} r_K (Kf - f),$$

where $\mathcal{K}$ is a countable collection of local probability kernels and $(r_K)_{K \in \mathcal{K}}$ are nonnegative rates.

Liggett (1972), Sullivan (1974,1976) gave sufficient conditions on the rates $(r_K)_{K \in \mathcal{K}}$ for the closure of G to generate a Feller semigroup. This yields *existence* and *distributional uniqueness*.

Starting with the work of **Harris (1972,1974)**, various authors have given constructions based on Poisson point sets called *graphical representations*. Such constructions yield a stochastic flow and *almost sure uniqueness*.

Continuous maps

Let $f : S^\Lambda \to S$. We say that a point $j \in \Lambda$ is *f-relevant* if

$$\exists x, y \in S^\Lambda \text{ s.t. } f(x) \neq f(y) \text{ and } x(k) = y(k) \ \forall k \neq j.$$

We write

$$\mathcal{R}(f) := \{j \in \Lambda : j \text{ is } f\text{-relevant}\}.$$

Lemma A function $f : S^\Lambda \to S$ is continuous iff

(i) $\mathcal{R}(f)$ is finite,

(ii) If $x, y \in S^\Lambda$ satisfy $x(j) = y(j)$ for all $j \in \mathcal{R}(f)$, then $f(x) = f(y)$.

Two discontinuous maps

Let $S = \{0,1\}$, $\Lambda = \mathbb{Z}$.

Example 1

$$f(x) = \begin{cases} 0 & \text{if } \inf\{j > 0 : x(j) = 1\} \in 2\mathbb{Z} \cup \{\infty\}, \\ 1 & \text{if } \inf\{j > 0 : x(j) = 1\} \in 2\mathbb{Z} + 1. \end{cases}$$

Now $\mathcal{R}(f) = \{1, 2, \ldots\}$ so condition (i) fails and f is discontinuous.

Example 2

$$f(x) = \begin{cases} 0 & \text{if } \{j > 0 : x(j) = 1\} \text{ is finite}, \\ 1 & \text{if } \{j > 0 : x(j) = 1\} \text{ is infinite}. \end{cases}$$

Now $\mathcal{R}(f) = \emptyset$ but condition (ii) fails so f is again discontinuous.

Local maps

For $m : S^\Lambda \to S^\Lambda$, we write $m(x) = \big(m[i](x)\big)_{i\in\Lambda}$ and

$$\mathcal{D}(m) := \big\{i \in \Lambda : m[i] \neq 1\big\}.$$

Def A map $m : S^\Lambda \to S^\Lambda$ is *local* if

(i) m is continuous,

(ii) $\mathcal{D}(m)$ is finite.

We will be interested in interacting particle systems with generator of the form

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m\big\{f\big(m(x)\big) - f\big(x\big)\big\} \qquad \text{(GEN)}$$

where $\mathcal{G}$ is a countable collection of local maps and $(r_m)_{m\in\mathcal{G}}$ are nonnegative rates.

Graphical representations

ρ measure on $\mathcal{G}$ defined by $\rho(\{m\}) := r_m$,
ℓ Lebesgue measure on $\mathbb{R}$,
π Poisson point subset of $\mathcal{G} \times \mathbb{R}$ with intensity $\rho \otimes \ell$.

$$\mathfrak{m}_t := \begin{cases} m & \text{if } (m,t) \in \pi, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem (Pathwise uniqueness) Assume

$$\sup_{i \in \Lambda} \sum_{\substack{m \in \mathcal{G} \\ \mathcal{D}(m) \ni i}} r_m\big(|\mathcal{R}(m[i])| + 1\big) < \infty. \qquad \text{(SUM)}$$

Then, almost surely, for all $s \in \mathbb{R}$ and $x \in S^\Lambda$, there exists a unique cadlag function $(X_t)_{t \geq s}$ that solves

$$X_s = x \quad \text{and} \quad X_t = \mathfrak{m}_t(X_{t-}) \qquad (t > s). \qquad \text{(EVOL)}$$

Theorem (Poisson construction) Assume (SUM). Then

$$\mathbb{X}_{s,u}(x) := X_u \quad \text{where} \quad (X_t)_{t\geq s} \text{ solves } (\mathrm{EVOL})$$

defines a stochastic flow $(\mathbb{X}_{s,u})_{s\leq u}$ with independent increments. Moreover, if $s \in \mathbb{R}$ and X_0 is an S^Λ-valued random variable, independent of π, then setting

$$X_t := \mathbb{X}_{s,s+t}(X_0) \qquad (t \geq 0)$$

defines a Feller process $(X_t)_{t\geq 0}$ with generator as in (GEN).

Three local maps

Let $S = \{0,1\}$. For all $i_1, i_2, i_3 \in \Lambda$, we define local maps by:

$$\mathtt{dth}_{i_1}(x)(j) := \begin{cases} 0 & \text{if } j = i_1, \\ x(j) & \text{otherwise.} \end{cases}$$

$$\mathtt{bra}_{i_1 i_2}(x)(k) := \begin{cases} x(i_1) \vee x(i_2) & \text{if } j = i_1, \\ x(j) & \text{otherwise.} \end{cases}$$

$$\mathtt{cob}_{i_1 i_2 i_3}(x)(j) := \begin{cases} x(i_1) \vee \big(x(i_2) \wedge x(i_3)\big) & \text{if } j = i_1 \\ x(j) & \text{otherwise.} \end{cases}$$

Three local maps

$$\mathcal{D}(\texttt{dth}_{i_1}) = \{i_1\}, \qquad \mathcal{R}(\texttt{dth}_{i_1}[i_1]) = \emptyset,$$

$$\mathcal{D}(\texttt{bra}_{i_1 i_2}) = \{i_1\}, \qquad \mathcal{R}(\texttt{dth}_{i_1 i_2}[i_1]) = \{i_1, i_2\},$$

$$\mathcal{D}(\texttt{cob}_{i_1 i_2 i_3}) = \{i_1\}, \qquad \mathcal{R}(\texttt{cob}_{i_1 i_2 i_3}[i_1]) = \{i_1, i_2, i_3\}.$$

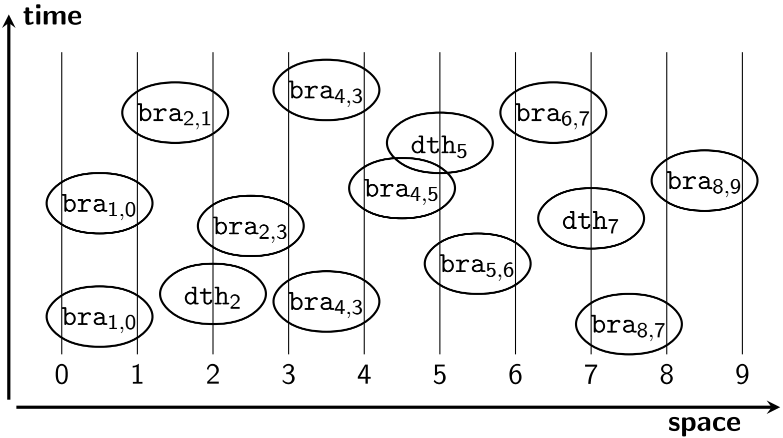
And in general $\mathcal{R}(m[j]) = \{j\}$ for $j \notin \mathcal{D}(m)$.

The contact process

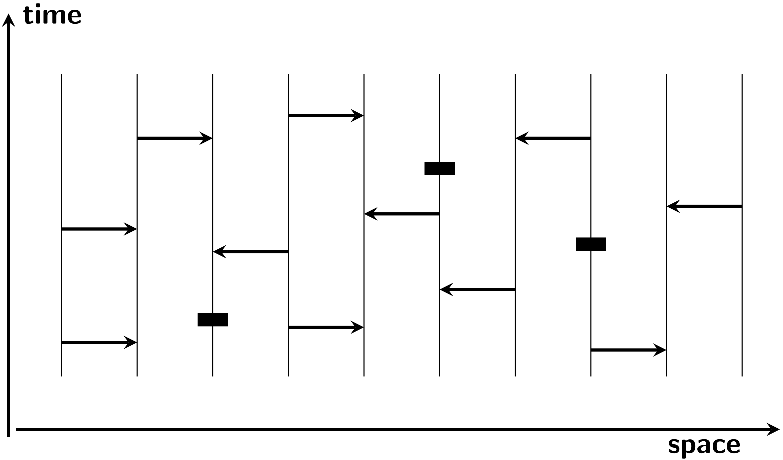
The *contact process* with *infection rate* λ on a graph $(\Lambda, \sim)$ is defined as follows:

- For each $i \in \Lambda$, with Poisson rate 1, we apply the map $\mathtt{dth}_i$.
- For each $i \in \Lambda$, with Poisson rate λ, we pick $j \sim i$ uniformly at random and apply the map $\mathtt{bra}_{ij}$.

The contact process



The contact process



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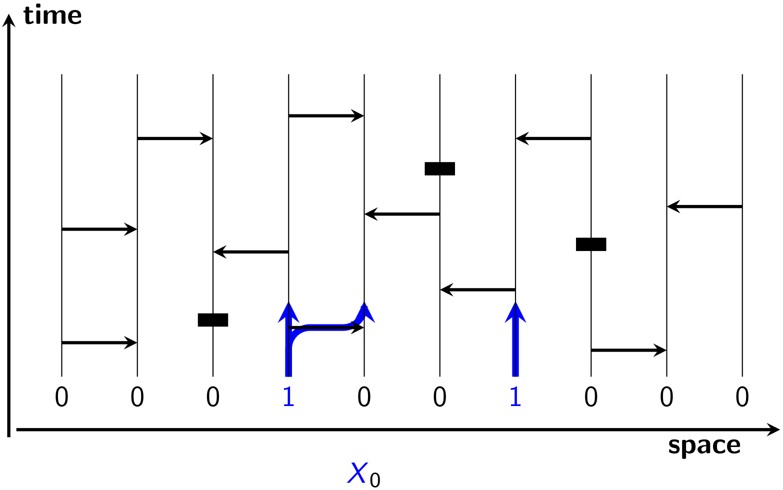

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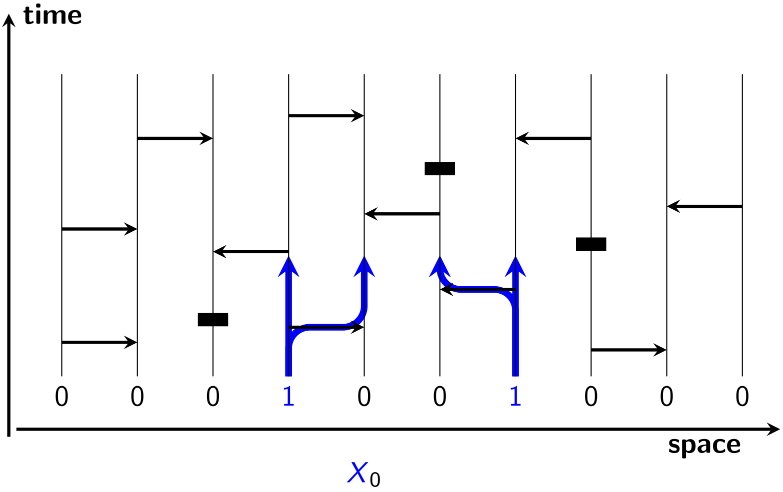

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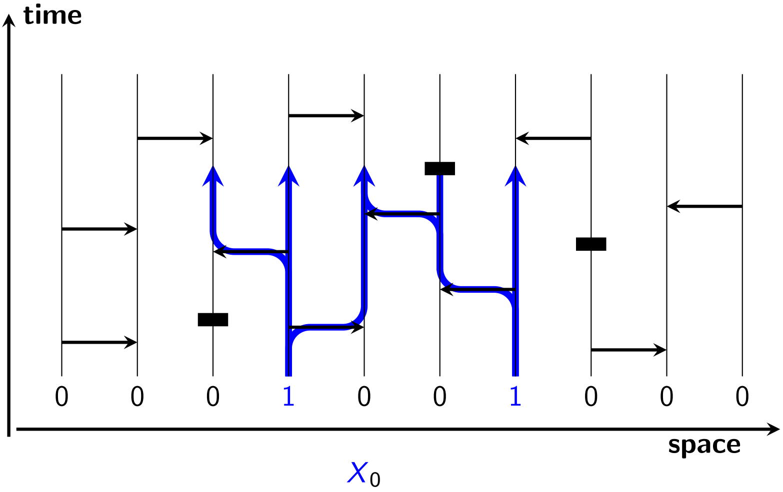

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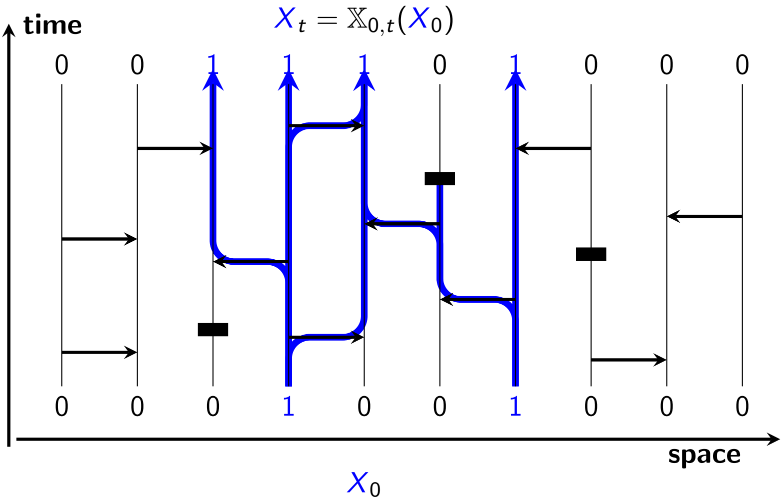
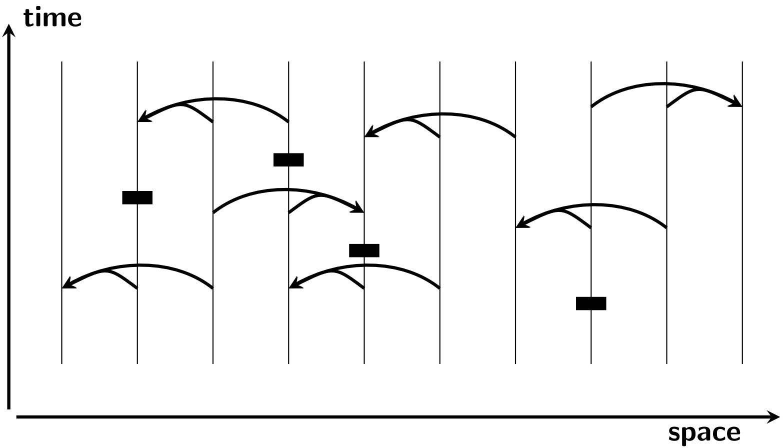


A cooperative contact process

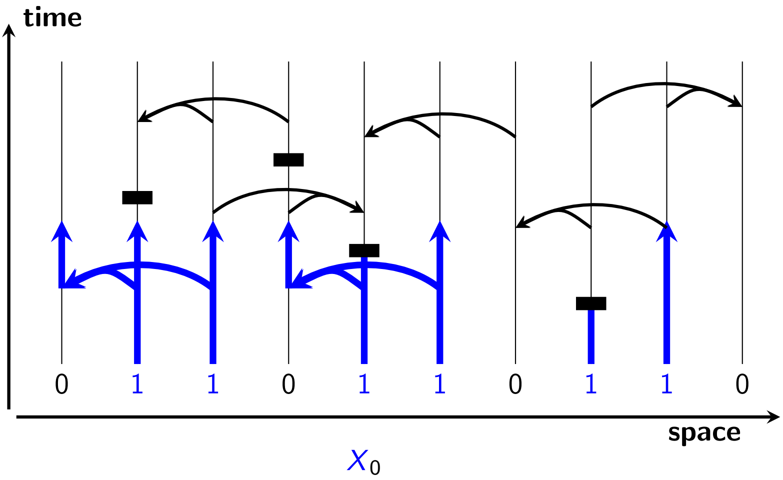
We define a *cooperative contact process* with *cooperative branching rate* λ on a graph $(\Lambda, \sim)$ as follows:

- For each $i \in \Lambda$, with Poisson rate 1, we apply the map $\texttt{dth}_i$.
- For each $i \in \Lambda$, with Poisson rate λ, we pick $i \sim j \sim k$ with $k \neq i$ uniformly at random and apply the map $\texttt{cob}_{ijk}$.

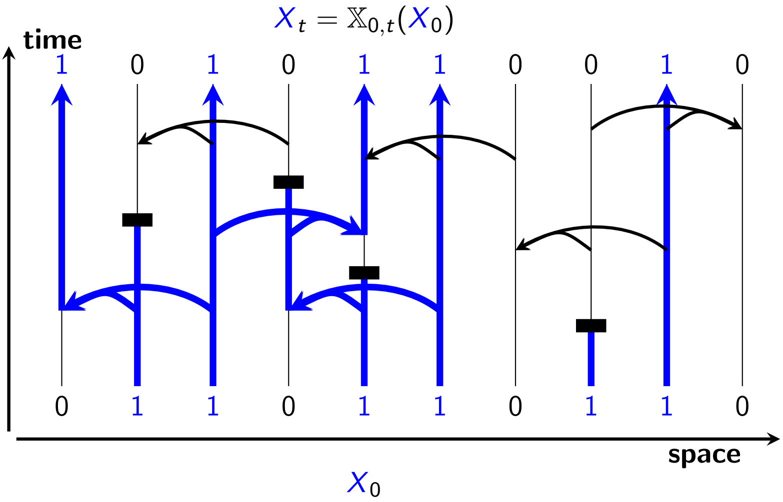
A cooperative contact process



A cooperative contact process



A cooperative contact process



Proof of the theorems

The condition (SUM) implies that the following constants are finite:

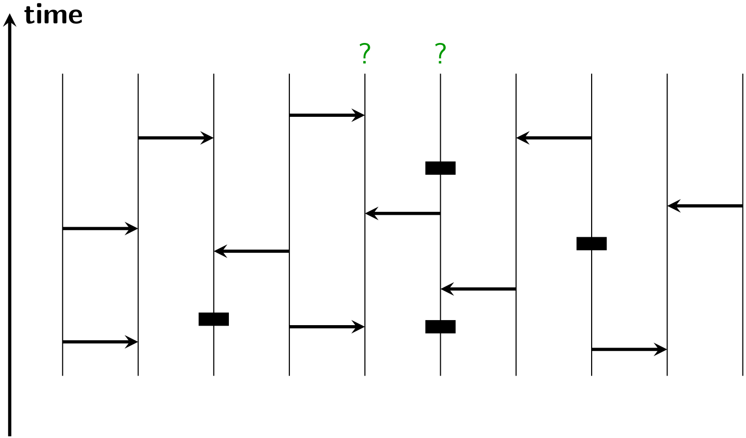
$$K_0 := \sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G} \\ \mathcal{D}(m)\ni i}} r_m,$$

$$K := \sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G} \\ \mathcal{D}(m)\ni i}} r_m\big(|\mathcal{R}(m[i])| - 1\big).$$

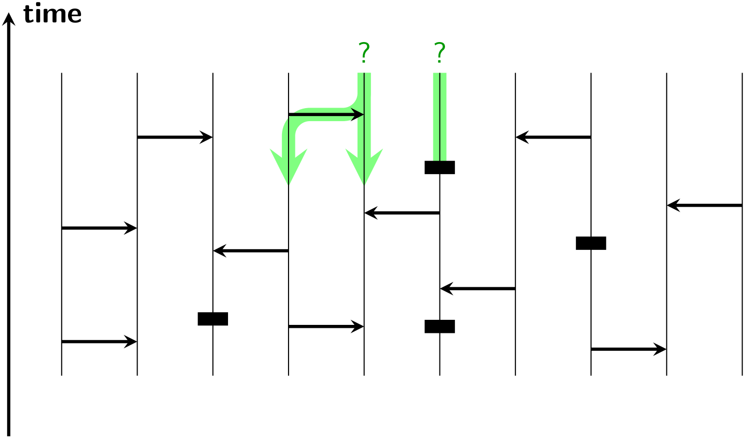
The finiteness of K_0 implies that in finite time intervals, only finitely many maps are applied that can change the state of a given site.

But $K_0 < \infty$ is not enough to conclude that the Poisson construction is well-defined.

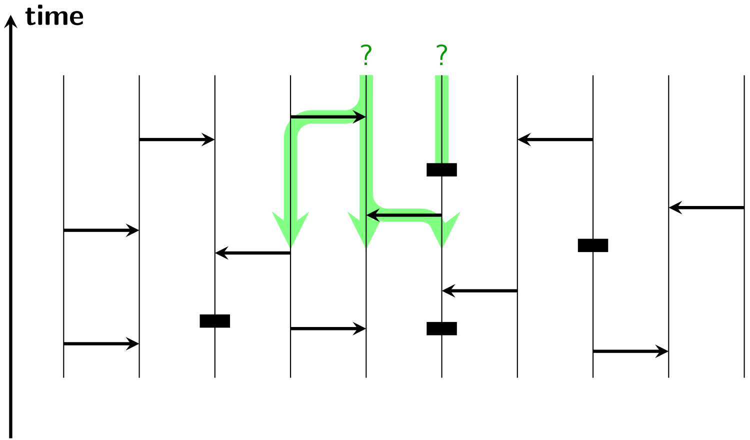
Proof of the theorems



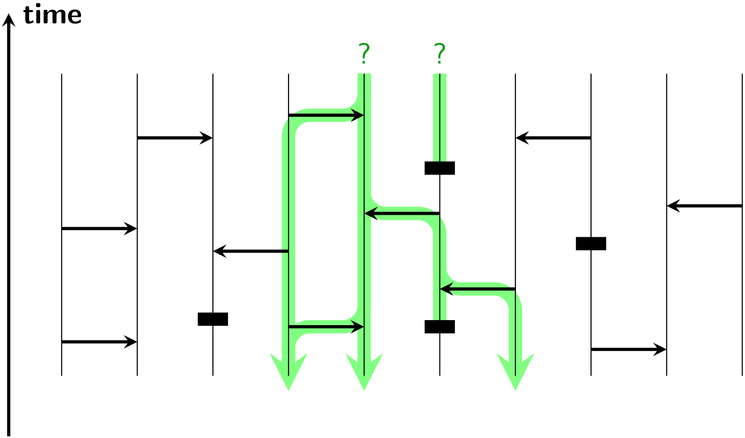
Proof of the theorems



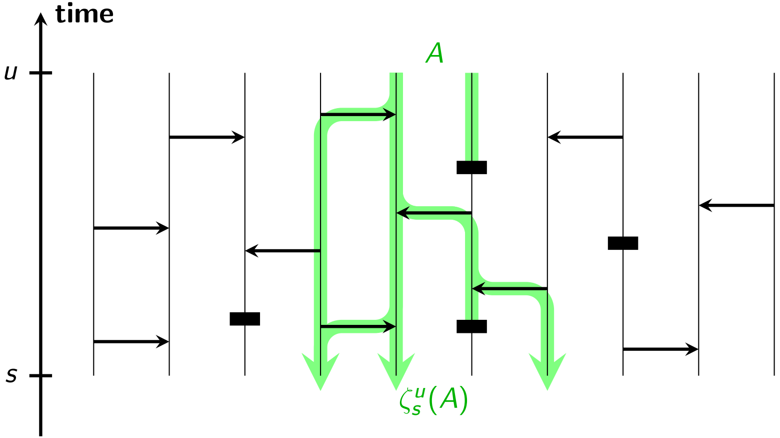
Proof of the theorems



Proof of the theorems



Proof of the theorems



For each $u \in \mathbb{R}$ and finite $A \subset \Lambda$, we define a set-valued process

$$\big(\zeta^u_s(A)\big)_{s\leq u}$$

with $\zeta^u_s(A) := A$ and

$$\zeta^u_{t-}(A) := \big(\zeta^u_t(A)\backslash\mathcal{D}(m)\big) \cup \bigcup_{i\in\mathcal{D}(m)\cap\zeta^u_t(A)} \mathcal{R}(m[i]).$$

Under the condition (SUM), it can be shown that

$$\mathbb{E}\big[\big|\zeta^u_s(A)\big|\big] \leq |A|e^{K(u-s)} \qquad (s \leq u).$$

It follows that $\mathbb{X}_{s,u}$ is a well-defined continuous map with

$$\mathcal{R}\big(\mathbb{X}_{s,u}[i]\big) \subset \zeta^u_s\big(\{i\}\big) \qquad (i \in \Lambda,\ s \leq u).$$

Proof of the theorems

It can now be shown that

$$P_{u-s}(x,\,\cdot\,) := \mathbb{P}\big[\mathbb{X}_{s,u}(x) \in \cdot\,\big] \qquad (x \in S^{\Lambda},\ s \leq u)$$

defines a Feller semigroup $(P_t)_{t \geq 0}$ with generator G.

To prove the continuity of $(x,t) \mapsto P_t(x,\,\cdot\,)$, we use *coupling from the past*. More precisely, we show that

$$\mathbb{X}_{-t_n,0}(x_n) \underset{n\to\infty}{\longrightarrow} \mathbb{X}_{-t,0} \quad \text{a.s.} \quad \text{as } t_n \to t \text{ and } x_n \to x.$$

For the details, see my lecture notes. ∎

Ergodicity

Theorem (Ergodicity) Assume that $K < 0$. Then

$$\lim_{s\to-\infty} \zeta^u_s\big(\{i\}\big) = \emptyset \quad \text{a.s.} \qquad (i \in \Lambda,\ u \in \mathbb{R}).$$

Moreover, the interacting particle system X has a unique invariant law ν, and the process started in any $X_0 = x$ satisfies

$$\mathbb{P}^x\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \nu \qquad (x \in S^\Lambda).$$

Proof Since $\mathcal{R}(\mathbb{X}_{s,u}[i]) \subset \zeta^u_s\big(\{i\}\big)$, the map $\mathbb{X}_{s,u}$ converges to a constant as $s \to -\infty$. As a consequence,

$$\lim_{t\to\infty} \mathbb{P}^x\big[X_t \in \cdot\,\big] = \lim_{t\to\infty} \mathbb{P}\big[\mathbb{X}_{-t,0}(x) \in \cdot\,\big]$$

does not depend on x and there exists a unique solution to

$$X_t = \mathfrak{m}_t(X_{t-}) \qquad (t \in \mathbb{R}).$$

The cooperative contact process

Application For $\lambda < 1/2$, the only invariant law of the cooperative contact process on $\mathbb{Z}$, which has the generator

$$\begin{aligned} Gf(x) := \tfrac{1}{2}\lambda \sum_{i\in\mathbb{Z}} \big\{ f\big(\mathtt{cob}_{i,i-1,i-2}(x)\big) - f\big(x\big)\big\} \\ +\tfrac{1}{2}\lambda \sum_{i\in\mathbb{Z}} \big\{ f\big(\mathtt{cob}_{i,i+1,i+2}(x)\big) - f\big(x\big)\big\} \\ +\sum_{i\in\mathbb{Z}} \big\{ f\big(\mathtt{dth}_{i}(x)\big) - f\big(x\big)\big\}, \end{aligned}$$

is the delta measure $\delta_{\underline{0}}$ on the all-zero state, and

$$\mathbb{P}^x\big[X_t(i) = 0\big] \underset{t\to\infty}{\longrightarrow} 1$$

for arbitrary initial states x.