Interacting Particle Systems: Almost sure uniqueness, pathwise duality, and the mean-field limit

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Lecture 2: Pathwise duality

Poisson construction

S	a finite set called *local state space*,
Λ	a countable set called *lattice*,
$\mathcal{G}$	a countable set of local maps $m : S^\Lambda \to S^\Lambda$,
$(r_m)_{m\in\mathcal{G}}$	nonnegative rates,
π	Poisson subset of $\mathcal{G}\times\mathbb{R}$ with intensity $r_m \mathrm{d}t$,
$(\mathbb{X}_{s,u})_{s\leq u}$	stochastic flow constructed from π.

Theorem Assuming (SUM), the process $(X_t)_{t\geq 0}$ defined as

$$X_t := \mathbb{X}_{s,s+t}(X_0) \quad (t \geq 0)$$

is the interacting particle system with generator

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\}. \qquad \text{(GEN)}$$

Ergodicity

Theorem Assume that the constant

$$K := \sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G}\\ \mathcal{D}(m)\ni i}} r_m\big(|\mathcal{R}(m[i])| - 1\big)$$

satisfies $K < 0$. Then the interacting particle system X has a unique invariant law ν, and the process started in any $X_0 = x$ satisfies

$$\mathbb{P}^x\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \nu \qquad (x \in S^\Lambda).$$

Proof For each $i \in \Lambda$, $u \in \mathbb{R}$, there exists a random $s > -\infty$ such that $\mathbb{X}_{s,u}[i](x)$ is constant as a function of x. In other words, if $u - s$ is large, then $\mathbb{X}_{s,u}[i]$ "forgets" the initial state. ∎

The backwards in time picture

We may view $\mathbb{X}_{s,u}[i]$ as a map from $S^{\mathcal{R}(\mathbb{X}_{s,u}[i])}$ to S.

Observation For fixed $u \in \mathbb{R}$ and $i \in \Lambda$, the process

$$\big(\mathcal{R}(\mathbb{X}_{u-t,u}[i]), \mathbb{X}_{u-t,u}[i]\big)_{t \geq 0}$$

is a Markov process with countable state space consisting of all pairs $(\Delta, \mathbb{X})$ where $\Delta \subset \Lambda$ is finite and $\mathbb{X} : S^{\Delta} \to S$.

The dual picture

One has

$$\mathcal{R}(\mathbb{X}_{u-t,u}[i]) \subset \zeta^u_{u-t}(\{i\}) \quad \text{with} \quad \mathbb{E}\big[\big|\zeta^u_{u-t}(\{i\})\big|\big] \leq e^{Kt}.$$

Therefore $K < 0$ implies that for t large enough, $\mathcal{R}(\mathbb{X}_{u-t,u}[i]) = \emptyset$ and hence $\mathbb{X}_{u-t,u}[i]$ is constant, which means that the interacting particle system "forgets" its initial state and is ergodic.

On the other hand, for $K > 0$, it may happen that the system is nonergodic and there are multiple invariant laws.
Much information is contained in the "backward in time" Markov process

$$\big(\mathcal{R}(\mathbb{X}_{u-t,u}[i]), \mathbb{X}_{u-t,u}[i]\big)_{t\geq 0}.$$

It depends on the details of the system under consideration how tractable this backward process is.

Additive systems

Let $\underline{0}$ denote the configuration that is identically zero.
A map $m : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ is *additive* if

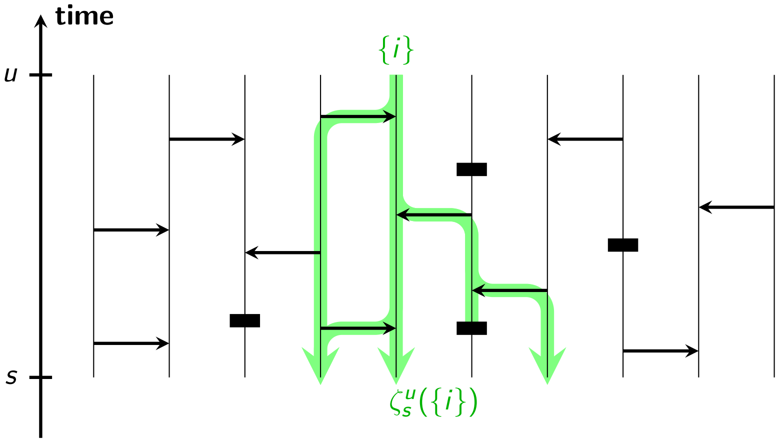
(i) $m(\underline{0}) = \underline{0}$,

(ii) $m(x \vee y) = m(x) \vee m(y) \qquad \big(x, y \in \{0,1\}^\Lambda\big)$.

Lemma If all maps in $\mathcal{G}$ are additive, then

$$\mathcal{R}\big(\mathbb{X}_{s,u}[i]\big) = \zeta_s^u(\{i\}) \quad \text{and} \quad \mathbb{X}_{s,u}[i](x) = \bigvee_{j \in \zeta_s^u(\{i\})} x(j).$$

The contact process



$\mathbb{X}_{s,u}(x)(i) = 1$ if and only if $x(j) = 1$ for at least one $j \in \zeta_s^u(\{i\})$.

Self-duality of the contact process

For the contact process, setting

$$Y_t(i) := \begin{cases} 1 & \text{if } i \in \zeta^u_{u-t}(A), \\ 0 & \text{otherwise.} \end{cases}$$

defines a "dual" contact process, albeit with left-continuous paths.

Generalising our construction, starting from a graphical representation π, we can define two stochastic flows $(\mathbb{X}^{\pm}_{s,u})_{s \leq u}$, where:

$$\mathbb{X}^{-}_{s,u} := \text{ the concatenation of all } m \text{ with } (m,t) \in \pi,\ \ t \in [s,u),$$
$$\mathbb{X}^{+}_{s,u} := \text{ the concatenation of all } m \text{ with } (m,t) \in \pi,\ \ t \in (s,u].$$

In particular, $\mathbb{X}^{+}_{s,u} = \mathbb{X}_{s,u}$ as before.

Additive systems duality

Let $\Psi : \{0,1\}^\Lambda \times \{0,1\}^\Lambda \to \{0,1\}$ be the duality function

$$\Psi(x,y) := \bigvee_{i\in\Lambda} x(i)y(i) \qquad \big(x,y \in \{0,1\}^\Lambda\big).$$

Lemma For each additive local map $m : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$, there exists a unique "dual" map $\hat m : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ such that

$$\Psi\big(m(x),y\big) = \Psi\big(x,\hat m(y)\big) \qquad \big(x,y \in \{0,1\}^\Lambda\big),$$

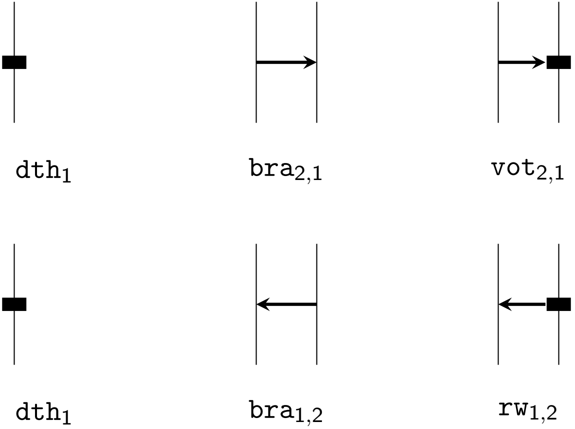
and $\hat m$ is also an additive local map.

Pathwise duality Let $\hat\pi := \big\{(\hat m,-t) : (m,t) \in \pi\big\}$.
Construct $(\mathbb{X}^\pm_{s,u})_{s\leq u}$ from π and $(\mathbb{Y}^\pm_{s,u})_{s\leq u}$ from $\hat\pi$. Then:

$$\Psi\big(\mathbb{X}^\pm_{s,u}(x),y\big) = \Psi\big(x,\mathbb{Y}^\mp_{-u,-s}(y)\big) \qquad \big(x,y \in \{0,1\}^\Lambda\big).$$

Dual local maps

Dual maps can be found using the recipe: reverse the arrows, keep the blocking symbols.



Cancellative systems

Let $(x \oplus y)(i) := x(i) + y(i) \mod(2)$.

A map $m : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ is *cancellative* if

(i) $m(\underline{0}) = \underline{0}$,

(ii) $m(x \oplus y) = m(x) \oplus m(y) \qquad \big(x, y \in \{0,1\}^\Lambda\big)$.

Let $|y| := \sum_{i\in\Lambda} y(i)$ and let Ψ be the duality function

$$\Psi(x,y) := \bigoplus_{i\in\Lambda} x(i)y(i) \qquad \big(x,y \in \{0,1\}^\Lambda,\ |y| < \infty\big).$$

Lemma For each cancellative local map $m : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$, there exists a unique "dual" map $\hat{m} : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ such that

$$\Psi\big(m(x), y\big) = \Psi\big(x, \hat{m}(y)\big) \qquad \big(x,y \in \{0,1\}^\Lambda,\ |y| < \infty\big),$$

and $\hat{m}$ is also a cancellative local map.

Cancellative duality

Pathwise duality Let $\hat{\pi} := \big\{(\hat{m}, -t) : (m, t) \in \pi\big\}$.
Construct $(\mathbb{X}^{\pm}_{s,u})_{s\leq u}$ from π and $(\mathbb{Y}^{\pm}_{s,u})_{s\leq u}$ from $\hat{\pi}$. Then:

$$\Psi\big(\mathbb{X}^{\pm}_{s,u}(x), y\big) = \Psi\big(x, \mathbb{Y}^{\mp}_{-u,-s}(y)\big) \qquad \big(x, y \in \{0,1\}^{\Lambda},\ |y| < \infty\big).$$

Now the "backward in time" Markov process

$$\big(\mathcal{R}(\mathbb{X}_{u-t,u}[i]), \mathbb{X}_{u-t,u}[i]\big)_{t\geq 0}$$

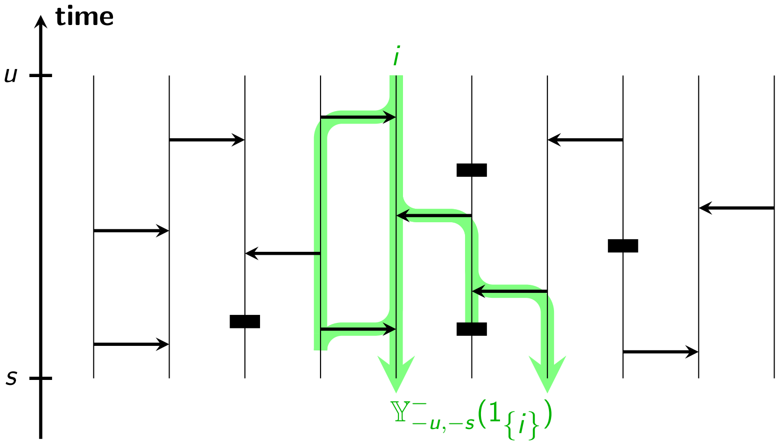
has the simple form

$$1_{\mathcal{R}(\mathbb{X}_{s,u}[i])} = \mathbb{Y}^{-}_{-u,-s}(1_{\{i\}})$$

and

$$\mathbb{X}_{s,u}[i](x) = \bigoplus_{i\in\mathcal{R}(\mathbb{X}_{s,u}[i])} x(j).$$

The annihilating branching process



$\mathbb{X}_{s,u}(x)(i) = 1$ if and only if $\big|\{j \in \mathcal{R}(\mathbb{X}_{s,u}[i]) : x(j) = 1\}\big|$ is odd.

Threshold voter model

For any graph $(\Lambda, \sim)$, $i \in \Lambda$, and $x \in \{0,1\}^\Lambda$, let

$$\mathtt{thresh}^0_i(x)(j) := \begin{cases} \bigwedge_{k:\, k\sim i} x(k) & \text{if } j = i, \\ x(j) & \text{otherwise.} \end{cases}$$

$$\mathtt{thresh}^1_i(x)(j) := \begin{cases} \bigvee_{k:\, k\sim i} x(k) & \text{if } j = i, \\ x(j) & \text{otherwise.} \end{cases}$$

The *threshold voter model* is defined by the generator

$$G_{\mathrm{thresh}} f(x) := \sum_{i\in\Lambda} \big\{ f\big(\mathtt{thresh}^0_i(x)\big) - f\big(x\big) \big\} + \sum_{i\in\Lambda} \big\{ f\big(\mathtt{thresh}^1_i(x)\big) - f\big(x\big) \big\}.$$

Note $\mathtt{thresh}^1_i$ is additive but $\mathtt{thresh}^0_i$ is only monotone.

Threshold voter model

Let $\mathcal{N}_i := \{i\} \cup \{k : k \sim i\}$, $N := |\mathcal{N}_i|$, and $\mathcal{D}_i := \{\Delta \subset \mathcal{N}_i : |\Delta| \text{ is even}\}$. We define

$$\texttt{flip}_i^{\Delta}(x)(j) := \begin{cases} \bigoplus_{k\in\Delta} x(k) & \text{if } j = i, \\ x(j) & \text{otherwise.} \end{cases}$$

Then

$$G_{\text{thresh}} f(x) = 2^{-N+2} \sum_{i\in\Lambda} \sum_{\Delta\subset\mathcal{D}_i} \{f(\texttt{flip}_i^{\Delta}(x)) - f(x)\}.$$

Note $\texttt{flip}_i^{\Delta}$ is cancellative but not monotone.

Threshold voter model

Trivially, $\delta_{\underline{0}}$ and $\delta_{\underline{1}}$ are invariant laws.

Def a *coexisting* probability law gives zero probability to $\{\underline{0}, \underline{1}\}$.

Complete convergence [Handjani 1999] The threshold voter model on $\mathbb{Z}^d$ with $d \geq 2$ has a unique coexisting invariant law ν. Setting

$$p_0 := \mathbb{P}\big[X_t = \underline{0} \text{ for some } t < \infty\big],$$
$$p_1 := \mathbb{P}\big[X_t = \underline{1} \text{ for some } t < \infty\big],$$

the process starting in a general initial law satisfies

$$\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} p_0\delta_{\underline{0}} + p_1\delta_{\underline{1}} + (1 - p_0 - p_1)\nu.$$

Threshold voter model

Cancellative duality gives

$$\Psi(X_t, y) = \Psi\big(\mathbb{X}^+_{0,t}(X_0), y\big) = \Psi\big(\mathbb{X}^+_{0,s}(X_0), \mathbb{Y}^-_{-t,-s}(y)\big).$$

Denote the survival probability of the dual by:

$$\rho(y) := \lim_{t\to\infty} \mathbb{P}\big[\mathbb{Y}^-_{-t,0}(y) \neq \underline{0}\big].$$

The proof consists of showing that

$$\lim_{t\to\infty} \mathbb{E}\big[\Psi(X_t, y)\big] = p_0\Psi(\underline{0}, y) + p_1\Psi(\underline{1}, y) + \tfrac{1}{2}(1 - p_0 - p_1)\rho(y).$$

Conditional on non-absorption of $\mathbb{Y}^-_{-t,-s}(y)$ in $\underline{0}$,
the 1's spread over all of space,
Conditional on non-absorption of $\mathbb{X}^+_{0,s}(X_0)$ in $\underline{0}$ or $\underline{1}$,
there are many 0's close to 1's.

Generalisations of additive and cancellative duality

Remark More generally, one can look at duality functions of the form

$$\Psi(x,y) := \sum_{i\in\Lambda} \psi\big(x(i), y(i)\big),$$

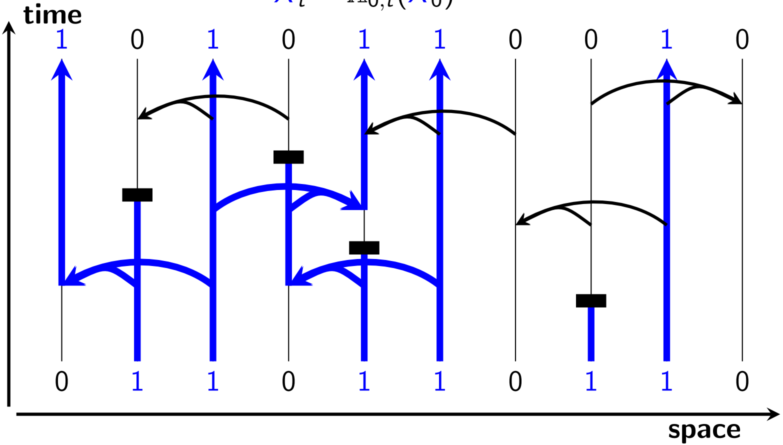
where $\psi : S \times S \to T$ is a "local" duality function taking values in a commutative monoid $(T,+)$, and $\sum_{i\in\Lambda}$ is the sum in $(T,+)$.

$S = T = \{0,1\}$ and $x + y := x \vee y$ gives additive duality.
$S = T = \{0,1\}$ and $x + y := x \oplus y$ gives cancellative duality.
In both cases $\psi(x,y) = x \cdot y$.

For local state spaces with two elements this seem to be the only two "reasonable" choices but already for $|S| = 3$ there are many more possibilities. See **[Latz & S '21]**.

The cooperative contact process

$X_t = \mathbb{X}_{0,t}(X_0)$



X_0

A particle system using the maps `cob` and `dth`.

The cooperative branching map is monotone but not additive.

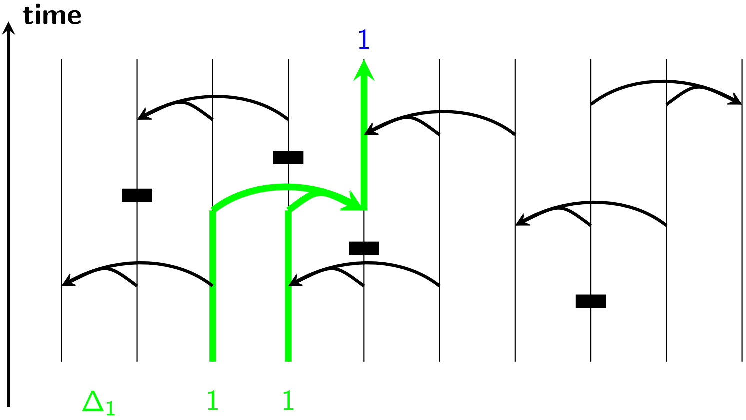
$$\mathrm{cob}_{i_1 i_2 i_3}(x)(j) := \begin{cases} x(i_1) \vee \big(x(i_2) \wedge x(i_3)\big) & \text{if } j = i_1 \\ x(j) & \text{otherwise.} \end{cases}$$

We can write

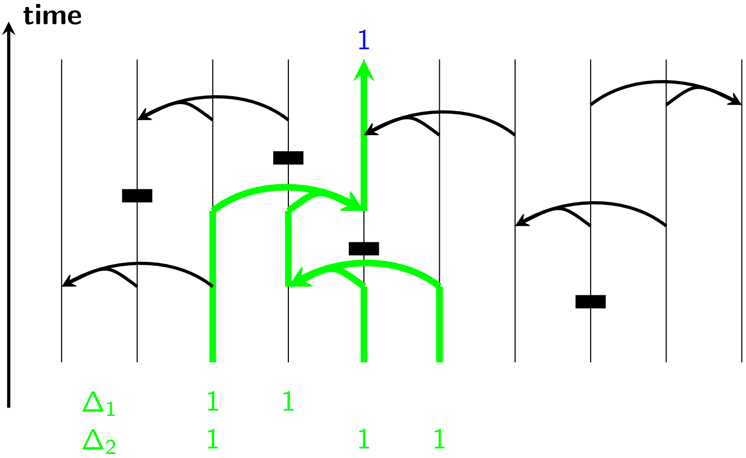
$$\mathbb{X}_{s,u}[i](x) = \bigvee_{\Delta \in \mathcal{Z}_{s,u}(i)} \bigwedge_{j \in \Delta} x(j),$$

where $\mathcal{Z}_{s,u}(i)$ is the set of "minimal configurations" Δ which need to be 1 in order for $\mathbb{X}_{s,u}[i](x)$ to be 1.

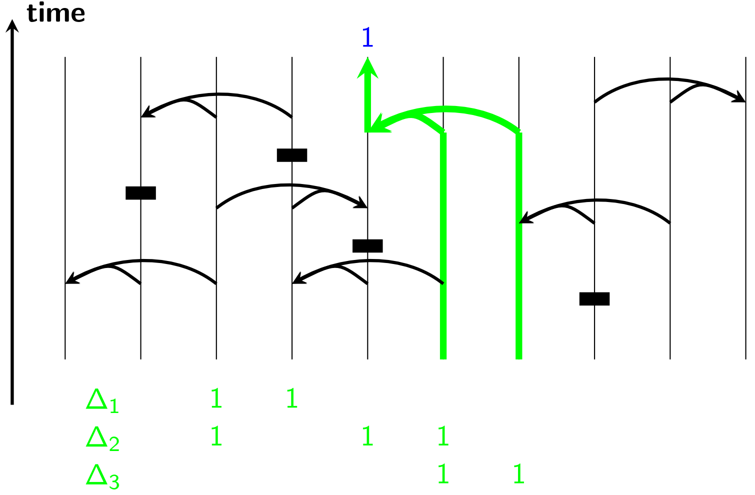
Minimal configurations



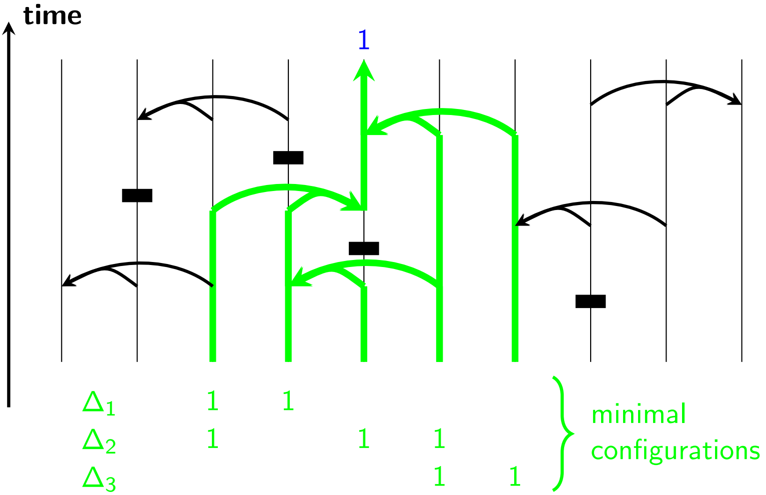
Minimal configurations



Minimal configurations



Minimal configurations



Monotone systems duality

Recall the "backward in time" Markov process

$$\big(\mathcal{R}(\mathbb{X}_{u-t,u}[i]), \mathbb{X}_{u-t,u}[i]\big)_{t\geq 0}.$$

For monotone systems,

$$\mathcal{R}\big(\mathbb{X}_{s,u}[i]\big) = \bigcup_{\Delta\in\mathcal{Z}_{s,u}(i)} \Delta,$$

$$\mathbb{X}_{s,u}[i](x) = \bigvee_{\Delta\in\mathcal{Z}_{s,u}(i)} \bigwedge_{j\in\Delta} x(j).$$

These sort of dual processes have first been used by **Gray (1986)**.

In general, these sort of dual processes are hard to control.
In the next lecture, we will study "backward in time" processes in the mean-field limit.