# Exam Quantum Probability

June 30th, 2017

*Hints:* You can use all results proved in the lecture notes (without proving them yourselves), as well as claims one is supposed to prove in exercises from the lecture notes. You can also use a claim you are supposed to prove in one excercise below to solve another excercise (even if you did not prove the claim). Partial solutions also yield points.

**Exercise 1 (Lie algebras)** Let $\mathcal{A}$ be an algebra. Recall that for any $A, B \in \mathcal{A}$, the *commutator* of $A$ and $B$ is defined as $[A, B] := AB - BA$. A linear subspace $\mathcal{B} \subset \mathcal{A}$ is a *Lie algebra* if $A, B \in \mathcal{B} \;\Rightarrow\; [A, B] \in \mathcal{B}$. Recall that $\mathcal{B}$ is itself an algebra (in fact, a subalgebra of $\mathcal{A}$) if $A, B \in \mathcal{B} \;\Rightarrow\; AB \in \mathcal{B}$.

**(a)** Let $\mathcal{V}$ be a (finite dimensional) linear space and let $\mathcal{L}(\mathcal{V})$ be the algebra of all linear operators $A : \mathcal{V} \to \mathcal{V}$. Let $\mathrm{tr}(A)$ denote the trace of an operator $A \in \mathcal{L}(\mathcal{V})$. Prove that $\mathcal{B} := \{A \in \mathcal{L}(\mathcal{V}) : \mathrm{tr}(A) = 0\}$ is a Lie algebra.

**(b)** Let $\mathcal{A}$ be a $*$-algebra. Prove that the real linear space $\mathcal{B} := \{A \in \mathcal{A} : A^* = -A\}$ is a Lie algebra.

**(c)** Show by example that not every Lie algebra is an algebra.

**Exercise 2 (A particle with spin 1)** For each $\alpha \in [0, 2\pi)$, let $T_\alpha$ denote the hermitian matrix

$$T_\alpha := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & e^{-i\alpha} & 0 \\ e^{i\alpha} & 0 & e^{-i\alpha} \\ 0 & e^{i\alpha} & 0 \end{pmatrix}.$$

**(a)** Show that there exist an orthonormal basis $\{\phi_\alpha, \eta_\alpha, \psi_\alpha\}$ of $\mathbb{C}^3$ such that

$$T_\alpha \phi_\alpha = -\phi_\alpha, \quad T_\alpha \eta_\alpha = 0, \quad \text{and} \quad T_\alpha \psi_\alpha = \psi_\alpha,$$

i.e., $\phi_\alpha, \eta_\alpha, \psi_\alpha$ are eigenvectors of $T_\alpha$ with eigenvalues $-1, 0, 1$, respectively.

**(b)** Show that $\psi_{\alpha+\pi} = \lambda \phi_\alpha$, where $\lambda \in \mathbb{C}$ (which may depend on $\alpha$) satisfies $|\lambda| = 1$.

**(c)** We prepare a physical system in the pure state $\rho_{\psi_\alpha}$ corresponding to the state vector $\psi_\alpha$, which corresponds to the eigenvalue $+1$ of $T_\alpha$, and then perform an ideal measurement of the observable $T_\beta$. Calculate the probability $\rho_{\psi_\alpha}(|\psi_\beta\rangle\langle\psi_\beta|)$ that this ideal measurement yields the value $+1$.

*Please turn over.*

**Exercise 3 (The original Bell inequality)** Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Q-algebras and let $\rho$ be a state on $\mathcal{A}_1 \otimes \mathcal{A}_2$. For projections $P \in \mathcal{A}_1$ and $Q \in \mathcal{A}_2$, let

$$\varepsilon_\rho(P,Q) := \rho(P \otimes Q) + \rho\big((1-P) \otimes (1-Q)\big)$$

denote the probability that in a simultaneous measurement, $P$ and $Q$ are either both true, or both false.

**(a)** Show that if $\rho = \rho_1 \otimes \rho_2$ is a product state, then for any projections $P, P' \in \mathcal{A}_1$ and $Q, Q' \in \mathcal{A}_2$, one has

$$\varepsilon_\rho(P,Q) \leq \varepsilon_\rho(P,Q') + \varepsilon_\rho(P',Q') + \varepsilon_\rho(P',Q). \tag{1}$$

*Hint* Set $\mu_+ := \rho_1(P)$, $\mu_- := \rho_1(1-P)$, $\nu_+ := \rho_2(Q)$, $\nu_- := \rho_2(1-Q)$, and define $\mu'_\pm, \nu'_\pm$ similarly with $P$ and $Q$ replaced by $P'$ and $Q'$. Show that the difference of the right- and left-hand sides of (1) can be written as

$$\sum_{\sigma_1} \sum_{\sigma'_1} \sum_{\sigma_2} \sum_{\sigma'_2} \mu_{\sigma_1} \mu'_{\sigma_1} \nu_{\sigma_2} \nu'_{\sigma_2} \big(1_{\{\sigma_1 = \sigma'_2\}} + 1_{\{\sigma'_1 = \sigma'_2\}} + 1_{\{\sigma'_1 = \sigma_2\}} - 1_{\{\sigma_1 = \sigma_2\}}\big),$$

where we sum over $\sigma_1 \in \{-,+\}$ etc.

**(b)** Show that (1) holds if $\rho$ is not entangled.

**Solutions**

**Ex 1**

**(a)** We need to check that $A, B \in \mathcal{B}$ implies $[A, B] \in \mathcal{B}$. We calculate $\mathrm{tr}([A, B]) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$ where we have used the property of the trace that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

**(b)** Again, we need to check that $A, B \in \mathcal{B}$ implies $[A, B] \in \mathcal{B}$. We calculate

$$\begin{aligned}[A, B]^* &= (AB - BA)^* = B^*A^* - A^*B^* \\ &= (-B)(-A) - (-A)(-B) = -(AB - BA) = -[A, B],\end{aligned}$$

proving that $[A, B] \in \mathcal{B}$.

**(c)** In view of parts (a) and (b), it suffices to show that either 1. $\mathrm{tr}(A) = 0$ and $\mathrm{tr}(B) = 0$ do not imply $\mathrm{tr}(AB) = 0$, or 2. $A^* = -A$ and $B^* = -B$ do not imply $(AB)^* = -(AB)$. A counterexample of type 1 is:

$$A = B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A counterexample of type 2 is:

$$A = B = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad AB = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(In fact, the latter example works in any dimension, including dimension one.)

**Ex 2**

**(a)** For a vector of the form

$$\psi = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

we calculate $T_\alpha \psi$ as

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & e^{-i\alpha} & 0 \\ e^{i\alpha} & 0 & e^{-i\alpha} \\ 0 & e^{i\alpha} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\alpha} y \\ e^{i\alpha} x + e^{-i\alpha} z \\ e^{i\alpha} y \end{pmatrix}.$$

The eigenvalue equation $T_\alpha \psi = \lambda \psi$ then becomes the system of equations

$$\begin{aligned} \tfrac{1}{\sqrt{2}} e^{-i\alpha} y &= \lambda x, \\ \tfrac{1}{\sqrt{2}} (e^{i\alpha} x + e^{-i\alpha} z) &= \lambda y, \\ \tfrac{1}{\sqrt{2}} e^{i\alpha} y &= \lambda z. \end{aligned}$$

Setting $w = 1/\sqrt{2}y$, we can simplify this to

$$\begin{aligned} e^{-i\alpha} w &= \lambda x, \\ e^{i\alpha} x + e^{-i\alpha} z &= 2\lambda w, \\ e^{i\alpha} w &= \lambda z. \end{aligned}$$

For $\lambda = -1, 0, 1$, these equations can easily be solved to find unnormalized eigenvectors

$$\begin{pmatrix} -e^{-i\alpha} \\ \sqrt{2} \\ -e^{i\alpha} \end{pmatrix}, \quad \begin{pmatrix} e^{-i\alpha} \\ 0 \\ -e^{i\alpha} \end{pmatrix}, \quad \begin{pmatrix} e^{-i\alpha} \\ \sqrt{2} \\ e^{i\alpha} \end{pmatrix},$$

Since $T_\alpha$ is hermitian, these eigenvectors are orthogonal. We can normalize them to get

$$\phi_\alpha = \frac{1}{2} \begin{pmatrix} -e^{-i\alpha} \\ \sqrt{2} \\ -e^{i\alpha} \end{pmatrix}, \quad \eta_\alpha = \frac{1}{2} \begin{pmatrix} e^{-i\alpha} \\ 0 \\ -e^{i\alpha} \end{pmatrix}, \quad \psi_\alpha = \frac{1}{2} \begin{pmatrix} e^{-i\alpha} \\ \sqrt{2} \\ e^{i\alpha} \end{pmatrix}.$$

**(b)** Since $e^{i(\alpha+\pi)} = -e^{i\alpha}$ and $e^{-i(\alpha+\pi)} = -e^{-i\alpha}$, we see that $\psi_{\alpha+\pi} = \phi_\alpha$, so in our case $\lambda = 1$. (Since there is some freedom in how one defines $\psi_\alpha$ and $\phi_\alpha$, the constant $\lambda$ may be something different for other definitions of $\psi_\alpha$ and $\phi_\alpha$.)

**(c)** We need to calculate

$$\rho_{\psi_\alpha}(|\psi_\beta\rangle\langle\psi_\beta|) = \langle\psi_\alpha|\psi_\beta\rangle\langle\psi_\beta|\psi_\alpha\rangle = \big|\langle\psi_\alpha|\psi_\beta\rangle\big|^2.$$

This yields

$$\begin{aligned} &\left\langle \frac{1}{2} \begin{pmatrix} e^{-i\alpha} \\ \sqrt{2} \\ e^{i\alpha} \end{pmatrix} \middle| \frac{1}{2} \begin{pmatrix} e^{-i\beta} \\ \sqrt{2} \\ e^{i\beta} \end{pmatrix} \right\rangle \\ &\quad = \frac{1}{4}\left(e^{i\alpha}e^{-i\beta} + 2 + e^{-i\alpha}e^{i\beta}\right) = \tfrac{1}{2} + \tfrac{1}{2}\cos(\alpha-\beta) = \cos\big(2(\alpha-\beta)\big)^2, \end{aligned}$$

so the requested probability is

$$\rho_{\psi_\alpha}(|\psi_\beta\rangle\langle\psi_\beta|) = \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(\alpha-\beta)\right)^2 = \cos\big(2(\alpha-\beta)\big)^4.$$

**Ex 3**

**(a)** We observe that

$$\begin{aligned} &\sum_{\sigma_1}\sum_{\sigma_1'}\sum_{\sigma_2}\sum_{\sigma_2'} \mu_{\sigma_1}\mu_{\sigma_1'}\nu_{\sigma_2}\nu_{\sigma_2'}1_{\{\sigma_1=\sigma_2'\}} \\ &\quad = \Big(\sum_{\sigma_1'}\mu_{\sigma_1'}\Big)\Big(\sum_{\sigma_2}\nu_{\sigma_2}\Big)\Big(\sum_{\sigma_1}\mu_{\sigma_1}\sum_{\sigma_2'}\nu_{\sigma_2'}1_{\{\sigma_1=\sigma_2'\}}\Big) \\ &\quad = 1\cdot 1\cdot(\mu_-\nu_-' + \mu_+\nu_+') \\ &\quad = \rho_1(P)\rho_2(Q') + \rho_1(1-P)\rho_2(1-Q') \\ &\quad = \rho\big(P\otimes Q'\big) + \rho\big((1-P)\otimes(1-Q)\big) = \varepsilon_\rho(P,Q'). \end{aligned}$$

Treating the other terms similarly, we see that

$$\begin{aligned} &\sum_{\sigma_1}\sum_{\sigma_1'}\sum_{\sigma_2}\sum_{\sigma_2'} \mu_{\sigma_1}\mu_{\sigma_1}'\nu_{\sigma_2}\nu_{\sigma_2}'\big(1_{\{\sigma_1=\sigma_2'\}} + 1_{\{\sigma_1'=\sigma_2\}} + 1_{\{\sigma_1'=\sigma_2'\}} - 1_{\{\sigma_1=\sigma_2\}}\big) \\ &\quad = \varepsilon_\rho(P,Q') + \varepsilon_\rho(P',Q') + \varepsilon_\rho(P',Q) - \varepsilon_\rho(P,Q). \end{aligned}$$

We need to prove that this expression is $\geq 0$. It suffices to show that

$$1_{\{\sigma_1=\sigma_2'\}} + 1_{\{\sigma_1'=\sigma_2'\}} + 1_{\{\sigma_1'=\sigma_2\}} - 1_{\{\sigma_1=\sigma_2\}} \geq 0$$

for any choice of $\sigma_1, \sigma_1', \sigma_2, \sigma_2' \in \{-,+\}$. The statement is clearly true if at least one of the events $\sigma_1 = \sigma_2'$, $\sigma_1' = \sigma_2'$, and $\sigma_1' = \sigma_2$ is true. But if $\sigma_1 \neq \sigma_2' \neq \sigma_1' \neq \sigma_2$, then $\sigma_1 \neq \sigma_2$, so we are fine in this case as well.

**(b)** If $\rho$ is not entangled, then it is a convex combination of product states, i.e.,

$$\rho = \sum_k p_k \rho_k \quad \text{where} \quad \rho_k = \rho_{1,k} \otimes \rho_{2,k}$$

are product states and $p_k \geq 0$ are constants such that $\sum_k p_k = 1$. We observe that

$$\varepsilon_\rho(P, Q) = \rho(P \otimes Q) + \rho\big((1-P) \otimes (1-Q)\big)$$

is linear as a function of $\rho$, so

$$\varepsilon_\rho(P, Q) = \sum_k p_k \varepsilon_{\rho_k}(P, Q).$$

Now

$$\begin{aligned} &\varepsilon_\rho(P, Q') + \varepsilon_\rho(P', Q') + \varepsilon_\rho(P', Q) - \varepsilon_\rho(P, Q) \\ &\quad = \sum_k p_k \big(\varepsilon_{\rho_k}(P, Q') + \varepsilon_{\rho_k}(P', Q') + \varepsilon_{\rho_k}(P', Q) - \varepsilon_{\rho_k}(P, Q)\big) \geq 0 \end{aligned}$$

by part (a).