# Exam Quantum Probability

July 4th, 2017

*Hints:* You can use all results proved in the lecture notes (without proving them yourselves), as well as claims one is supposed to prove in exercises from the lecture notes. You can also use a claim you are supposed to prove in one excercise below to solve another excercise (even if you did not prove the claim). Partial solutions (especially in Exercise 3 (b) and (c)) also yield points.

**Exercise 1 (Skew symmetric operators)** Let $\mathcal{H}$ be an inner product space. By definition, an operator $A \in \mathcal{L}(\mathcal{H})$ is *skew symmetric* if $A^* = -A$.

**(a)** Prove that every skew symmetric operator is normal.

**(b)** Prove that the operator

$$B := e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

is unitary if and only if $A$ is skew symmetric.

**Exercise 2 (An adjoint operation)** Let $\mathcal{L}(\mathbb{C}^2)$ denote the space of all complex $2 \times 2$ matrices. Let $A \mapsto A^\circ$ be the map defined by

$$A^\circ := MA^*M \qquad \big(A \in \mathcal{L}(\mathbb{C}^2)\big),$$

where $M$ is the matrix

$$M := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $A^*$ denotes the adjoint of $A$ with respect to the usual inner product

$$\left\langle \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \middle| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\rangle := \phi_1^* \psi_1 + \phi_2^* \psi_2.$$

**(a)** Show that $A \mapsto A^\circ$ is an adjoint operation on the algebra $\mathcal{L}(\mathbb{C}^2)$. (You do not have to prove that $\mathcal{L}(\mathbb{C}^2)$ is an algebra.)

**(b)** Show that $\langle \phi | M | \phi \rangle = 0$ implies $(|\phi\rangle\langle\phi|)^\circ \, (|\phi\rangle\langle\phi|) = 0$.

**(c)** Show that the adjoint operation $A \mapsto A^\circ$ is not positive in the sense defined in Section 2.1 of the lecture notes.

*Please turn over.*

**Exercise 3 (W state)** Let $\mathcal{H}$ be a two-dimensional inner product space with orthonormal basis $\{e(1), e(2)\}$. Let $\psi \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ be defined by

$$\psi := \frac{1}{\sqrt{3}}\big(e(1) \otimes e(1) \otimes e(2) + e(1) \otimes e(2) \otimes e(1) + e(2) \otimes e(1) \otimes e(1)\big)$$

and let $\rho = \rho_\psi$ be the associated pure state on $\mathcal{L}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \cong \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H})$.

**(a)** Let $P_2 := |e(2)\rangle\langle e(2)|$. Calculate $\rho(1 \otimes 1 \otimes P_2)$.

**(b)** Let $\rho_{12}$ be the marginal describing the first two subsystems, i.e.,

$$\rho_{12}(A_1 \otimes A_2) := \rho(A_1 \otimes A_2 \otimes 1).$$

Show that

$$\rho_{12} = \tfrac{1}{3}\rho_\xi + \tfrac{2}{3}\rho_\eta,$$

where

$$\xi := e(1) \otimes e(1) \quad \text{and} \quad \eta := \frac{1}{\sqrt{2}}\big(e(1) \otimes e(2) + e(2) \otimes e(1)\big).$$

*Hint* To simplify your calculations, you can use the physicist's informal notation

$$|112\rangle := \big|e(1) \otimes e(1) \otimes e(2)\big\rangle, \quad \langle 121| := \big\langle e(1) \otimes e(2) \otimes e(1)\big|, \quad \text{etc.}$$

**(c)** *Bonus question:* Do you think $\rho_{12}$ is entangled?

**Solutions**

**Ex 1**

One possible solution is to observe that $A$ is skew symmetric if and only if it is of the form $A = iB$ with $B$ a hermitian operator, as follows by writing $(iB)^* = -i \cdot -B = iB$. By Exercise 1.2.8 of the lecture notes, an operator is hermitian if and only if it is normal and all its eigenvalues are real. Since $B$ normal $\Rightarrow iB$ normal this proves part (a). By Exercise 1.2.7 of the lecture notes, an operator $U \in \mathcal{L}(\mathcal{H})$ is unitary if and only if it is normal and all its eigenvalues $\lambda_i$ satisfy $|\lambda_i| = 1$. Any normal operator $B$ can be written as

$$B = \sum_j \lambda_j |e(j)\rangle\langle e(j)|$$

where $\{e(1), \ldots, e(n)\}$ is an orthonormal basis and $\lambda_j$ are the eigenvalues of $B$. By Exercise 1.2.13 of the lecture notes,

$$e^{iB} = \sum_j e^{i\lambda_j} |e(j)\rangle\langle e(j)|$$

Since $|e^{i\lambda_j}| = 1$ if and only if $\lambda_j \in \mathbb{R}$, this proves $e^{iB}$ unitary $\Leftrightarrow$ $B$ hermitian $\Leftrightarrow$ $A$ skew symmetric. (Compare also Exercise 1.2.14 where one is supposed to prove the first implication $\Leftarrow$.)

Alternatively, since $A^*A = -A^2 = AA^*$, we see that each skew symmetric operator is normal and hence can be diagonalized with respect to an orthonormal basis. Since

$$\Big(\sum_j \lambda_j |e(j)\rangle\langle e(j)|\Big)^* = \sum_j \lambda_j^* |e(j)\rangle\langle e(j)|,$$

we see that $A$ is skew symmetric if and only if it is normal and all its eigenvalues are strictly imaginary. The solution then proceeds as above.

**Ex 2**

We observe that $M = M^*$ and $M^2 = 1$. Since

(i) $(A^\circ)^\circ = M(MA^*M)^*M = MM^*AM^*M = A$,

(ii) $(aA + bB)^\circ = M(aA + bB)^*M = a^*MA^*M + b^*MB^*M = a^*A^\circ + b^*B^\circ$,

(iii) $(AB)^\circ = M(AB)^*M = MB^*A^*M = MB^*MMA^*M = B^\circ A^\circ$,

we see that $A \mapsto A^\circ$ is an adjoint operation, solving part (a).

Part (b) follows simply by observing that $\langle\phi|M|\phi\rangle = 0$ implies

$$M(|\phi\rangle\langle\phi|)^*M|\phi\rangle\langle\phi| = M|\phi\rangle\langle\phi|M|\phi\rangle\langle\phi| = 0.$$

In view of part (b), to solve part (c), it suffices to find a nonzero $\phi \in \mathbb{C}^2$ such that $\langle\phi|M|\phi\rangle = 0$. Here

$$\langle\phi|M|\phi\rangle = \sum_{ij} \phi_i^* M_{ij}\phi_j = \phi_1^*\phi_1 - \phi_2^*\phi_2,$$

so setting $\phi_1 = \phi_2 = 1$ solves the exercise.

**Ex 3**
**(a)** Since

$$(1 \otimes 1 \otimes P_2)(\phi \otimes \psi \otimes \chi) = \phi \otimes \psi \otimes (P_2\chi),$$

and $P_2 e(1) = 0$, $P_2 e(2) = e(2)$, we see that

$$(1 \otimes 1 \otimes P_2)\psi = \frac{1}{\sqrt{3}} e(1) \otimes e(1) \otimes e(2)$$

and hence

$$\rho_\psi(1 \otimes 1 \otimes P_2) = \langle \psi | 1 \otimes 1 \otimes P_2 | \psi \rangle = \frac{1}{3},$$

where we have used that vectors of the form $e(i) \otimes e(j) \otimes e(k)$ form an orthonormal basis for $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$.

**(b)** Using the physicist's simplified notation, we have that

$$\begin{aligned}\rho_{12}(A \otimes B) &= \langle \psi | A \otimes B \otimes 1 | \psi \rangle \\ &= \frac{1}{3}\big(\langle 112| + \langle 121| + \langle 211|\big)\big(A \otimes B \otimes 1\big)\big(|112\rangle + |121\rangle + |211\rangle\big).\end{aligned}$$

We observe that in our new notation

$$\begin{aligned}(A \otimes 1 \otimes 1)|112\rangle &= (A \otimes 1 \otimes 1)\big(e(1) \otimes e(1) \otimes e(2)\big) = Ae(1) \otimes e(1) \otimes e(2) \\ &= A_{11} e(1) \otimes e(1) \otimes e(2) + A_{21} e(2) \otimes e(1) \otimes e(2) = A_{11}|112\rangle + A_{21}|212\rangle,\end{aligned}$$

and similarly

$$\begin{aligned}(A \otimes B \otimes 1)|112\rangle &= A_{11}B_{11}|112\rangle + A_{21}B_{11}|212\rangle + A_{11}B_{21}|122\rangle + A_{21}B_{21}|222\rangle, \\ (A \otimes B \otimes 1)|121\rangle &= A_{11}B_{12}|111\rangle + A_{21}B_{12}|211\rangle + A_{11}B_{22}|121\rangle + A_{21}B_{22}|221\rangle, \\ (A \otimes B \otimes 1)|211\rangle &= A_{12}B_{11}|111\rangle + A_{22}B_{11}|211\rangle + A_{12}B_{21}|121\rangle + A_{22}B_{21}|221\rangle.\end{aligned}$$

Using the fact that vectors of the form $|111\rangle, |112\rangle, \ldots$ are orthonormal, we see that

$$\begin{aligned}\rho_{12}(A \otimes B) &= \frac{1}{3}\big(A_{11}B_{11} + A_{21}B_{12} + A_{11}B_{22} + A_{22}B_{11} + A_{12}B_{21}\big) \\ &= \frac{1}{3}\big(\langle 11|A \otimes B|11\rangle + \langle 21|A \otimes B|12\rangle + \langle 12|A \otimes B|12\rangle \\ &\qquad + \langle 21|A \otimes B|21\rangle + \langle 12|A \otimes B|21\rangle\big) \\ &= \frac{1}{3}\langle 11|A \otimes B|11\rangle + \frac{2}{3}\langle \eta|A \otimes B|\eta\rangle.\end{aligned}$$

A simpler way to arrive at the same answer is to note that

$$\psi = \frac{1}{\sqrt{3}}\big\{\xi \otimes e(2) + 2\eta \otimes e(1)\big\}.$$

Let $P_i := |e(i)\rangle\langle e(i)|$. Performing the ideal measurement $\{P_1, P_2\}$ on the third subsystem changes $\rho_\psi$ into

$$\tfrac{1}{3}\rho_\xi \otimes \rho_{e(2)} + \tfrac{2}{3}\rho_\eta \otimes \rho_{e(1)}.$$

Since performing a operation on the third system does not change the joint law of the first two systems, this also proves the claim. More precisely, this argument goes as follows. We define an operation $T'$ by

$$(T'\rho)(A) := \rho\big((1 \otimes 1 \otimes P_1)A(1 \otimes 1 \otimes P_1)\big) + \rho\big((1 \otimes 1 \otimes P_2)A(1 \otimes 1 \otimes P_2)\big)$$

$\big(A \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})\big)$. By Proposition 8.4.1 of the lecture notes, this formula defines an operation. By Lemma 8.4.3, $\rho_{12} = (T'\rho)_{12}$, i.e., performing an operation on the third subsystem does not change the marginal describing the first and second subsystem. Now for $\rho = \rho_\psi$, we get

$$(T'\rho)(A) := \langle\psi|(1 \otimes 1 \otimes P_1)A(1 \otimes 1 \otimes P_1)|\psi\rangle + \langle\psi|(1 \otimes 1 \otimes P_2)A(1 \otimes 1 \otimes P_2)|\psi\rangle.$$

Using the fact that

$$(1 \otimes 1 \otimes P_1)\psi = \tfrac{2}{\sqrt{3}}\eta \otimes e(1) \quad \text{and} \quad (1 \otimes 1 \otimes P_2)\psi = \tfrac{1}{\sqrt{3}}\xi \otimes e(2),$$

we see that

$$T'\rho = \tfrac{1}{3}\rho_{\xi\otimes e(2)} + \tfrac{2}{3}\rho_{\eta\otimes e(1)}.$$

Since $\rho_{\xi\otimes e(2)} = \rho_\xi \otimes \rho_{e(2)}$ and $\rho_{\eta\otimes e(1)} = \rho_\eta \otimes \rho_{e(1)}$, it is easy to see that the marginal $\rho_{12} = (T'\rho)_{12}$ is given by

$$\rho_{12} = \tfrac{1}{3}\rho_\xi + \tfrac{2}{3}\rho_\eta.$$

**(c)** We know that states of the form $\eta$ are entangled; see Section 7.3 in the lecture notes. However, this does not necessarily mean that $\rho_{12}$ is entangled. Nevertheless, it is claimed on Wikipedia that this is the case, but I do not see an easy way to prove this.