# An introduction to free independence

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## 1 $*$-Algebras

By definition, an *algebra* is a linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that

$$\begin{array}{rll} \text{(i)} & (AB)C = A(BC) & (A,B,C \in \mathcal{A}), \\ \text{(ii)} & A(bB + cC) = bAB + cAC & (A,B,C \in \mathcal{A},\ b,c \in \mathbb{K}), \\ \text{(iii)} & (aA + bB)C = aAC + bBC & (A,B,C \in \mathcal{A},\ a,b \in \mathbb{K}) \end{array}$$

Often, it is assumed that $\mathcal{A}$ contains a (necessarily unique) element 1 such that

$$\text{(iv)} \quad 1A = A = A1 \qquad (A \in \mathcal{A}).$$

An algebra is *abelian* if

$$AB = BA \qquad (A,B \in \mathcal{A}).$$

An *adjoint operation* is a map $A \mapsto A^*$ such that

$$\begin{array}{rll} \text{(v)} & (A^*)^* = A & (A \in \mathcal{A}), \\ \text{(vi)} & (aA + bB)^* = \overline{a}A^* + \overline{b}B^* & (A,B \in \mathcal{A},\ a,b \in \mathbb{C}), \\ \text{(vii)} & (AB)^* = B^*A^* & (A,B \in \mathcal{A}). \end{array}$$

In what follows, we reverse the term $*$-*algebra* for an algebra over $\mathbb{C}$ that is equipped with an adjoint operation such that (i)–(vii) hold. A $C*$-*algebra* is a $*$-algebra equipped with a norm $\| \cdot |$ such that

$$\begin{array}{rl} \text{(viii)} & \mathcal{A} \text{ is complete in the norm } \| \cdot \|, \\ \text{(ix)} & \|AB\| \leq \|A\|\|B\| \qquad (A,B \in \mathcal{A}), \\ \text{(x)} & \|A^*A\| = \|A\|^2. \end{array}$$

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the space of all bounded linear operators $A : \mathcal{H} \to \mathcal{H}$, equipped with the operator norm $\|A\| := \sup_{\|x\| \leq 1} \|Ax\|$. Let $\mathcal{A} \subset \mathcal{H}$ be a linear subspace of $\mathcal{L}(\mathcal{H})$ such that

- $A, B \in \mathcal{A} \ \Rightarrow \ AB \in \mathcal{A}$,
- $A \in \mathcal{A} \ \Rightarrow \ A^* \in \mathcal{A}$,
- $\mathcal{A}$ is closed in the norm $\| \ \cdot \|$.

Then $\mathcal{A}$ is a $C*$-*algebra.* The *Gelfand-Naimark theorem* says that each $C*$-*algebra* is isomorphic to a $C*$-algebra of this form. If $\mathcal{A}$ is separable, then $\mathcal{H}$ can be taken separable too.

A map $\tau : \mathcal{A} \to \mathbb{C}$ is a *linear form* if

$$\text{(xi)} \quad \tau(aA + bB) = a\tau(A) + b\tau(B) \quad (A,B \in \mathcal{A},\ a,b \in \mathbb{C}).$$

It is called *real* if

$$\text{(xii)} \quad \tau(A^*) = \overline{\tau(A)} \quad (A \in \mathcal{A}).$$

A *positive* linear form is a real linear form such that

$$\text{(xiii)} \quad \tau(A^*A) \geq 0 \quad (A \in \mathcal{A}).$$

If moreover

- $\tau(A^*A) = 0 \ \Rightarrow \ A = 0,$

then we say that $\tau$ is *faithful.* A positive linear form that is normalized in the sense that

$$\text{(xiv)} \quad \tau(1) = 1$$

is called a *state*. If moreover

- $\tau(AB) = \tau(BA) \qquad (A, B \in \mathcal{A}),$

then $\tau$ is called a *pseudotrace*. It can be shown that every positive linear form is continuous, and in fact satisfies

$$|\tau(A)| \leq |\tau(1)| \, \|A\|.$$

*Example 1* We can take $\mathcal{A} = M_n(\mathbb{C})$, the space of all complex $n \times n$ matrices, equipped with the usual adjoint and the *normalized trace* $\tau(A) := \frac{1}{n}\mathrm{tr}(A)$. Then $\tau$ is a state, and moreover a faithful pseudotrace.

*Example 2a* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{L}^{\infty-}$ be the space of all measurable maps $X : \Omega \to \mathbb{C}$ such that $\mathbb{E}[|X|^k] < \infty$ for all $k \geq 0$. Then $\mathcal{A} := \mathcal{L}^{\infty-}$, equipped with the pointwise product $(XY)(\omega) := X(\omega)Y(\omega)$ and adjoint operation $X^*(\omega) := \overline{X(\omega)}$ is an abelian $*$-algebra, and $\tau(X) := \mathbb{E}[X]$ is a state and moreover a pseudotrace. If we replace $\mathcal{L}^{\infty-}$ by the space $L^{\infty-}$ of equivalence classes of a.s. equal elements of $\mathcal{L}^{\infty-}$, then $\tau$ is moreover faithful.

*Example 2b* If in the preceding example we let $\Omega$ be a compact metrizable space and replace $\mathcal{L}^{\infty-}$ by the space $\mathcal{C}(\Omega)$ of all continuous functions $X : \Omega \to \mathbb{C}$ equipped with the supremum-norm $\|X\| := \sup_{\omega \in \Omega} |X(\omega)|$, then we obtain an abelian $C*$-algebra. It can be proved that each abelian separable $C*$-algebra is isomorphic to a $C*$-algebra of this form.

*Example 3* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{A}$ be the space of all measurable maps $X : \Omega \to M_n(\mathbb{C})$ such that $X_{ij} \in \mathcal{L}^{\infty-}$ for all $i, j$. Equip $\mathcal{A}$ with the product $(XY)_{ik}(\omega) := \sum_j X_{ij}(\omega)Y_{jk}(\omega)$ and adjoint operation $X^*_{ij}(\omega) := \overline{X_{ji}(\omega)}$. Then $\mathcal{A}$ is a $*$-algebra. Moreover, $\tau(X) := \frac{1}{n}\mathbb{E}[\mathrm{tr}(X)]$ is a normalized pseudotrace. If we replace $\mathcal{L}^{\infty-}$ by the space $L^{\infty-}$ of equivalence classes of a.s. equal elements of $\mathcal{L}^{\infty-}$, then $\tau$ is moreover faithful.

By definition, an element $X \in \mathcal{A}$ is *normal* if $XX^* = X^*X$. An $n \times n$ matrix $X \in M_n(\mathbb{C})$ is normal if and only if it is diagonal with respect to an orthonormal basis of $\mathbb{C}^n$. Equivalently, this says that there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ such that

$$X = \sum_{i=1}^{n} \lambda_i P_{e_i}, \tag{1}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $X$ and $P_{e_i}$ denotes the orthogonal projection operator on $e_i$. We can define a *spectral measure* $\pi_X$ by

$$\pi_X(D) = \sum_{i:\, \lambda_i \in D} P_{e_i} \qquad (D \in \mathcal{B}(\mathbb{C})),$$

where $\mathcal{B}(\mathbb{C})$ denotes the Borel-$\sigma$-algebra on $\mathbb{C}$. Then (1) can formally be written

$$X = \int \lambda \, \pi_X(\mathrm{d}\lambda).$$

More generally,

$$X^k(X^*)^l = \int \lambda^k \overline{\lambda}^l \, \pi_X(\mathrm{d}\lambda). \tag{2}$$

By the complex version of the Stone-Weierstrass theorem, this formula determines $\pi_X$ uniquely. It turns out that (2) can be generalized to any normal element of a $C*$-algebra $\mathcal{A}$. More precisely, if $\mathcal{A}$ is a $C*$-algebra and $X \in \mathcal{A}$ satisfies $XX^* = X^*X$, then there exists a unique compactly supported, projection operator-valued measure $\pi_X$ on $\mathbb{C}$ such that (2) holds for each $k, l$. The measure $\pi_X$ is called the *spectral measure* of $X$. More generally than in (2), one has

$$F(X) = \int F(\lambda) \, \pi_X(\mathrm{d}\lambda) \tag{3}$$

for any continuous function $F : \mathbb{C} \to \mathbb{C}$, provided one defines $F(X)$ in the right way.[1] A normal operator is self-adjoint if and only if its spectrum is real, which means that $\pi_X$ is concentrated on $\mathbb{R}$.

Let $\mathcal{A}$ be a $C*$-algebra, let $\tau$ be a state on $\mathcal{A}$, and $X \in \mathcal{A}$ be normal. Then we can define a probability measure $\mu_X$ on $\mathbb{C}$ by setting

$$\int F(\lambda) \, \mu_X(\mathrm{d}\lambda) := \tau\big(F(X)\big) = \tau\Big(\int F(\lambda) \, \pi_X(\mathrm{d}\lambda)\Big)$$

for any continuous $F : \mathbb{C} \to \mathbb{C}$. This is equivalent to

$$\int \lambda^k \overline{\lambda}^l \, \mu_X(\mathrm{d}\lambda) = \tau(X^k(X^*)^l).$$

Informally, $\mu_X(\mathrm{d}\lambda) = \tau(\pi_X(\mathrm{d}\lambda))$. In the special case that $X$ is self-adjoint, $\mu_X$ is the unique compactly supported probability measure on $\mathbb{R}$ such that

$$\int \lambda^k \, \mu_X(\mathrm{d}\lambda) = \tau(X^k) \qquad (k \geq 1). \tag{4}$$

One does not always need $C*$-algebras:

> **Thm 2.5.8** Let $\mathcal{A}$ be a $*$-algebra and let $\tau$ be a normalized positive linear form. Assume that $X \in \mathcal{A}$ satisfies $X^* = X$. Then the limit
>
> $$\rho(X) := \lim_{k \to \infty} |\tau(X^{2k})|^{1/2k}$$
>
> exists. If $\rho(X) < \infty$, then there exists a unique probability measure $\mu_X$ on $[-\rho(X), \rho(X)]$ such that (4) holds.

*Example 1* In our first example, $\mu_X$ is the *empirical spectral distribution* of a normal matrix $X$.

*Example 2* In our second example, $\mu_X$ is the *law* of a random variable $X$.

*Example 3* In our third example, $\mu_X$ is the *mean of the empirical spectral distribution* of a random matrix $X$.

**Remark 1** The last two examples show that in many cases, one would like to allow self-adjoint $X$ for which $\mu_X$ has unbounded support. This is technically rather difficult. If $X$ is a bounded self-adjoint operator, then $U_t := e^{itX}$ defines a one-parameter group of unitary operators. More generally, strongly continuous one-parameter group of unitary operators

[1] One way is to use (3) as a definition of $F(X)$ but alternatively, using the Stone-Weierstrass theorem, one can approximate any $F$ by a polynomial of $X$ and $X^*$ and then take the limit.

have a generator that is a possibly unbounded self-adjoint operator. Unbounded self-adjoint operators are best treated via their associated unitary groups.

**Remark 2** A pair $(\mathcal{A}, \tau)$ where $\mathcal{A}$ is a $C*$-algebra and $\tau$ is a state on $\mathcal{A}$ is a *quantum probability space*. Here $\tau$ plays more or less the role of a probability measure. Self-adjoint operators correspond to observables and $\mu_X$ is the law of $X$.

## 2 Independence

Let $\mathcal{A}$ be a $C*$-algebra and let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ be sub-$C*$-algebras (i.e., linear spaces that are closed under the product and adjoint operation, and are closed in the norm). We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ *commute* if

$$A_1 A_2 = A_2 A_1 \qquad (A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2).$$

If $\mathcal{A}_1$ and $\mathcal{A}_2$ commute, then

$$\mathcal{A}_1\mathcal{A}_2 := \overline{\mathrm{span}\{A_1 A_2 : A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2\}} \tag{5}$$

is a sub-$C*$-algebra of $\mathcal{A}$. (Here $\overline{\mathcal{B}}$ denotes the closure of $\mathcal{B}$ in the norm.) Let $\tau$ be a state on $\mathcal{A}$. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *independent* if they commute and

$$\tau(A_1 A_2) = \tau(A_1)\tau(A_2) \qquad (A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2).$$

Let $\tau_i$ denote the restriction of $\tau$ to $\mathcal{A}_i$ and let $\tau_{12}$ denote the restriction of $\tau$ to $\mathcal{A}_1\mathcal{A}_2$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent, then in view of (5), using the linearity and continuity of states, we see that $\tau_{12}$ is uniquely determined by $\tau_1$ and $\tau_2$. If moreover $\mathcal{A}_1$ and $\mathcal{A}_2$ are *logically independent* in the sense that

- $\{A^1_k A^2_l : 1 \leq k \leq n,\ 1 \leq l \leq m\}$ are linearly independent whenever $\{A^1_1, \ldots, A^1_n\} \subset \mathcal{A}_1$ and $\{A^2_1, \ldots, A^2_m\} \subset \mathcal{A}_2$ are linearly independent,

then one can show that given states $\tau_i$ on $\mathcal{A}_i$ $(i = 1, 2)$, there always exists a unique state $\tau_{12}$ on $\mathcal{A}_1\mathcal{A}_2$ such that $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent and the restriction of $\tau_{12}$ to $\mathcal{A}_i$ is $\tau_i$. We can view $\tau_{12}$ as a non-commutative generalization of the product measure.

Each $X \in \mathcal{A}$ generates a sub-$C*$-algebra

$$\alpha(X) := \overline{\mathrm{span}\Big\{\prod_{i=1}^{n} Y_i : Y_i \in \{X, X^*\}\Big\}}.$$

If $X$ is normal (i.e., $X$ commutes with $X^*$), this simplifies to

$$\alpha(X) := \overline{\mathrm{span}\{X^k (X^*)^l : k, l \geq 0\}}.$$

If $X$ is self-adjoint, this simplifies even more to

$$\alpha(X) := \overline{\mathrm{span}\{X^k : k \geq 0\}}$$

(with $X^0 := 1$). Alternatively, for any normal operator $X$, we have the formula

$$\alpha(X) = \{F(X) : F : \mathbb{C} \to \mathbb{C} \text{ continuous}\}.$$

Note that the law $\mu_X$ of a self-adjoint operator uniquely determines $\tau(F(X))$ for each $F$, and hence uniquely determines the restriction of $\tau$ to $\alpha(X)$.

We say that two self-adjoint operators $X_1$ and $X_2$ are independent if they generate independent sub-$C*$-algebras. Then clearly, the laws $\mu_{X_i}$ $(i = 1, 2)$ uniquely determine $\tau$ on the sub-$C*$-algebra $\alpha(X_1)\alpha(X_2)$. In particular, $X_1 + X_2 \in \alpha(X_1)\alpha(X_2)$, so $\mu_{X_1+X_2}$ should be a function of $\mu_{X_1}$ and $\mu_{X_2}$. Indeed,

$$\mu_{X_1+X_2} = \mu_{X_1} * \mu_{X_2},$$

where $*$ denotes convolution of probability measures.

## 3 Free independence

Let $\mathcal{A}$ be a $C*$-algebra and let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ be sub-$C*$-algebras. Let $1, A^1_1, \ldots, A^1_n \in \mathcal{A}_1$ be linearly independent and likewise let $1, A^2_1, \ldots, A^2_m \in \mathcal{A}_2$ be linearly independent. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *free* if all elements of the form

$$1,\ A^1_i,\ A^2_i,\ A^1_i A^2_j,\ A^2_i A^1_j,\ A^1_i A^2_j A^1_k,\ A^2_i A^1_j A^2_k,\ A^1_i A^2_j A^1_k A^2_l, \ldots \tag{6}$$

are linearly independent. Note that this says that in a sense, the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are maximally noncommuting. Let $\alpha(\mathcal{A}_1 \cup \mathcal{A}_2)$ denote the smallest $C*$-algebra containing $\mathcal{A}_1$ and $\mathcal{A}_2$, i.e., the closure of the linear span of all elements of the form (6). Note that $\alpha(\mathcal{A}_1 \cup \mathcal{A}_2)$ is infinite dimensional as soon as $\mathcal{A}_1$ and $\mathcal{A}_2$ each have dimension $\geq 2$.

Let $\tau$ be a state on $\mathcal{A}$ and let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ be sub-$C*$-algebras. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *freely independent* if

$$\tau(A^1_i A^2_j) = 0,\ \tau(A^2_j A^1_i) = 0,\ \tau(A^1_i A^2_j A^1_k) = 0,\ \tau(A^2_j A^1_i A^2_l) = 0,\ \tau(A^1_i A^2_j A^1_k A^2_l) = 0, \ldots \tag{7}$$

whenever $A^1_i, A^1_k, \ldots \in \mathcal{A}_1$ and $A^2_j, A^2_l, \ldots \in \mathcal{A}_2$ satisfy $\tau(A^1_i) = 0$, $\tau(A^1_k) = 0$, $\tau(A^2_j) = 0$, $\tau(A^2_l) = 0$, etc.

**Proposition 1 (Free product measure)** *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be free and let $\tau_i$ be states on $\mathcal{A}_i$ $(i = 1, 2)$. Then there exists a unique state $\tau_{12}$ on $\alpha(\mathcal{A}_1 \cup \mathcal{A}_2)$ whose restriction to $\mathcal{A}_i$ is $\tau_i$ $(i = 1, 2)$ such that under $\tau_{12}$, the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are freely independent. If we drop the assumption that $\mathcal{A}_1$ and $\mathcal{A}_2$ are free, then the uniqueness statement still holds (but $\tau_{12}$ may fail to exist in general).*

**Proof** See Exercises 2.5.17 and 2.5.18. Let $X \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$. Then $X - \tau(X)1$ has trace zero and hence

$$0 = \tau\big((X - \tau(X))(Y - \tau(Y))\big) = \tau(XY) - \tau(X)\tau(Y),$$

from which we see that

$$\tau(XY) = \tau(X)\tau(Y). \tag{8}$$

Similarly,

$$\begin{aligned}
0 &= \tau\big((X - \tau(X))(Y - \tau(Y))(X - \tau(X))\big) \\
&= \tau(XYX) - \tau(XY)\tau(X) - \tau(X^2)\tau(Y) - \tau(YX)\tau(X) \\
&\quad +3\tau(X)\tau(Y)\tau(X) - \tau(X)\tau(Y)\tau(X) \\
&= \tau(XYX) - \tau(X^2)\tau(Y),
\end{aligned}$$

where in the last step we have used (8). It follows that

$$\tau(XYX) = \tau(X^2)\tau(Y), \tag{9}$$

which in fact we would also have if $X$ and $Y$ were independent (and would commute). In a similar way

$$\tau(X^2 Y) = \tau(X^2)\tau(Y), \tag{10}$$

which is again the same as we would get in the independent case. However, continuing in the same spirit, we find that

$$\begin{aligned}
0 &= \tau\big((X - \tau(X))(Y - \tau(Y))(X - \tau(X))(Y - \tau(Y))\big) \\
&= \tau(XYXY) - \tau(XYX)\tau(Y) - \tau(XY^2)\tau(X) - \tau(X^2Y)\tau(Y) - \tau(YXY)\tau(X) \\
&\quad +\tau(XY)\tau(X)\tau(Y) + \tau(X^2)\tau(Y)^2 + \tau(YX)\tau(X)\tau(Y) + \tau(Y^2)\tau(X)^2 \\
&\quad -4\tau(X)^2\tau(Y)^2 + \tau(X)^2\tau(Y)^2,
\end{aligned}$$

from which using (8), (9), and (10) we obtain

$$\tau(XYXY) = \tau(X^2)\tau(Y^2) - \tau(X^2)\tau(Y)^2 - \tau(X)^2\tau(Y^2). \tag{11}$$

This time, we get something different from the independent case. Nevertheless, it is not hard to show by induction that using (7), one can express $\tau$ of any mixed moment of elements of $\mathcal{A}_1$ and $\mathcal{A}_2$ in moments of elements of $\mathcal{A}_1$ and $\mathcal{A}_2$ separately. ∎

We say that two self-adjoint operators $X_1$ and $X_2$ are freely independent if they generate freely independent sub-$C*$-algebras. It follows from Proposition 1 that if $X_1$ and $X_2$ are freely independent, then the law of $\mu_{X_1+X_2}$ (and in fact any reasonable function of $X_1$ and $X_2$) is uniquely determined by the marginal laws $\mu_{X_1}$ and $\mu_{X_2}$, so we can write

$$\mu_{X_1+X_2} = \mu_{X_1} \boxplus \mu_{X_2},$$

where $\boxplus$ is called *free convolution* of probability measures.

Free independence of three or more algebras is defined in a similar way as for two algebras, i.e., algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are free if, roughly speaking, a product $A_1, \ldots, A_k$ of operators from $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$ cannot be simplified if $A_i$ and $A_{i+1}$ are from different sub-algebras for each $i$. (More precisely, this should be formulated in terms of linear independence as in (6).) It is not hard to see that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are freely independent if and only if $\mathcal{A}_{i+1}$ is freely independent of $\alpha(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_i)$ for each $i$.

## 4 The Free Central Limit Theorem

We note that if $X$ and $Y$ are freely independent with mean

$$\tau(X) = \int \lambda\, \mu_X(\mathrm{d}\lambda) = 0 \quad \text{and} \quad \tau(Y) = \int \lambda\, \mu_Y(\mathrm{d}\lambda) = 0,$$

then by (8),

$$\int \lambda^2\, \mu_{X+Y}(\mathrm{d}\lambda) = \tau\big((X+Y)^2\big) = \tau(X^2) + \tau(Y^2).$$

More generally, the variance of $X + Y$ is the sum of the variances of $X$ and $Y$. We recall that the (standard) semicircle law has mean zero and variance $C_{2/2} = 1$. More generally, we can define semicircle laws with any mean and variance by adding a constant and scaling. The following proposition and theorem show that free independence is indeed very similar to classical independence.

**Proposition 2 (Stability of the semicircle law)** *Assume that $X_1, \ldots, X_k$ are freely independent and that $X_i$ has a semicircle law with mean $\tau(X_i)$ and variance $\mathrm{Var}(X_i) := \tau\big((X_i - \tau(X_i))^2\big)$. Then $\sum_{i=1}^k X_i$ has a semicircle law with mean $\sum_{i=1}^k \tau(X_i)$ and variance $\sum_{i=1}^k \mathrm{Var}(X_i)$.*

**Theorem 3 (Free Central Limit Theorem)** *Let $(X_i)_{i\geq 1}$ bee freely independent and identically distributed with mean zero and variance 1. Then the law of $\frac{1}{\sqrt{n}}\sum_{i=1}^k X_i$ converges weakly to the semicircle law.*

Before we give some idea of how Proposition 2 and Theorem 3 can be proved, we first discuss the relation of free independence to random matrix theory. Recall that the space $\mathcal{A}_n$ of all random $n \times n$ matrices with finite moments of all orders on a given probability space forms a $*$-algebra, and that $\tau_n(X) := \frac{1}{n}\mathbb{E}[\mathrm{tr}(X)]$ defines a normalized pseudotrace. Let $M_n$ be a Wigner matrix where all upper diagonal entries are i.i.d. with mean zero, variance 1 (!), and finite moments of all orders. Set $X_n := n^{-1/2}M_n$. Note that $\mu_{X_n}$ is the mean of the empirical distribution of $X_n$. Since the empirical distribution of $X_n$ converges in expectation to the semicircle law, $\mu_{X_n}$ converges to the semicircle law.

Let $X_n'$ be an independent copy of $X_n$. We claim that $X_n$ and $X_n'$ are *asymptotically free* in the sense that

$$\tau\big(\{F(X_n) - \tau(F(X_n))\}\{G(X_n') - \tau(G(X_n'))\}\big) \underset{n\to\infty}{\longrightarrow} 0$$

for any fixed polynomials $F, G$. (See Proposition 2.5.1 in the book for a sketch of a proof.) Now it is actually not so hard to show (see Section 2.5.2 in the book) that, going to a subsequence if necessary, we can find a $*$-algebra and pseudotrace $(\mathcal{A}_\infty, \tau_\infty)$ as well as elements $X_\infty, X_\infty' \in \mathcal{A}_\infty$ such that all mixed moments of $X_n$ and $X_n'$ converge to those of $X_\infty$ and $X_\infty'$. In particular, this implies that $X_\infty$ and $X_\infty'$ are freely independent. We know that $\mu_{X_n}$ and $\mu_{X_n'}$ converge to the standard semicircle law while $\mu_{X_n+X_n'}$ converges to the semicircle law with mean zero and variance 2. This explains why the limit law we find in random matrix theory (the semicircle law) should be stable under free convolution.

**Proof of Proposition 2 (sketch)** We only consider the case of two freely independent $X_1, X_2$ that each have a standard semicircle law. Consider the Hilbert space $\ell^2(\mathbb{N})$ of square integrable functions $f : \mathbb{N} \to \mathbb{C}$. Let $e_0, e_1, \ldots$ denote the usual basis. Set $\mathcal{A} := \ell^2(\mathbb{N})$ and define

$$\tau(A) := \langle e_0, Ae_0 \rangle.$$

It is not hard to check this is a normalized positive linear form (but not faithful and not a pseudotrace). Define $U : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $Ue_n := e_{n+1}$ $(n \geq 0)$. It is not hard to see that $U$ is a unitary operator and $U^* e_n = e_{n-1}$ $(n \geq 1)$ while $U^* e_0 = 0$. Set $X = U + U^*$, which is self-adjoint. We claim that $\mu_X$ is the semicircle law. This can be proved by showing that the $k$-th moment of $X$ equals the number of walks $\omega : \{0, \ldots, k\} \to \mathbb{N}$ such that $\omega_{i+1} = \omega_i \pm 1$ for all $i$ and $\omega_0 = \omega_k = 0$. As we have already seen, this is zero for odd $k$ and equal to the Catalan number $C_{k/2}$ for even $k$. (See Exercises 2.5.12 and 2.5.13.) ∎