Exam Quantum Probability

September 5th, 2017

Hints: You can use all results proved in the lecture notes (without proving them yourselves), as well as claims one is supposed to prove in exercises from the lecture notes. You can also use a claim you are supposed to prove in one excercise below to solve another excercise (even if you did not prove the claim). Partial solutions also yield points; this is in particular true for Exercise 2 (d).

Exercise 1 (Logarithm of an operator) Let $\mathcal{H}$ be an inner product space (complex, finite dimensional). For each operator $A \in \mathcal{L}(\mathcal{H})$, we define

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

(a) Show that if $A \in \mathcal{L}(\mathcal{H})$ is a normal operator, and 0 is not an eigenvalue of A, then there exists a normal operator $\log(A) \in \mathcal{L}(\mathcal{H})$ such that

$$e^{\log(A)} = A. \tag{1}$$

(b) Show that the operator $\log(A)$ is in general not uniquely defined by (1).

(c) Show that if A is a positive operator, then $\log(A)$ can be taken hermitian, and with this extra condition it is unique.

Exercise 2 (Repeated measurements) Let α, β be real numbers with $\alpha < \beta$. Fix $n \geq 1$ and for $k = 0, 1, \ldots, n$, define

$$\alpha_k := \tfrac{n-k}{n}\alpha + \tfrac{k}{n}\beta.$$

We perform, in sequence, $n+1$ ideal measurements on the same photon, measuring its polarization along the directions $\alpha_0, \alpha_1, \ldots, \alpha_n$. The polarization of the photon is described by a Q-algebra of the form $\mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ has dimension 2 and orthonormal basis $\{e(1), e(2)\}$. For $\alpha \in \mathbb{R}$, define $\eta(\alpha), \xi(\alpha) \in \mathcal{H}$ by

$$\begin{aligned} \eta(\alpha) &:= \cos(\alpha)e(1) + \sin(\alpha)e(2), \\ \xi(\alpha) &:= -\sin(\alpha)e(1) + \cos(\alpha)e(2) = \eta(\alpha + \pi/2). \end{aligned}$$

Let $P_\alpha := |\eta(\alpha)\rangle\langle\eta(\alpha)|$ and $Q_\alpha := |\xi(\alpha)\rangle\langle\xi(\alpha)|$ denote the projections on the orthogonal subspaces spanned by $\eta(\alpha)$ and $\xi(\alpha)$, respectively. For $k = 0, \ldots, n$ we perform, in sequence, the ideal measurements corresponding to the partitions of the identity $\{P_{\alpha_k}, Q_{\alpha_k}\}$.

(a) Conditional on the event that the first measurement yields the outcome P_{α_0}, give an expression for the probability that the following n measurements yield the outcomes $P_{\alpha_1}, \ldots, P_{\alpha_n}$.

(b) Calculate the limit of the expression in (a) as $n \to \infty$, for fixed α and β.

(c) For $k = 0, \ldots, n$, let $T_k : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ denote the map

$$T_k(A) := P_{\alpha_k} A P_{\alpha_k} + Q_{\alpha_k} A Q_{\alpha_k},$$

and define $T'_k : \mathcal{L}(\mathcal{H})' \to \mathcal{L}(\mathcal{H})'$ by $(T'_k \rho)(A) := \rho(T_k(A))$. Show that T'_k is an operation, i.e., cite the right proposition from the lecture notes and check that its conditions are satisfied.

(d) Show that

$$\lim_{n\to\infty} T_n \circ \cdots \circ T_0 = T,$$

where $T' : \mathcal{L}(\mathcal{H})' \to \mathcal{L}(\mathcal{H})'$ is the operation defined by

$$T(A) := |\eta(\beta)\rangle\langle\eta(\alpha)|A|\eta(\alpha)\rangle\langle\eta(\beta)| + |\xi(\beta)\rangle\langle\xi(\alpha)|A|\xi(\alpha)\rangle\langle\xi(\beta)|.$$

Exercise 3 (Correlation versus anticorrelation) Let $\mathcal{H}$ be a two-dimensional inner product space with orthonormal basis $\{e(1), e(2)\}$. Let $\alpha, \beta \in \mathbb{R}$ and let $\{f(1), f(2)\}$ be the orthonormal basis of $\mathcal{H}$ given by

$$\begin{aligned} f(1) &:= \cos(\alpha)e(1) + e^{i\beta}\sin(\alpha)e(2), \\ f(2) &:= -e^{-i\beta}\sin(\alpha)e(1) + \cos(\alpha)e(2). \end{aligned}$$

Consider the vectors $\psi, \phi \in \mathcal{H} \otimes \mathcal{H}$ defined as

$$\begin{aligned} \psi &:= \frac{1}{\sqrt{2}}\big(e(1)\otimes e(1) + e(2)\otimes e(2)\big), \\ \phi &:= \frac{1}{\sqrt{2}}\big(e(1)\otimes e(2) - e(2)\otimes e(1)\big). \end{aligned}$$

While answering the following questions, it may be convenient to use the physicist's shortened notation

$$|11\rangle = e(1)\otimes e(1), \quad |12\rangle = e(1)\otimes e(2), \quad \text{etc.}$$

(a) Show that ϕ looks the same with respect to the orthonormal basis $\{f(1), f(2)\}$, i.e.,

$$\phi = \frac{1}{\sqrt{2}}\big(f(1)\otimes f(2) - f(2)\otimes f(1)\big).$$

(b) Show that if $\beta = 0$ or $\beta = \pi$, then ψ also looks the same, i.e.,

$$\psi = \frac{1}{\sqrt{2}}\big(f(1)\otimes f(1) + f(2)\otimes f(2)\big).$$

(c) Let $P \in \mathcal{L}(\mathcal{H})$ be a projection on a one-dimensional subspace and let ρ_ϕ be the pure state associated with ϕ. Show that

$$\rho_\phi(P \otimes P) = 0.$$

(d) Show that there exists a projection $P \in \mathcal{L}(\mathcal{H})$ on a one-dimensional subspace such that

$$\rho_\psi\big(P \otimes (1 - P)\big) \neq 0.$$

Solutions

Ex 1

(a) If A is normal, then there exists an orthonormal basis $\{e(1), \dots, e(n)\}$ of $\mathcal{H}$ such that

$$A = \sum_{k=1}^{n} \lambda_k |e(k)\rangle\langle e(k)|,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A (not necessarily all different). Exercise 1.2.13 of the lecture notes tells us that

$$e^A = \sum_{k=1}^{n} e^{\lambda_k} |e(k)\rangle\langle e(k)|.$$

We recall that if $z = re^{i\phi}$ is a nonzero complex number written in radial form, with $r, \phi \in \mathbb{R}$ and $-\pi < \phi \leq \pi$, then $\log(z) := \log(r) + i\phi$ has the property that $e^{\log(z)} = z$. However, $\log(z)$ (defined in this way) is not the only complex number with this property, since

$$e^{\log(z) + i2\pi m} = z$$

for each $m \in \mathbb{Z}$. Using these observations, we see that if we define $\log(A)$ using the functional calculus for normal operators, i.e., if we set

$$\log(A) := \sum_{k=1}^{n} \log(\lambda_k) |e(k)\rangle\langle e(k)|,$$

then $e^{\log(A)} = A$.

(b) By our earlier remarks, for any $m_1, \dots m_n \in \mathbb{Z}$, the operator

$$B := \sum_{k=1}^{n} \big(\log(\lambda_k) + i2\pi m_k\big) |e(k)\rangle\langle e(k)|$$

has the property that $e^B = A$.

(c) A normal operator is hermitian if and only if its eigenvalues are real (see Exercise 1.2.8 in the lecture notes). Since $\log(r) \in \mathbb{R}$ for all $r \in \mathbb{R}_+$, it follows that $\log(A)$ (the way we have defined it) is hermitian if A is positive. Since for $r \in \mathbb{R}_+$, there is only one real number s such that $e^s = r$, namely $s = \log r$, it follows that $\log(A)$ is the only hermitian matrix such that $e^{\log(A)} = A$.

Ex 2

(a) Conditional on the event that the first measurement yields the outcome $P_{\alpha_0} = P_\alpha$, our system is described by the pure state $\rho_{\eta(\alpha)}$. This is true regardless of the initial state ρ and follows from point 5° of our interpretation of quantum probability spaces, which says that after we observe P_α, the new state is

$$\rho'(A) = \frac{\rho\big(|\eta(\alpha)\rangle\langle\eta(\alpha)|A|\eta(\alpha)\rangle\langle\eta(\alpha)|\big)}{\rho\big(|\eta(\alpha)\rangle\langle\eta(\alpha)|} = \langle\eta(\alpha)|A|\eta(\alpha)\rangle = \rho_{\eta(\alpha)}(A).$$

Similarly, after the $(k-1)$-th measurement has yielded the outcome $P_{\alpha_{k-1}}$, we have to describe our system with the pure state $\rho_{\eta(\alpha_{k-1})}$. In the state $\rho_{\eta(\alpha_{k-1})}$, the probability that an ideal measurement yields the outcome P_{α_k} is

$$\rho_{\eta(\alpha_{k-1})}(P_{\alpha_k}) = \langle \eta(\alpha_{k-1}) | \eta(\alpha_k) \rangle \langle \eta(\alpha_k) | \eta(\alpha_{k-1}) \rangle = \cos^2(\alpha_k - \alpha_{k-1}) = \cos^2\big(\frac{\beta-\alpha}{n}\big).$$

(See Exercise 2.3.2 in the lecture notes.) In view of this, the probability that n consecutive ideal measurements yield the outcomes $P_{\alpha_1}, \ldots, P_{\alpha_n}$, given that the initial state is $\rho_{\eta(\alpha_0)}$, is given by

$$\prod_{k=1}^{n} \cos^2\big(\tfrac{\beta-\alpha}{n}\big) = \Big(\cos\big(\tfrac{\beta-\alpha}{n}\big)\Big)^{2n}.$$

(b) Since

$$\cos\big(\tfrac{\beta-\alpha}{n}\big) = 1 - \tfrac{1}{2}(\beta-\alpha)^2 n^{-2} + O(n^{-4})$$

as $n \to \infty$, we see that

$$\Big(\cos\big(\tfrac{\beta-\alpha}{n}\big)\Big)^{2n} \underset{n\to\infty}{\longrightarrow} 1.$$

More formally, there exists a constant $K < \infty$ such that

$$\log\big(\cos\big(\tfrac{\beta-\alpha}{n}\big)\big) \le K n^{-2}$$

for all $n \ge 1$, and hence

$$2n \log\Big(\cos\big(\tfrac{\beta-\alpha}{n}\big)\Big) \underset{n\to\infty}{\longrightarrow} 0.$$

(c) By Proposition 8.4.1, $T_k' : \mathcal{L}(\mathcal{H})' \to \mathcal{L}(\mathcal{H})'$ is an operation if and only if T_k is of the form

$$T_k(A) = \sum_{m=1}^{n} V(m) A V(m)^*$$

where $V(1), \ldots, V(n) \in \mathcal{L}(\mathcal{H})$ satisfy $\sum_{m=1}^{n} V(m)V(m)^* = 1$. In our case, $V(1) = V(1)^* = P_{\alpha_k}$ and $V(2) = V(2)^* = Q_{\alpha_k}$, and

$$\sum_{m=1}^{n} V(m)V(m)^* = P_{\alpha_k}^2 + Q_{\alpha_k}^2 = P_{\alpha_k} + Q_{\alpha_k} = 1$$

since $\{P_\alpha, Q_\alpha\}$ is a partition of the identity.

(d) Let us write $V_k(1) := P_{\alpha_k}$ and $V_k(2) := Q_{\alpha_k}$. Then

$$T_n \circ \cdots \circ T_0(A) = \sum_{m_0=1}^{2} \cdots \sum_{m_n=1}^{2} V_n(m_n) \cdots V_0(m_0) A V_0(m_0) \cdots V_n(m_n).$$

This is a sum with 2^{n+1} terms. We claim that in the limit $n \to \infty$, only two terms contribute. These are the terms with either $m_0 = m_1 = \cdots = m_n = 1$ or $m_0 = m_1 = \cdots = m_n = 2$. Indeed, for $m_0 = m_1 = \cdots = m_n = 1$, we obtain

$$\begin{aligned} V_0(1) \cdots V_n(1) &= |\eta(\alpha_0)\rangle\langle\eta(\alpha_0)|\eta(\alpha_1)\rangle\langle\eta(\alpha_1) \cdots |\eta(\alpha_n)\rangle\langle\eta(\alpha_n)| \\ &= \Big(\prod_{k=1}^{n} \langle\eta(\alpha_{k-1})|\eta(\alpha_k)\rangle\Big)|\eta(\alpha_0)\rangle\langle\eta(\alpha_n)| \\ &= \Big(\cos\big(\tfrac{\beta-\alpha}{n}\big)\Big)^n |\eta(\alpha)\rangle\langle\eta(\beta)| \end{aligned}$$

By what we have already proved under (b), in the limit $n \to \infty$, this converges to $|\eta(\beta)\rangle\langle\eta(\alpha)|$. Likewise

$$V_0(2) \cdots V_n(2) \underset{n\to\infty}{\longrightarrow} |\xi(\alpha)\rangle\langle\xi(\beta)|.$$

To complete the proof, we must show that the combined effect of all other terms tends to zero. Let $\{1,2\}^{n+1}$ denote the space of all sequences $\vec{m} = (m_0, \ldots, m_n)$ of 1's and 2's and write

$$U_n(\vec{m}) := V_n(m_n) \cdots V_0(m_0).$$

Then

$$T_n \circ \cdots \circ T_0(A) = \sum_{\vec{m} \in \{1,2\}^{n+1}} U_n(\vec{m}) A U_n(\vec{m})^*.$$

Let $\underline{1}, \underline{2} \in \{1,2\}^{n+1}$ denote the sequences that consist only of 1's resp. 2's. We have just seen that

$$U_n(\underline{1}) \underset{n\to\infty}{\longrightarrow} W(1) \quad \text{and} \quad U_n(\underline{2}) \underset{n\to\infty}{\longrightarrow} W(2),$$

where

$$W(1) := |\eta(\beta)\rangle\langle\eta(\alpha)| \quad \text{and} \quad W(2) := |\xi(\beta)\rangle\langle\xi(\alpha)|$$

It follows that for any $A \in \mathcal{L}(\mathcal{H})$

$$U_n(\underline{1}) A U_n(\underline{1})^* + U_n(\underline{2}) A U_n(\underline{2})^* \underset{n\to\infty}{\longrightarrow} W(1) A W(1)^* + W(2) A W(2)^* = T(A).$$

To complete the proof, we must show that

$$\sum_{\vec{m} \neq \underline{1},\underline{2}} U_n(\vec{m}) A U_n(\vec{m})^* \underset{n\to\infty}{\longrightarrow} 0. \tag{2}$$

We start by observing that (as claimed in the exercise) T' is an operation, i.e.,

$$\sum_{m=1}^{2} W(m) W(m)^* = 1.$$

Indeed

$$\begin{aligned} \sum_{m=1}^{2} W(m) W(m)^* &= |\eta(\beta)\rangle\langle\eta(\alpha)|\eta(\alpha)\rangle\langle\eta(\beta)| + |\xi(\beta)\rangle\langle\xi(\alpha)|\xi(\alpha)\rangle\langle\xi(\beta)| \\ &= |\eta(\beta)\rangle\langle\eta(\beta)| + |\xi(\beta)\rangle\langle\xi(\beta)| = 1. \end{aligned}$$

Likewise, $(T_n \circ \cdots \circ T_0)'$ is an operation, so

$$\sum_{\vec{m} \in \{1,2\}^{n+1}} U_n(\vec{m}) U_n(\vec{m})^* = 1.$$

By what we have already proved

$$\begin{aligned} \sum_{\vec{m} \neq \underline{1}, \underline{2}} U_n(\vec{m}) U_n(\vec{m})^* &= 1 - U_n(\underline{1}) U_n(\underline{1})^* - U_n(\underline{2}) U_n(\underline{2})^* \\ &\underset{n\to\infty}{\longrightarrow} 1 - W(1)W(1)^* - W(2)W(2)^* = 0. \end{aligned}$$

This proves (2) in the special case that $A = 1$. We next observe that if P is a projection, then

$$U(\vec{m})U(\vec{m})^* - U(\vec{m})PU(\vec{m})^* = U(\vec{m})(1-P)U(\vec{m})^* = \big(U(\vec{m})(1-P)\big)\big(U(\vec{m})(1-P)\big)^*$$

is a positive operator by Exercise 1.2.16, so using the partial order on the space of operators introduced on page 14 of the lecture notes, we can estimate

$$0 \leq \sum_{\vec{m} \neq \underline{1}, \underline{2}} U_n(\vec{m}) P U_n(\vec{m})^* \leq \sum_{\vec{m} \neq \underline{1}, \underline{2}} U_n(\vec{m}) U_n(\vec{m})^* \underset{n\to\infty}{\longrightarrow} 0.$$

This proves (2) in the special case that A is a projection. By Exercise 4.1.8 of the lecture notes, the projections span the space of all operators, so we conclude that (2) holds in general.

Ex 3

(a) Using the physicist's shortened notation, we have

$$\begin{aligned} f(1) \otimes f(2) &= -e^{-i\beta} \cos\alpha \sin\alpha |11\rangle + \cos^2\alpha |12\rangle - \sin^2\alpha |21\rangle + e^{i\beta} \cos\alpha \sin\alpha |22\rangle, \\ f(2) \otimes f(1) &= -e^{-i\beta} \cos\alpha \sin\alpha |11\rangle - \sin^2\alpha |12\rangle + \cos^2\alpha |21\rangle + e^{i\beta} \cos\alpha \sin\alpha |22\rangle, \end{aligned}$$

which gives

$$f(1) \otimes f(2) - f(2) \otimes f(1) = (\cos^2\alpha + \sin^2\alpha)|12\rangle - (\cos^2\alpha + \sin^2\alpha)|21\rangle,$$

and hence

$$\frac{1}{\sqrt{2}}\big(f(1) \otimes f(2) - f(2) \otimes f(1)\big) = \frac{1}{\sqrt{2}}\big(|12\rangle - |21\rangle\big) = \phi.$$

(b) We have

$$\begin{aligned} f(1) \otimes f(1) &= \cos^2\alpha \sin\alpha |11\rangle + e^{i\beta} \cos\alpha \sin\alpha |12\rangle + e^{i\beta} \cos\alpha \sin\alpha |21\rangle + e^{i2\beta} \sin^2\alpha |22\rangle, \\ f(2) \otimes f(2) &= e^{-i2\beta} \sin^2\alpha |11\rangle - e^{-i\beta} \cos\alpha \sin\alpha |12\rangle - e^{-i\beta} \cos\alpha \sin\alpha |21\rangle + \cos^2\alpha |22\rangle, \end{aligned}$$

which in the special case $\beta \in \{0, \pi\}$ gives

$$f(1) \otimes f(1) + f(2) \otimes f(2) = (\cos^2\alpha + \sin^2\alpha)|11\rangle + (\cos^2\alpha + \sin^2\alpha)|22\rangle,$$

and hence

$$\frac{1}{\sqrt{2}}\big(f(1)\otimes f(1)+f(2)\otimes f(2)\big)=\frac{1}{\sqrt{2}}\big(|11\rangle+|22\rangle\big)=\psi.$$

(c) Let $P=|\eta\rangle\langle\eta|$, where $\eta\in\mathcal{H}$ is a vector of norm one. A general vector $\eta\in\mathcal{H}$ of norm one is of the form

$$\eta=e^{i\gamma}\big(\cos(\alpha)e(1)+e^{i\beta}\sin(\alpha)e(2)\big)$$

with $\alpha,\beta,\gamma\in\mathbb{R}$, which gives

$$P=|e^{i\gamma}f(1)\rangle\langle e^{i\gamma}f(1)|=|f(1)\rangle\langle f(1)|.$$

By part (a), it follows that

$$(P\otimes P)\phi=\frac{1}{\sqrt{2}}\Big(Pf(1)\otimes Pf(2)-Pf(2)\otimes Pf(1)\Big)=0,$$

and hence

$$\rho_\phi(P\otimes P)=\langle\phi|P\otimes P|\phi\rangle=0.$$

(d) As we have seen in part (c), without loss of generality, we can take $P=|f(1)\rangle\langle f(1)|$. Using the physicist's shortened notation

$$|1'1'\rangle=f(1)\otimes f(1),\quad |1'2'\rangle=f(1)\otimes f(2),\quad \text{etc.},$$

we then have

$$P\otimes(1-P)=|f(1)\rangle\langle f(1)|\otimes|f(2)\rangle\langle f(2)|=|1'2'\rangle\langle 1'2'|.$$

In view of this, we need to choose α and β such that if we write $\psi=\frac{1}{\sqrt{2}}(|11\rangle+|22\rangle)$ in terms of the basis $|1'1'\rangle,|1'2'\rangle,|2'1'\rangle,|2'2'\rangle$, the constant in front of $|1'2'\rangle$ is nonzero. We observe that

$$\begin{aligned}e(1)&:=\cos\alpha f(1)-e^{i\beta}\sin\alpha f(2),\\ e(2)&:=e^{-i\beta}\sin(\alpha)f(1)+\cos(\alpha)f(2),\end{aligned}$$

which implies that

$$\begin{aligned}|11\rangle&=\cos^2\alpha|1'1'\rangle-e^{i\beta}\cos\alpha\sin\alpha|1'2'\rangle-e^{i\beta}\cos\alpha\sin\alpha|2'1'\rangle+e^{i2\beta}\sin^2\alpha|2'2'\rangle,\\ |22\rangle&=e^{-i2\beta}\sin^2\alpha|1'1'\rangle+e^{-i\beta}\cos\alpha\sin\alpha|1'2'\rangle+e^{-i\beta}\cos\alpha\sin\alpha|2'1'\rangle+\cos^2\alpha|2'2'\rangle.\end{aligned}$$

In view of part (b), setting $\beta=0$ or $\beta=\pi$ will not work. We see that choosing $\alpha\in\{-\pi/2,0,\pi/2,\pi\}$ will also not work. Instead, we set $\alpha=\pi/4$ and $\beta=\pi/2$. In this case, $\sin\alpha=\cos\alpha=1/\sqrt{2}$ and $e^{i\beta}=i$, $e^{-i\beta}=-i$, so

$$\begin{aligned}|11\rangle&=\tfrac{1}{2}|1'1'\rangle-i\tfrac{1}{2}|1'2'\rangle-i\tfrac{1}{2}|2'1'\rangle-\tfrac{1}{2}|2'2'\rangle,\\ |22\rangle&=-\tfrac{1}{2}|1'1'\rangle-i\tfrac{1}{2}|1'2'\rangle-i\tfrac{1}{2}|2'1'\rangle+\tfrac{1}{2}|2'2'\rangle,\end{aligned}$$

and

$$\psi=-\frac{i}{\sqrt{2}}\big(|1'2'\rangle+|2'1'\rangle\big).$$

(Note the + sign; in view of part (c), ψ never takes the form $\frac{1}{\sqrt{2}}(|1'2'\rangle-|2'1'\rangle)$.) It then follows that

$$\rho_\psi\big(P\otimes(1-P)\big)=\langle\psi|1'2'\rangle\langle 1'2'|\psi\rangle=\tfrac{1}{2}\neq 0.$$