Interacting Particle Systems: Almost sure uniqueness, pathwise duality, and the mean-field limit

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Lecture 4: The cooperative branching process

The cooperative contact process

Let $S := \{0,1\}$. For all $i_1, i_2, i_3 \in \Lambda$, we define local maps by:

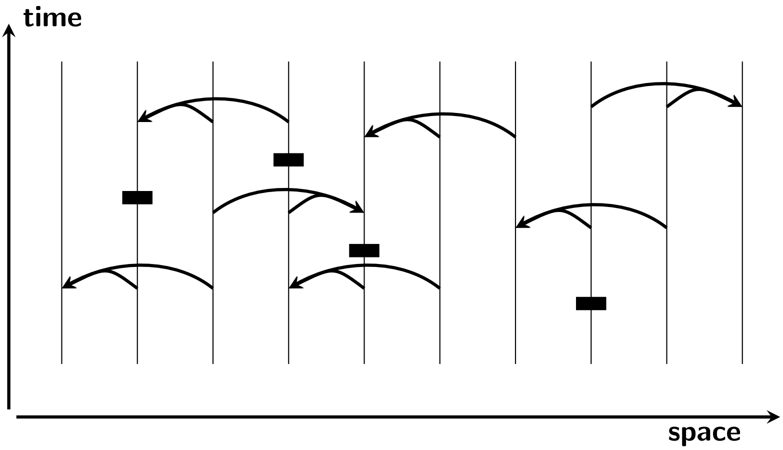
$$\mathtt{dth}_{i_1}(x)(j) := \begin{cases} 0 & \text{if } j = i_1, \\ x(j) & \text{otherwise.} \end{cases}$$

$$\mathtt{cob}_{i_1 i_2 i_3}(x)(j) := \begin{cases} x(i_1) \vee \big(x(i_2) \wedge x(i_3)\big) & \text{if } j = i_1 \\ x(j) & \text{otherwise.} \end{cases}$$

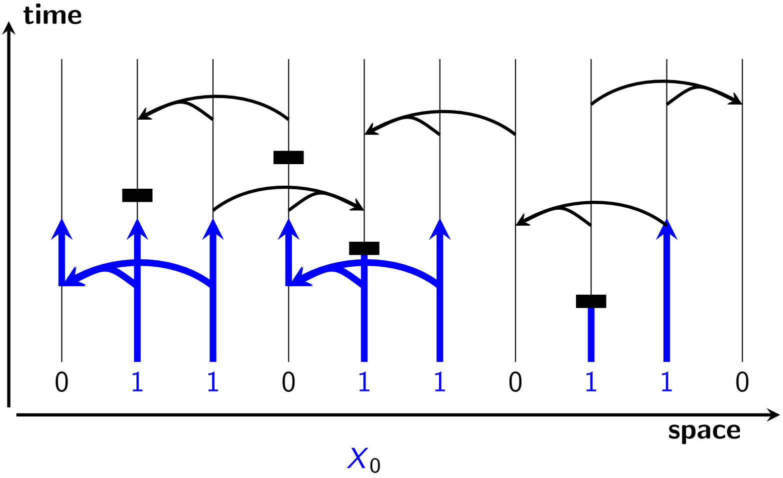
The *cooperative contact process* with *cooperative branching rate* λ on a graph $(\Lambda, \sim)$ evolves as follows:

- For each $i_1 \in \Lambda$, with Poisson rate 1, we apply the map $\mathtt{dth}_{i_1}$.
- For each $i_1 \in \Lambda$, with Poisson rate λ, we pick $i_1 \sim i_2 \sim i_3$ with $i_3 \neq i_1$ uniformly at random and apply the map $\mathtt{cob}_{i_1 i_2 i_3}$.

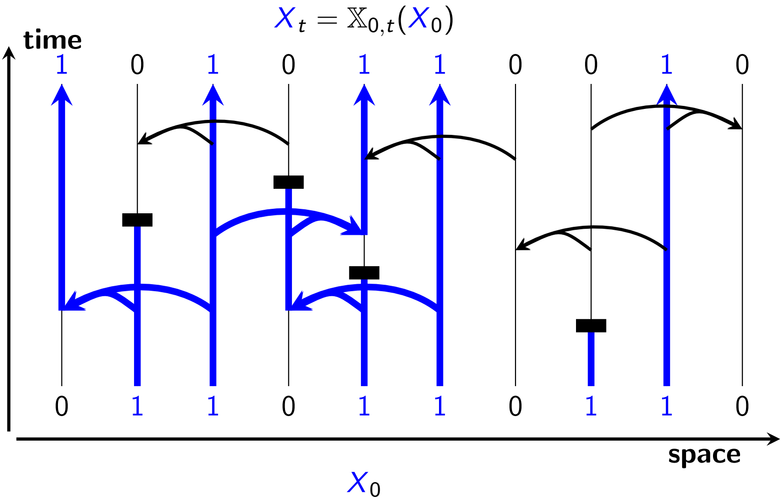
A cooperative contact process



A cooperative contact process



A cooperative contact process



The backward picture

We are interested in the "backward in time" process

$$\big(\mathcal{R}(\mathbb{X}_{-t,0}[i]), \mathbb{X}_{-t,0}[i]\big)_{t\geq 0}.$$

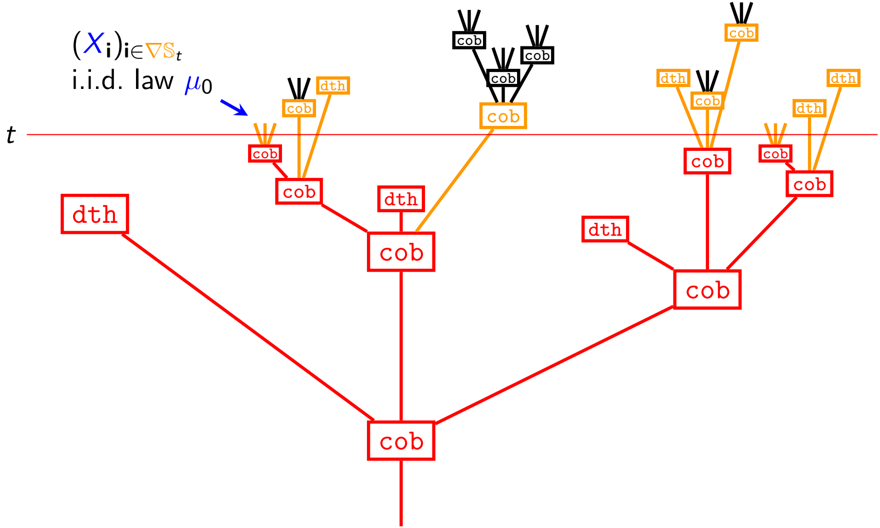
In the mean-field limit, the "backward in time" process converges, in an appropriate sense, to the process

$$\big(\Gamma_{\mathbb{S}_t}\big)_{t\geq 0},$$

where $(\mathbb{S}_t)_{t\geq 0}$ is the family tree of a branching process and $\Gamma_{\mathbb{S}_t}$ is the concatenation of maps attached to the nodes of this tree.

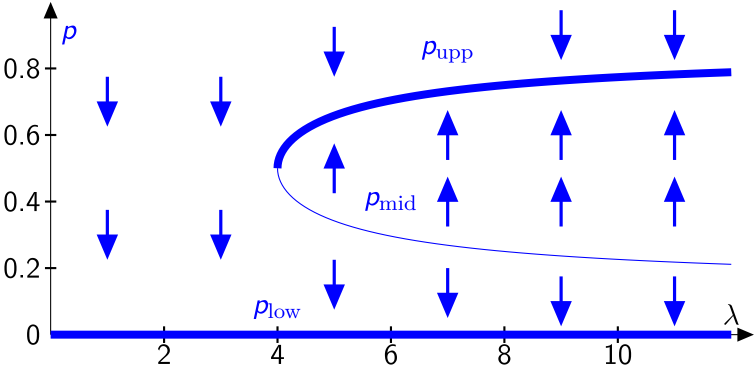
Baake, Cordero, & Hummel '21 have studied $\Gamma_{\mathbb{S}_t}$ from a biological point of view, motivated by the family tree of a diploid organism carrying a recessive advantageous gene.

A recursive tree representation



$$X_\varnothing = \Gamma_{\mathbb{S}_t}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}\big) \qquad \text{law } \mu_t = \mathbf{T}_t(\mu_0)$$

Fixed points of the mean-field equation



Density $p_{\rm xxx} := \nu_{\rm xxx}(\{1\})$ of fixed points $\nu_{\rm xxx}$ of $(\mathbf{T}_t)_{t\geq 0}$.

The ergodic regime

The system is ergodic in all of the following cases:
One has $K < 0$ for $\lambda < 1/2$.
The branching process $(\nabla\mathbb{S}_t)_{t\geq 0}$ dies our a.s. iff $\lambda \leq 1/2$.
The functions $\Gamma_{\mathbb{S}_t}$ are constant for t large enough iff $\lambda < 4$.

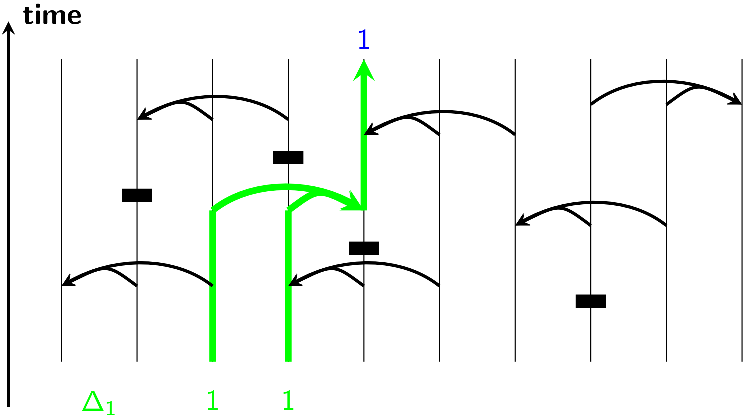
On the other hand, for $\lambda \geq 4$, there are multiple invariant laws and the functions $\Gamma_{\mathbb{S}_t}$ do not a.s. converge to a constant as $t \to \infty$.

Recall that we can write

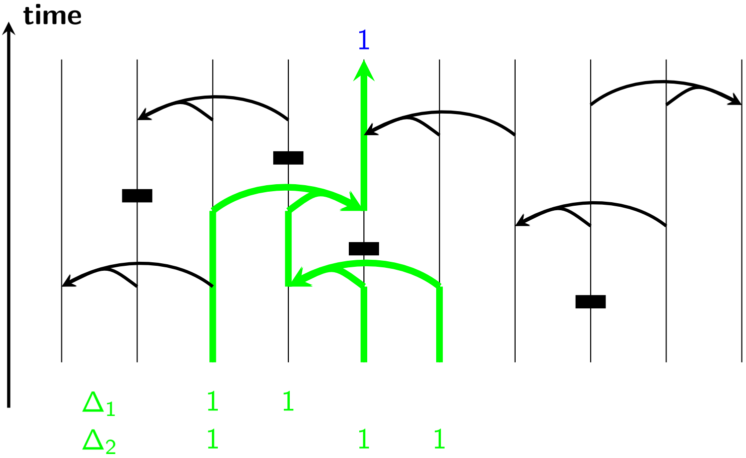
$$\mathbb{X}_{s,u}[i](x) = \bigvee_{\Delta \in \mathcal{Z}_{s,u}(i)} \bigwedge_{j \in \Delta} x(j),$$

where $\mathcal{Z}_{s,u}(i)$ is the set of "minimal configurations" Δ which need to be 1 in order for $\mathbb{X}_{s,u}[i](x)$ to be 1.

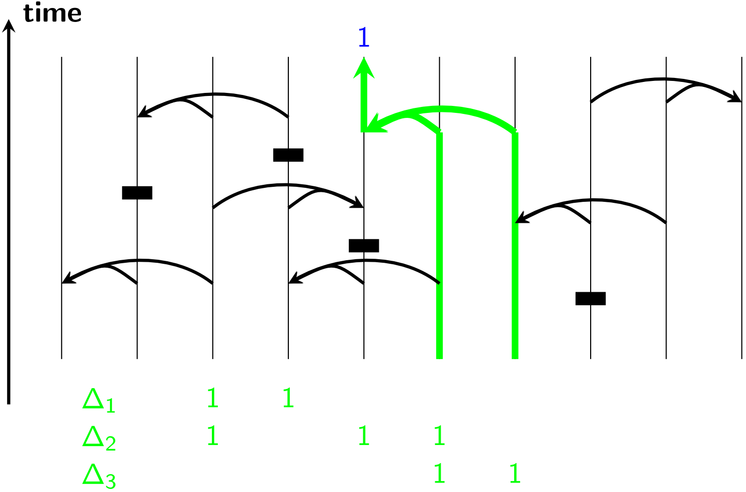
Minimal configurations



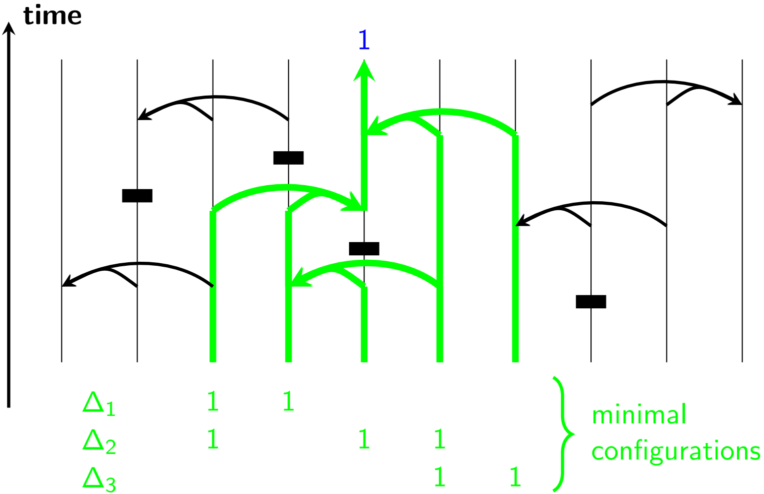
Minimal configurations



Minimal configurations



Minimal configurations

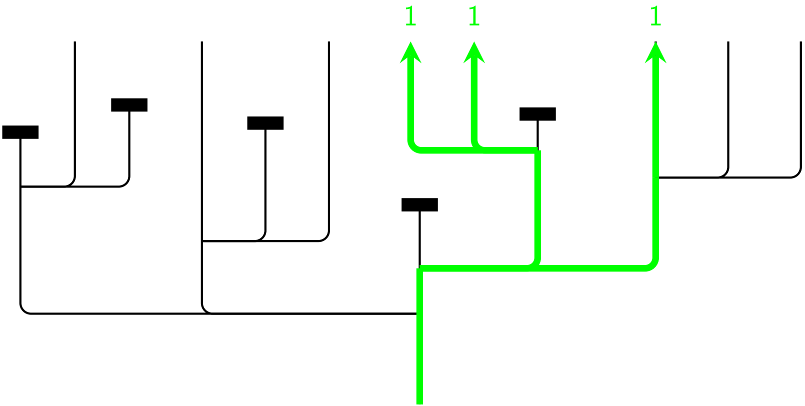


Minimal configurations

In the mean-field limit, minimal configurations correspond to subtrees $\mathbb{V} \subset \mathbb{S}_t \cup \nabla\mathbb{S}_t$ with the property that for each $\mathbf{i} \in \mathbb{V} \cap \mathbb{S}_t$ such that $\gamma[\omega_{\mathbf{i}}] = \mathtt{cob}$,

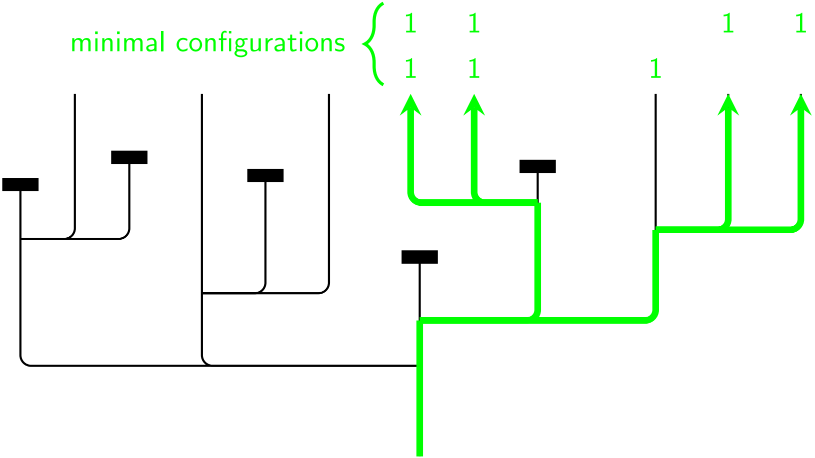
$$\text{either } \mathbf{i}1 \in \mathbb{V} \text{ or } \{\mathbf{i}2, \mathbf{i}3\} \subset \mathbb{V} \text{ (but not both).}$$

Minimal configurations



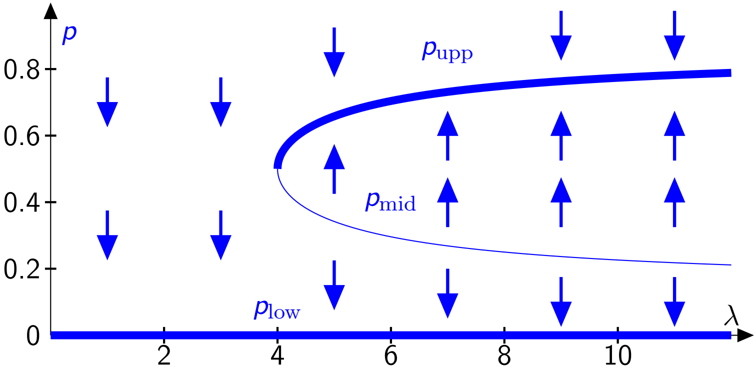
Minimal configurations for the map $\Gamma_{\mathbb{S}_t}$.

Minimal configurations



Minimal configurations for the map $\Gamma_{\mathbb{S}_t}$.

Fixed points of the mean-field equation



In the limit, p_{upp} is the probability that $\Gamma_{\mathbb{S}_t}$ is not constant.

p_{mid} is the minimal density that a product measure needs to ensure that at least one minimal configuration is completely filled with 1's.

Spatial models

For other graphs than the complete graph, much less is known.

Let

$$\lambda_{\rm c} := \sup\{\lambda > 0 : \delta_{\underline{0}} \text{ is the only invariant law}\},$$
$$\lambda'_{\rm c} := \sup\{\lambda > 0 : \mathbb{P}^{1_{\{i,j\}}}[X_t = \underline{0}] \underset{t\to\infty}{\longrightarrow} 1\} \quad (i \sim j).$$

In simulations, the model on $\mathbb{Z}^d$ seems to have $\lambda_{\rm c} = \lambda'_{\rm c}$ and the phase transition is continuous, similar to the contact process.

On the other hand, trivially $\lambda'_{\rm c} = 0$ if we change the rules so that:

- For each $i_1 \in \Lambda$, with Poisson rate λ, we pick $i_1 \sim i_2$ and $i_1 \sim i_3$ with $i_2 \neq i_3$ uniformly at random and apply the map $\mathtt{cob}_{i_1 i_2 i_3}$,

Spatial models

A crucial question seems to be: How does the "backward in time" process survive?

- Are all minimal configurations very large as in the mean-field case?
- Or are there also small minimal configurations consisting of just two neighbouring sites?

It seems that depending on the details of the model, both can happen, and this influences the shape of the phase diagram.

Question If $(\Lambda, \sim)$ is a regular tree, then does there exist an intermediate invariant law $\nu_{\rm mid}$?