Interacting Particle Systems: Almost sure uniqueness, pathwise duality, and the mean-field limit

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Lecture 5: Recursive tree process

Recursive Tree Processes

A *Recursive Distributional Equation* is an equation of the form

$$X \stackrel{\mathcal{D}}{=} \gamma[\omega](X_1, \ldots, X_{\kappa(\omega)}) \qquad \text{(RDE)},$$

where $X_1, X_2, \ldots$ are i.i.d. copies of X, independent of ω.

A law ν solves (RDE) iff

$$\text{(i)} \quad \mathbf{T}_t(\nu) = \nu \quad (t \geq 0) \qquad \text{or} \qquad \text{(ii)} \quad \mathbf{T}(\nu) = \nu.$$

Solutions to the RDE are the equivalent of invariant laws in the mean-field setting.
For the cooperative contact process, solutions to the RDE are the Bernoulli distributions $\nu_{\rm low}$, $\nu_{\rm mid}$, $\nu_{\rm upp}$ with density $p_{\rm low}$, $p_{\rm mid}$, $p_{\rm upp}$.

Recursive Tree Processes

For any rooted subtree $\mathbb{U} \subset \mathbb{T}$, let

$$\partial\mathbb{U} := \big\{ i_1 \cdots i_n \in \mathbb{T} : i_1 \cdots i_{n-1} \in \mathbb{U},\ i_1 \cdots i_n \notin \mathbb{U} \big\}$$

denote the boundary of $\mathbb{U}$ relative to $\mathbb{T}$.

For each solution ν of (RDE), there exists a *Recursive Tree Process (RTP)* $(\omega_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$, unique in law, such that:

(i) $(\omega_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ are i.i.d. with law $|\mathbf{r}|^{-1}\mathbf{r}$.

(ii) For finite $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(X_{\mathbf{i}})_{\mathbf{i} \in \partial\mathbb{U}}$ are i.i.d. with ν and independent of $(\omega_{\mathbf{i}})_{\mathbf{i} \in \mathbb{U}}$.

(iii) $X_{\mathbf{i}} = \gamma[\omega_{\mathbf{i}}]\big(X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}\kappa(\omega_{\mathbf{i}})}\big) \qquad (\mathbf{i} \in \mathbb{T}).$

If we add independent exponentially distributed lifetimes, then:

- Conditional on $\mathcal{F}_t$, the r.v.'s $(X_{\mathbf{i}})_{\mathbf{i} \in \nabla\mathbb{S}_t}$ are i.i.d. with law ν.

n-Variate processes

For each $n \geq 1$, a measurable map $g : S^k \to S$ gives rise to *n-variate map* $g^{(n)} : (S^n)^k \to S^n$ defined as

$$g^{(n)}(x_1, \ldots, x_k) = g^{(n)}(x^1, \ldots, x^n) := \big(g(x^1), \ldots, g(x^n)\big),$$

with $x = (x_i^m)_{i=1,\ldots,k}^{m=1,\ldots,n}$, $x_i = (x_i^1, \ldots, x_i^n)$, $x^m = (x_1^m, \ldots, x_k^m)$.

We define an *n-variate map*

$$\mathbf{T}^{(n)}(\mu^{(n)}) := |\mathbf{r}|^{-1} \int_\Omega \mathbf{r}(\mathrm{d}\omega) \mathbf{T}_{\gamma^{(n)}[\omega]}(\mu^{(n)}),$$

which acts on probability measures $\mu^{(n)}$ on S^n.

The *n-variate mean-field equation*

$$\tfrac{\partial}{\partial t}\mu_t^{(n)} = \int_\Omega \mathbf{r}(\mathrm{d}\omega)\big\{\mathbf{T}_{\gamma^{(n)}[\omega]}(\mu_t^{(n)}) - \mu_t^{(n)}\big\} \qquad (t \geq 0).$$

describes the mean-field limit of n coupled processes that are constructed using the same stochastic flow $(\mathbb{X}_{s,u})_{s\leq u}$.

n-Variate processes

$\mathcal{P}_{\rm sym}(S^n)$	space of probability measures on S^n that are symmetric under a permutation of the coordinates.
$S^n_{\rm diag}$	$\{x \in S^n : x_1 = \cdots = x_n\}$
$\mathcal{P}(S^n)_\mu$	space of probability measures on S^n whose one-dimensional marginals are all equal to μ.

- If $(\mu^{(n)}_t)_{t\geq 0}$ solves the *n*-variate equation, then its *m*-dimensional marginals solve the *m*-variate equation.
- $\mu^{(n)}_0 \in \mathcal{P}_{\rm sym}(S^n)$ implies $\mu^{(n)}_t \in \mathcal{P}_{\rm sym}(S^n)$ $(t \geq 0)$.
- $\mu^{(n)}_0 \in \mathcal{P}(S^n_{\rm diag})$ implies $\mu^{(n)}_t \in \mathcal{P}(S^n_{\rm diag})$ $(t \geq 0)$.
- If $\mathbf{T}(\nu) = \nu$, then $\mu^{(n)}_0 \in \mathcal{P}(S^n)_\nu$ implies $\mu^{(n)}_t \in \mathcal{P}(S^n)_\nu$.

n-Variate processes

If $\nu = \mathbb{P}[X \in \cdot\,]$ solves the RDE $\mathbf{T}(\nu) = \nu$, then

$$\overline{\nu}^{(n)} := \mathbb{P}\big[(\underbrace{X, \ldots, X}_{n \text{ times}}) \in \cdot\,\big]$$

solves the *n-variate RDE* $\mathbf{T}^{(n)}(\nu^{(n)}) = \nu^{(n)}$.

Questions:

- Is $\overline{\nu}^{(n)}$ a stable fixed point of the *n*-variate equation?
- Is $\overline{\nu}^{(n)}$ the only solution in $\mathcal{P}_{\rm sym}(S^n)_\nu$ of the *n*-variate RDE?

Endogeny

Aldous and Bandyopadyay (2005) say that an RTP is *endogenous* if

$$X_\varnothing \text{ is measurable w.r.t. the } \sigma\text{-field generated by } (\boldsymbol{\omega}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}.$$

Theorem [AB '05 & MSS '20] The following statements are equivalent:

(i) The RTP corresponding to ν is endogenous.

(ii) $\mathbf{T}^{(n)}_t(\mu) \underset{t\to\infty}{\Longrightarrow} \overline{\nu}^{(n)}$ for all $\mu \in \mathcal{P}(S^n)_\nu$ and $n \geq 1$.

(iii) $\overline{\nu}^{(2)}$ is the only solution in $\mathcal{P}_{\rm sym}(S^2)_\nu$ of the bivariate RDE.

In our example, the RTPs for $\nu_{\rm low}, \nu_{\rm upp}$ are endogenous, but the RTP corresponding to $\nu_{\rm mid}$ is not.

Endogeny

Proof (main idea) Conditional on $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, let $(X'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an independent copy of $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. Then

$$\underline{\nu}^{(2)} := \mathbb{P}\big[(X_\varnothing, X'_\varnothing) \in \cdot\,\big]$$

solves the bivariate RDE and endogeny is equivalent to $\underline{\nu}^{(2)} = \overline{\nu}^{(2)}$. ∎

Endogeny

Let $(\omega_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to γ and ν.

Define random measures $\underline{\eta}_{\mathbf{i}}$ and $\overline{\eta}_{\mathbf{i}}$ ($\mathbf{i} \in \mathbb{T}$) by

$$\underline{\eta}_{\mathbf{i}} := \mathbb{P}\big[X_{\mathbf{i}} \in \cdot\,\big|(\omega_{\mathbf{ij}})_{\mathbf{j}\in\mathbb{T}}\big] \quad \text{and} \quad \overline{\eta}_{\mathbf{i}} := \delta_{X_{\mathbf{i}}}.$$

Let $\underline{\nu} := \mathbb{P}[\underline{\eta}_\varnothing \in \cdot\,]$ and $\overline{\nu} := \mathbb{P}[\overline{\eta}_\varnothing \in \cdot\,]$.

$(\omega_{\mathbf{i}}, \underline{\eta}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is the RTP corresponding to $\check{\gamma}$ and $\underline{\nu}$.
$(\omega_{\mathbf{i}}, \overline{\eta}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is the RTP corresponding to $\check{\gamma}$ and $\overline{\nu}$.

Here, for any measurable map $g : S^k \to S$, we define $\check{g} : \mathcal{P}(S)^k \to \mathcal{P}(S)$ by

$$\check{g} := \text{ the law of } g(X_1, \ldots, X_k), \\ \text{where } X_1, \ldots, X_k \text{ are independent with laws } \mu_1, \ldots, \mu_k.$$

Endogeny

Define *n-th moment measures*

$$\rho^{(n)} := \mathbb{E}\big[\underbrace{\eta \otimes \cdots \otimes \eta}_{n\text{ times}}\big] \quad \text{where } \eta \text{ has law } \rho.$$

Now

$$(\omega_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}} \text{ is endogenous} \quad \Leftrightarrow \quad \underline{\eta}_\varnothing = \overline{\eta}_\varnothing$$
$$\Leftrightarrow \quad \underline{\nu} = \overline{\nu} \quad \Leftrightarrow \quad \underline{\nu}^{(2)} = \overline{\nu}^{(2)}.$$

Proposition [MSS '20] $(\rho_t)_{t\geq 0}$ solves the *higher-level mean-field equation*

$$\tfrac{\partial}{\partial t}\rho_t = \int_\Omega \mathbf{r}(\mathrm{d}\omega)\big\{\mathbf{T}_{\check\gamma[\omega]}(\rho_t) - \rho_t\big\} \qquad (t \geq 0).$$

if and only if its n-th moment measures $(\rho_t^{(n)})_{t\geq 0}$ solve the n-variate mean-field equation.

Pivotals

Define $\mathbf{j}$ is *pivotal* if

$$\Gamma_{\mathbb{U}}\big(X_{\mathbf{j}},(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}\backslash\{\mathbf{j}\}}\big)\neq\Gamma_{\mathbb{U}}\big(x,(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}\backslash\{\mathbf{j}\}}\big).$$

For some $x\neq X_{\mathbf{j}}$ and $\mathbb{U}$ such that $\mathbf{j}\in\nabla\mathbb{U}$.

Johnson, Podder & Skerman (2020) observe that

$$J_n:=\big\{\mathbf{j}\in\nabla\mathbb{S}_{(n)}:\mathbf{j}\text{ is pivotal}\big\}\qquad(n\geq 0)$$

is a branching process. In a special setting, they prove $(J_n)_{n\geq 0}$ subcritical $\Rightarrow$ endogeny. For a more restrictive class, endogeny is equivalent to extinction of $(J_n)_{n\geq 0}$.