Interacting Particle Systems: Almost sure uniqueness, pathwise duality, and the mean-field limit

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Lecture 6: Frozen percolation

Frozen percolation

Let (V, E) be a countably infinite, connected graph with *vertex set* V and *edge set* E. For $x \in \{0,1\}^E$ and $v \in V$, we write $v \stackrel{x}{\longrightarrow} \infty$ if there exists $v = v_0, v_1, v_2, \ldots$, all different, such that $x(\{v_{k-1}, v_k\}) = 0$ for all $k \geq 1$.
For each $\{v, w\} \in E$, we define $\chi_{\{v,w\}} : \{0,1\}^E \to \{0,1\}$ by

$$\chi_{\{v,w\}}(x) := \begin{cases} 1 & \text{if } v \stackrel{x}{\longrightarrow} \infty \text{ or } w \stackrel{x}{\longrightarrow} \infty, \\ 0 & \text{otherwise.} \end{cases}$$

For each edge $e \in E$, we define $\mathtt{act}_e : \{0,1\}^E \to \{0,1\}^E$ by

$$\mathtt{act}_e(x)(f) := \begin{cases} \chi_e(x) & \text{if } f = e, \\ x(f) & \text{if } f \neq e. \end{cases}$$

Frozen percolation $(X_t)_{t\geq 0}$ is defined by $X_0 := \underline{1}$ and the generator

$$Gf(x) := \sum_{e\in E} \big\{ f\big(\mathtt{act}_e\big) - f\big(x\big) \big\}.$$

Frozen percolation

Wait a minute! There is *no way* that $\mathtt{act}_e$ is a local map!

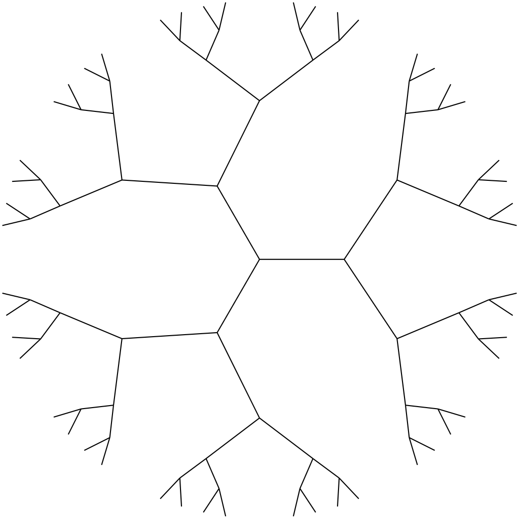
So it is not clear if frozen percolation *exists*, if it is *unique in law*, or even *almost surely unique*.

David Aldous (2000) has shown that frozen percolation on the infinite 3-regular tree exists, and under suitable additional assumptions, is unique in law.

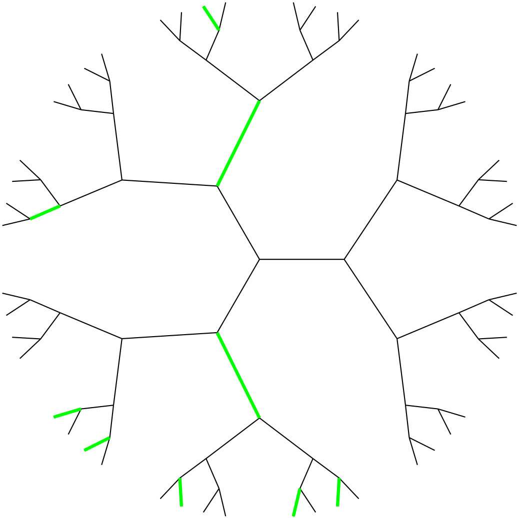
Itai Benjamini & Oded Schramm (2001) have shown that, by contrast, on $\mathbb{Z}^2$, frozen percolation does not exist.

The problem is wide open on $\mathbb{Z}^d$ for $d \geq 3$.

Frozen percolation on the 3-regular tree



Frozen percolation on the 3-regular tree



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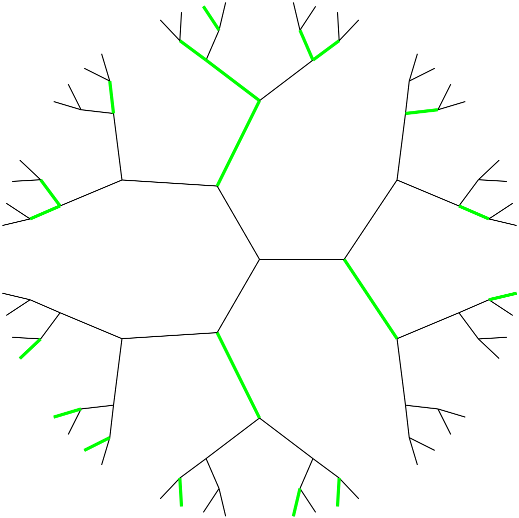

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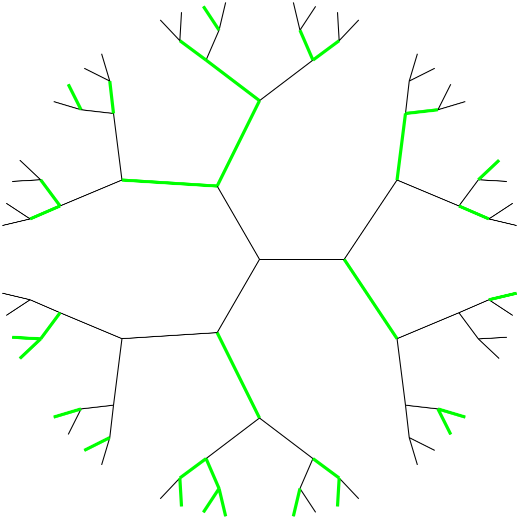

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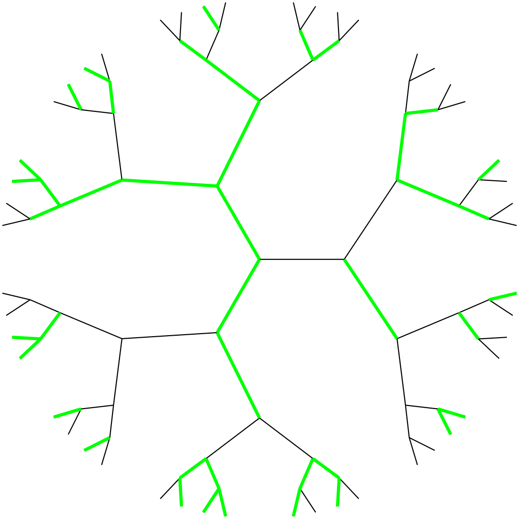

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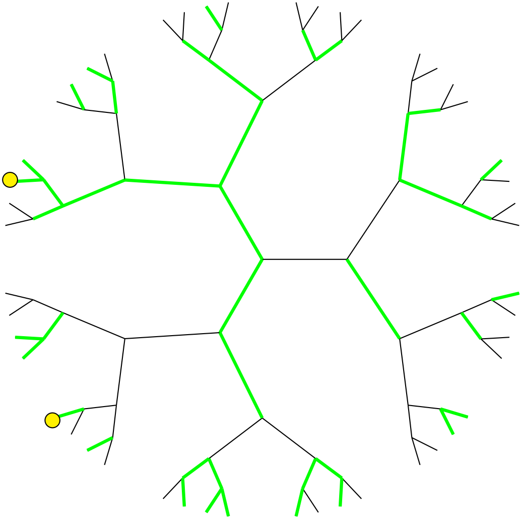

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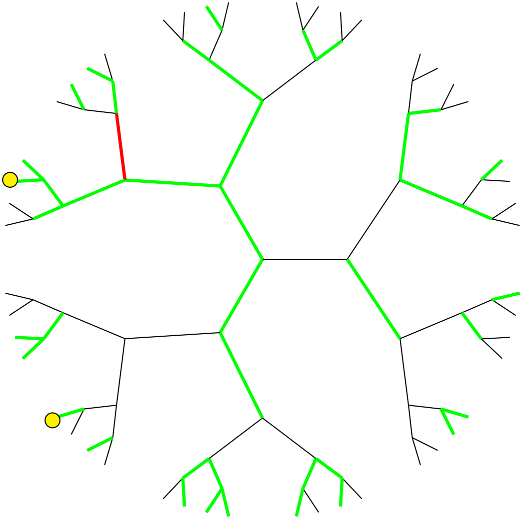

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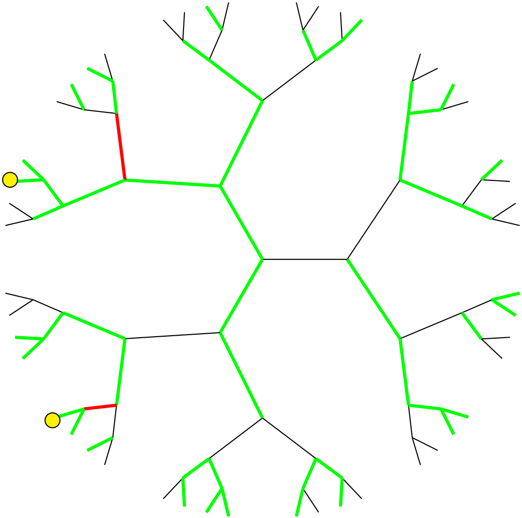

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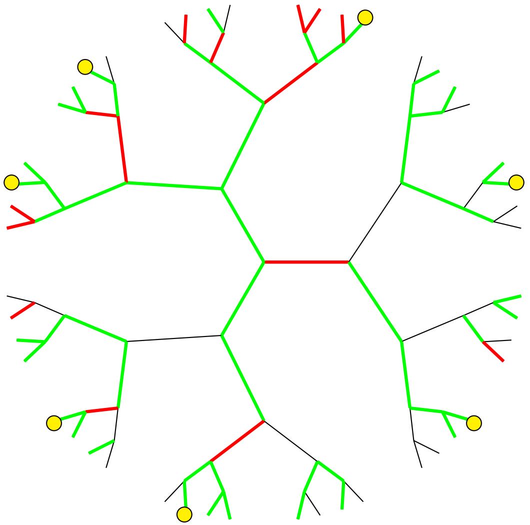

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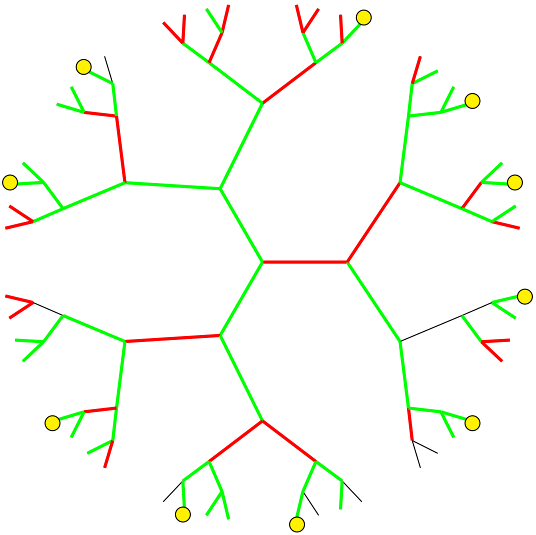

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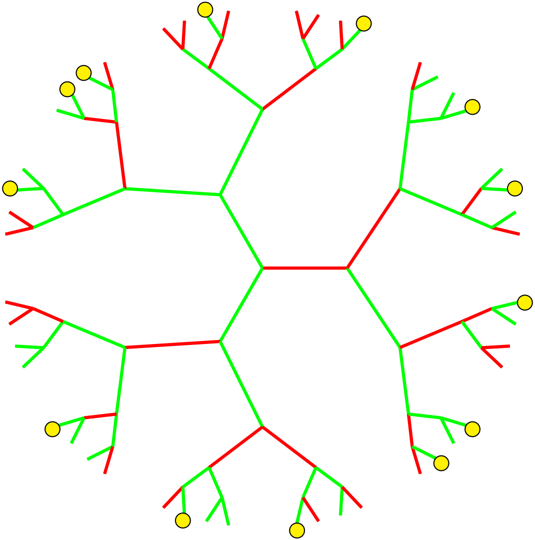

The oriented binary tree

Frozen percolation on the (unoriented) 3-regular tree can be translated into frozen percolation on the oriented binary tree $\mathbb{T}$ consisting of all words $\mathbf{i} = i_1 \cdots i_n$ made of the alphabet $\{1, 2\}$.

We assign i.i.d. Unif$[0, 1]$ *activation times* $(\tau_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ to the oriented edges.

We fix a set $\Xi \subset (0, 1]$ of *freezing times*.

- Initially, all edges are closed.

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- At each freezing time $t \in \Xi$, all closed edges for which the tree above it percolates are frozen.

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- At its activation time, an edge opens, provided it is not frozen.
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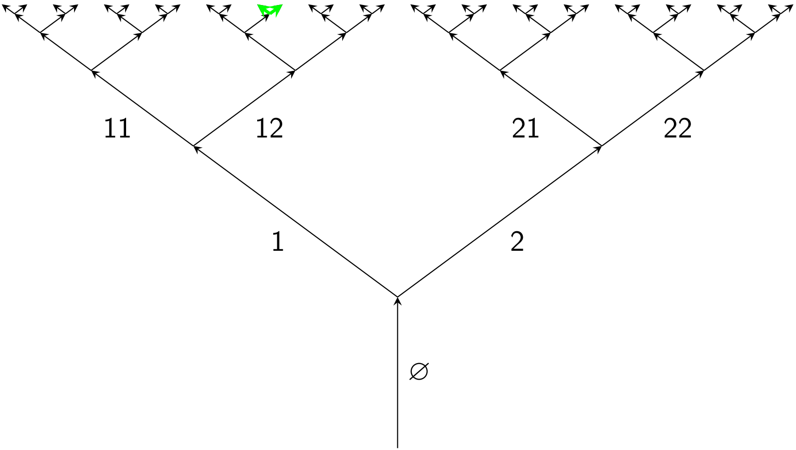
Special case $\Xi = (0, 1]$ means edges are frozen as soon as the tree above them percolates.

Frozen percolation on the oriented binary tree



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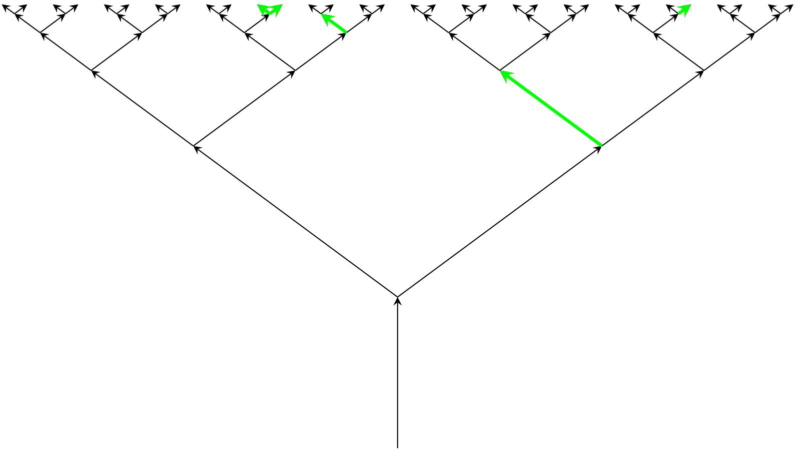

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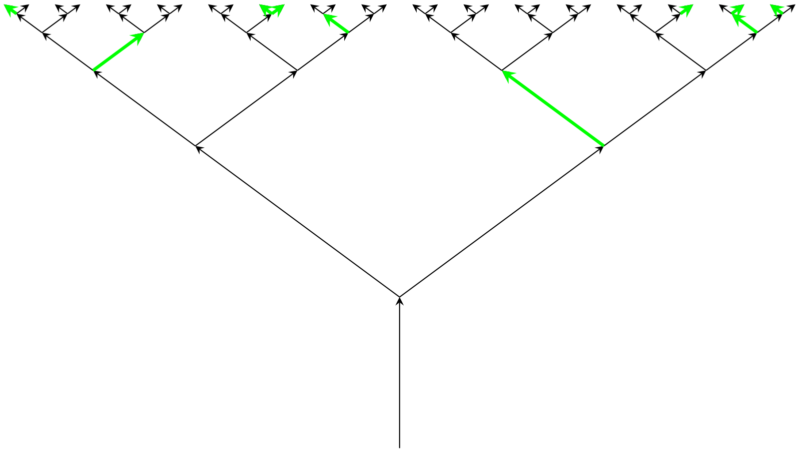

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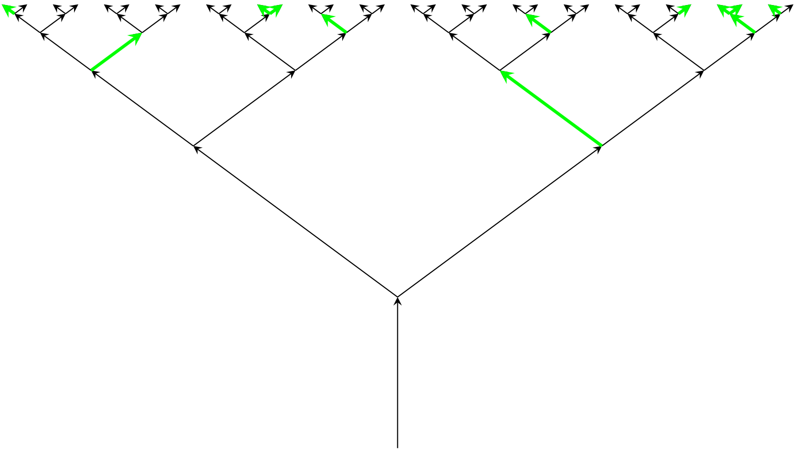

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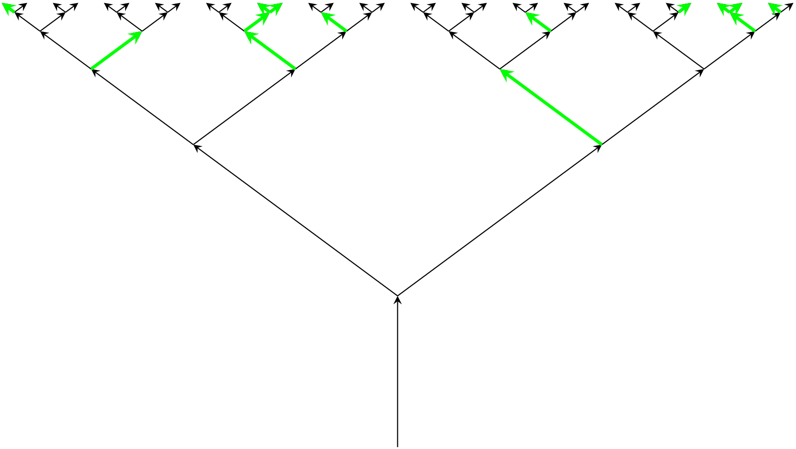

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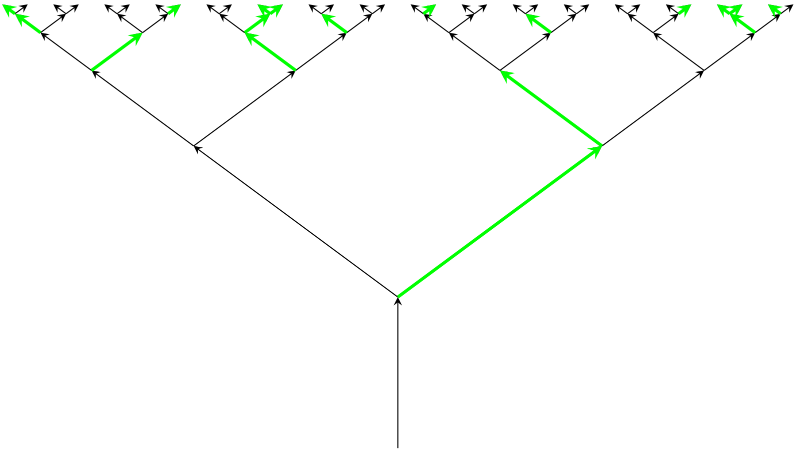

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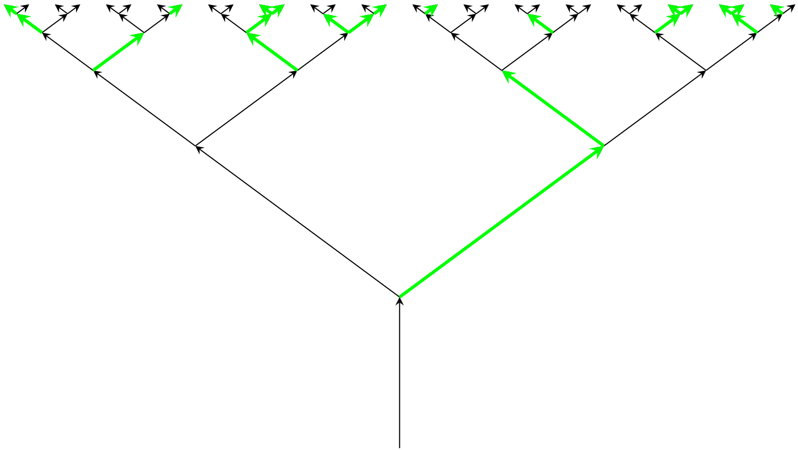

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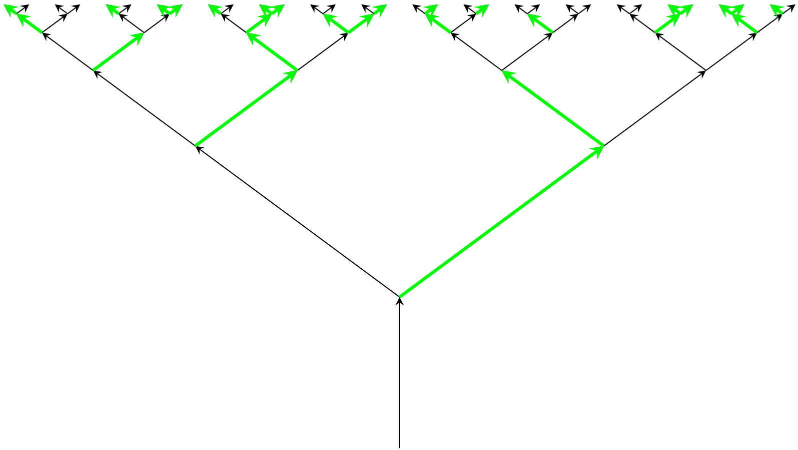

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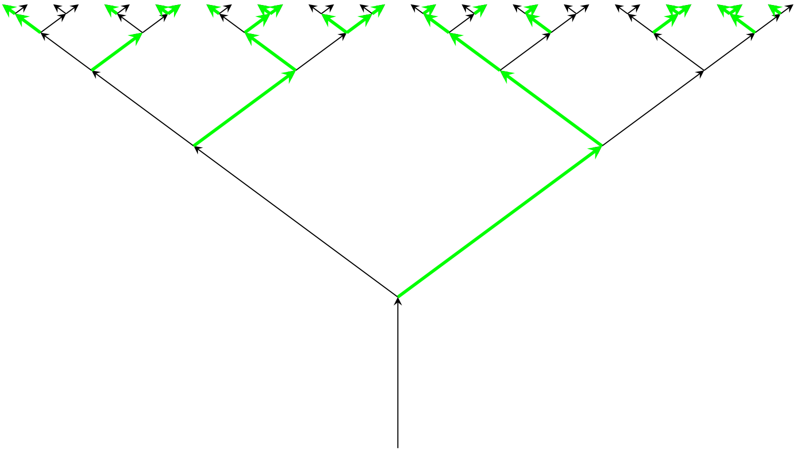

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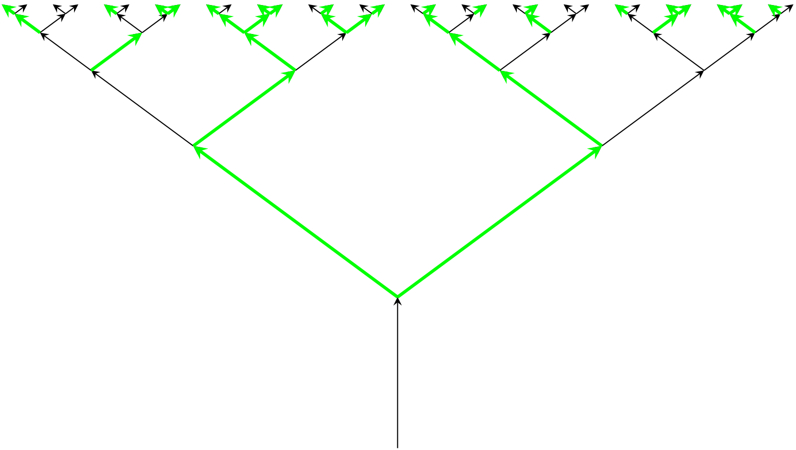

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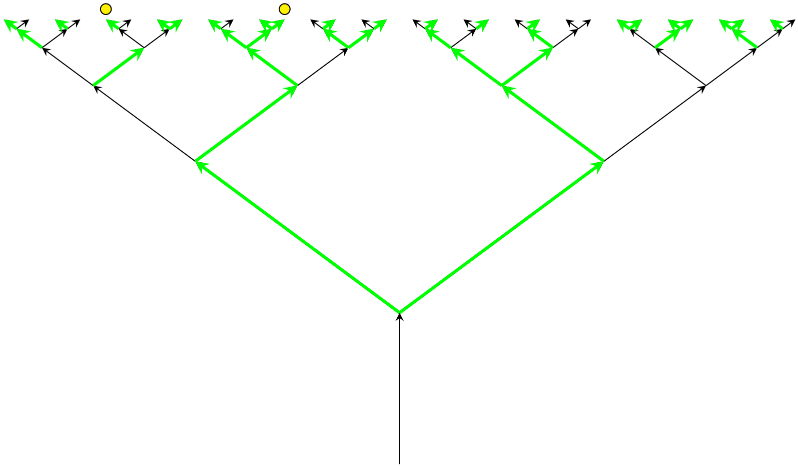

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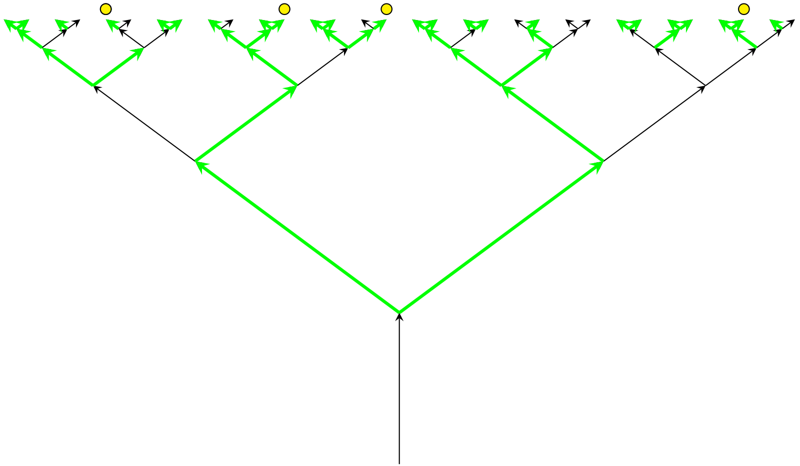

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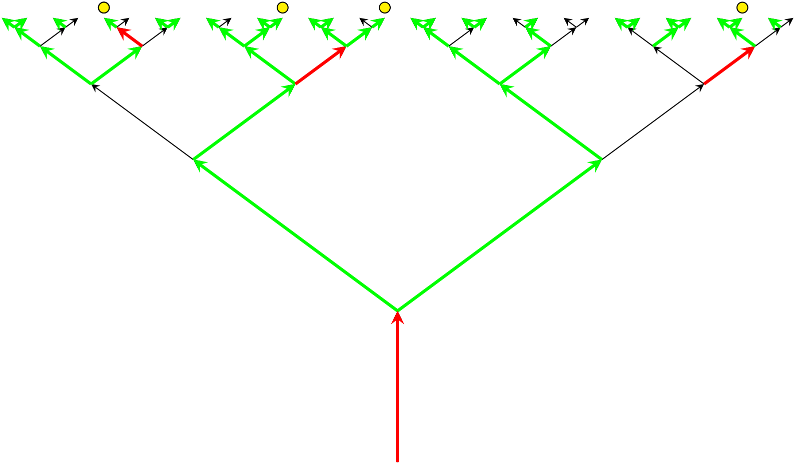
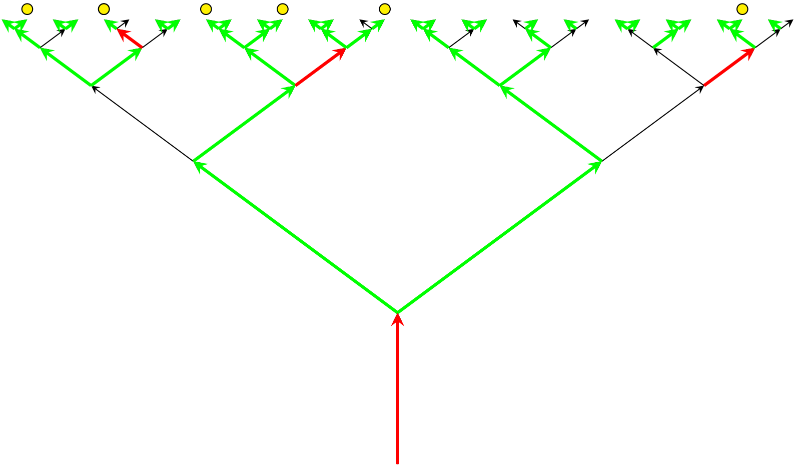


First time $t \in \Xi$.

Frozen percolation on the oriented binary tree



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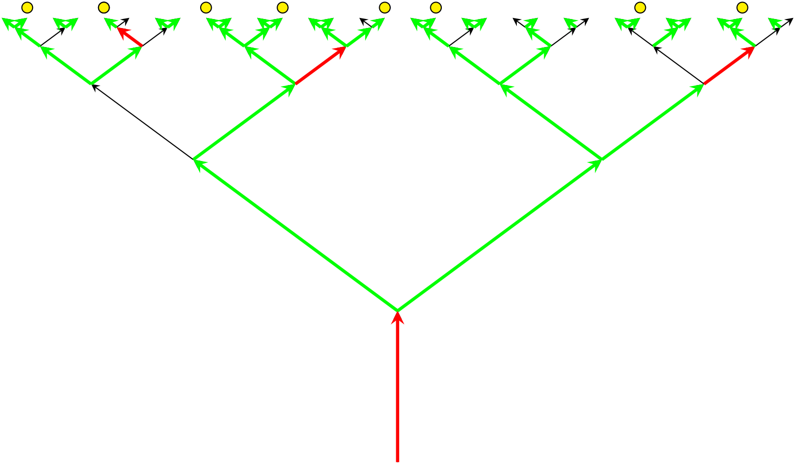

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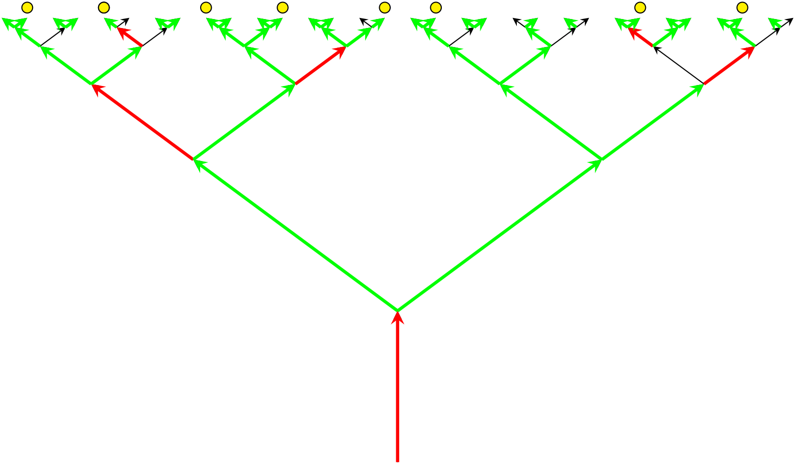
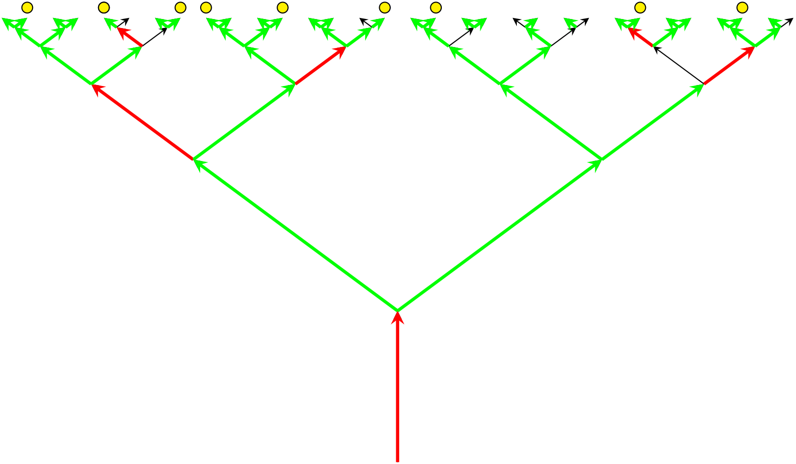
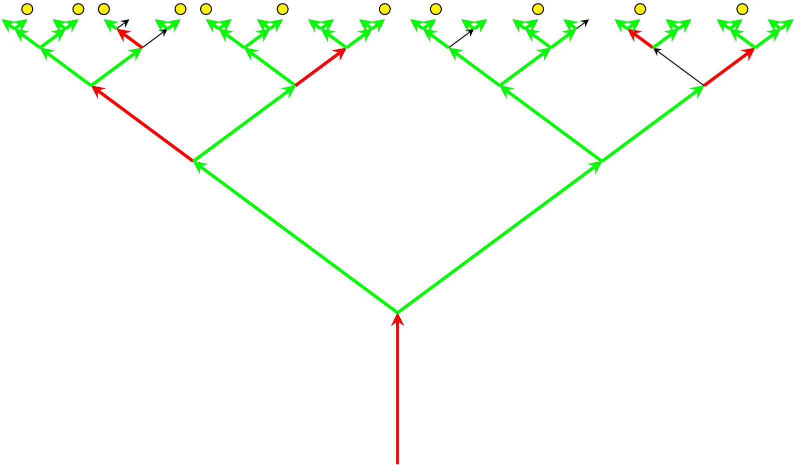


Second time $t \in \Xi$.

Frozen percolation on the oriented binary tree



Frozen percolation on the oriented binary tree



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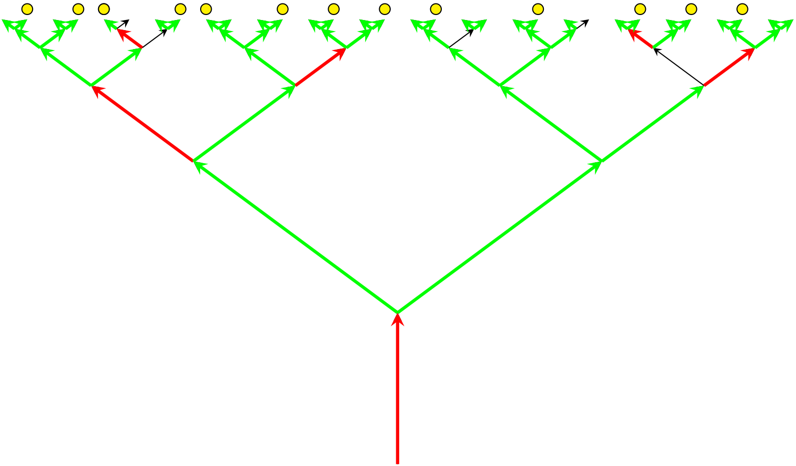

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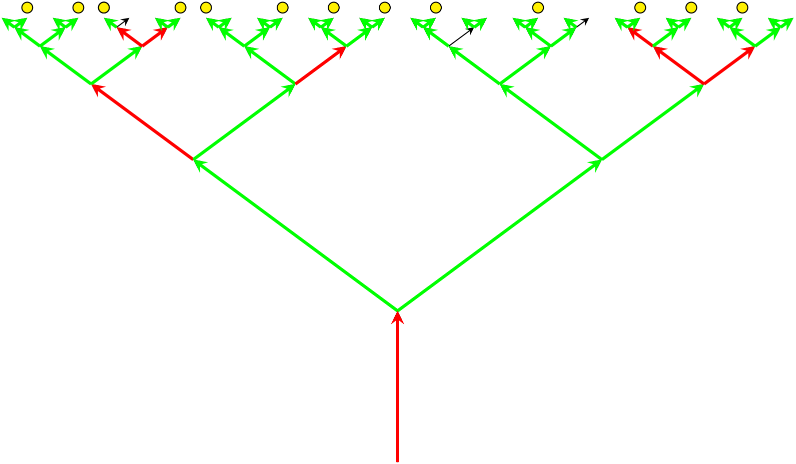
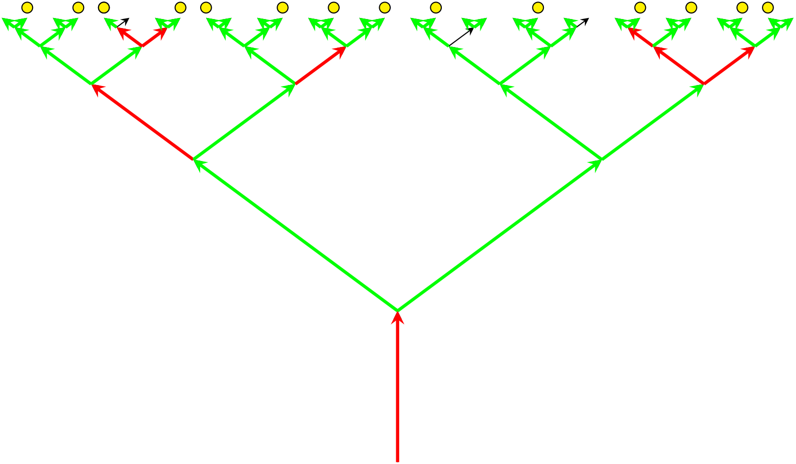
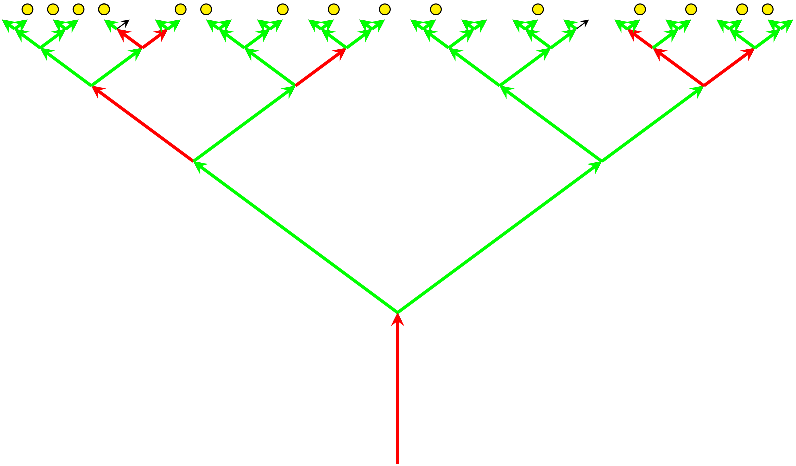


Third time $t \in \Xi$.

Frozen percolation on the oriented binary tree



Frozen percolation on the oriented binary tree



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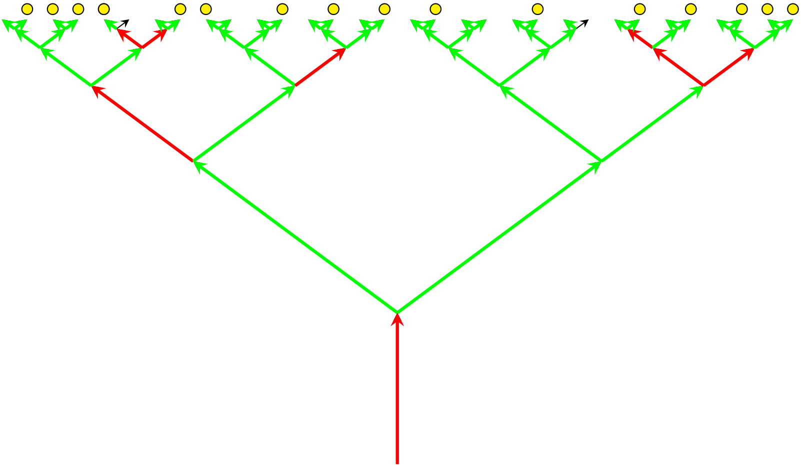


Notation

$$\mathbb{A}_t := \{\mathbf{i} \in \mathbb{T} : \tau_{\mathbf{i}} \leq t\},$$
$$\mathbb{F} := \{\mathbf{i} \in \mathbb{T} : \mathbf{i} \text{ frozen at the final time } 1\}.$$

Then $\mathbb{A}_t \backslash \mathbb{F}$ are the open edges at time t.

We write $\mathbf{i} \xrightarrow{\mathbb{A}_t \backslash \mathbb{F}} \infty$ if the tree above $\mathbf{i}$ percolates at time t.

The Frozen Percolation Equation

The *Frozen Percolation Equation* (FPE) reads:

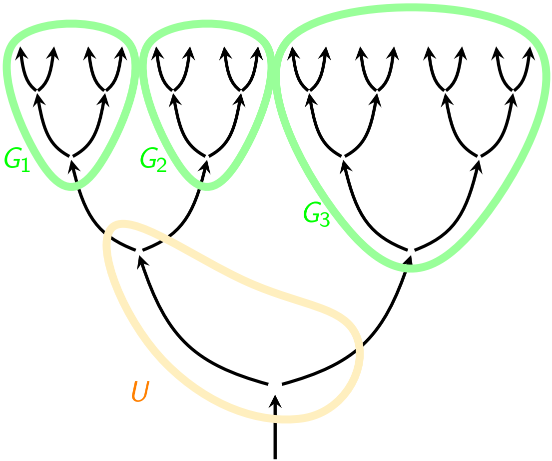
$$\mathbb{F} = \big\{\mathbf{i} \in \mathbb{T} : \mathbf{i} \stackrel{\mathbb{A}_t \backslash \mathbb{F}}{\longrightarrow} \infty \text{ for some } t \in (0, \tau_{\mathbf{i}}] \cap \Xi\big\}.$$

If Ξ is finite, then (FPE) has a solution, which is a.s. unique.

Questions for infinite Ξ:

- Existence of solutions?
- Uniqueness of solutions?
- Uniqueness in which sense?

Distributional uniqueness



On the oriented binary tree, we impose *natural conditions*:
The subtrees G_1, G_2, G_3 should be i.i.d., equally distributed with the original tree G, and independent of U.

Distributional uniqueness

[Ráth, S., Szőke '21] For each closed $\Xi \subset (0,1]$, (FPE) has a solution that satisfies the natural conditions, and the joint law of $\big((\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}, \mathbb{F}\big)$ is uniquely determined.

Note The case $\Xi = (0,1]$ was essentially treated in **[Aldous '00]**.

How about almost sure uniqueness?

Freezing times

Let [**i**] denote the starting vertex of the edge **i**. The freezing time of the root

$$Y_{[\varnothing]} := \inf\left\{t \in \Xi : [\varnothing] \stackrel{\mathbb{A}_t \setminus \mathbb{F}}{\longrightarrow} \infty\right\}$$

solves the *Recursive Distributional Equation* (RDE)

$$Y_{[\varnothing]} \stackrel{\mathcal{D}}{=} \gamma(\tau_\varnothing, Y_{[1]}, Y_{[2]}) := \begin{cases} Y_{[1]} \wedge Y_{[2]} & \text{if } \tau_\varnothing < Y_{[1]} \wedge Y_{[2]}, \\ \infty & \text{otherwise.} \end{cases}$$

For each closed $\Xi \subset (0,1]$, there exists a unique solution ρ_Ξ to (RDE) that yields a solution $\mathbb{F}$ of (FPE).

Aldous (2000) proved

$$\rho_{(0,1]}(\mathrm{d}y) = \frac{\mathrm{d}y}{2y^2} 1_{[\frac{1}{2},1]}(y) \qquad \rho(\{\infty\}) = \tfrac{1}{2}.$$

Almost sure uniqueness

For $0 < \theta < 1$, set $\Xi_\theta := \{\theta^n : n \geq 0\}$ and set $\Xi_1 := (0, 1]$.

[Ráth, S., Szőke, Terpai '19 & '21] There exists a parameter $\theta^* = 0.636\ldots$ such that the RTP corresponding to ρ_{Ξ_θ} is endogenous iff $\theta < \theta^*$.

Consequence Under the natural conditions, the set of frozen edges $\mathbb{F}$ is a.s. uniquely determined by the freezing times $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ iff $\theta < \theta^*$.

In particular, the process constructed by Aldous (2000) is not a.s. unique.