Exam Quantum Probability

June 2nd, 2020

Hints: You can use all results proved in the lecture notes (without proving them yourselves), as well as claims one is supposed to prove in exercises from the lecture notes. You can also use a claim you are supposed to prove in one excercise below to solve another excercise (even if you did not prove the claim). Partial solutions also yield points.

Exercise 1 (A normal operator) Let $\mathcal{H}$ be an inner product space (complex, finite dimensional). Let $A \in \mathcal{L}(\mathcal{H})$ be a normal operator such that $A^2 + 1 = 0$, where 1 denotes the identity operator.

(a) Show that A is unitary.

(b) Show that there exists a unitary operator B such that $B^2 = A$.

The Pauli matrices are defined on page 39 of the lecture notes. Using the physicist's notation

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we have

$$\begin{aligned} S_{\mathrm{x}}|0\rangle &= |1\rangle, & S_{\mathrm{y}}|0\rangle &= i|1\rangle, & S_{\mathrm{z}}|0\rangle &= |0\rangle, \\ S_{\mathrm{x}}|1\rangle &= |0\rangle, & S_{\mathrm{y}}|1\rangle &= -i|0\rangle, & S_{\mathrm{z}}|1\rangle &= -|1\rangle. \end{aligned}$$

For each $\theta = (\theta_{\mathrm{x}}, \theta_{\mathrm{y}}, \theta_{\mathrm{z}}) \in \mathbb{R}^3$ with $\|\theta\| = 1$, we set

$$S_\theta := \theta_{\mathrm{x}} S_{\mathrm{x}} + \theta_{\mathrm{y}} S_{\mathrm{y}} + \theta_{\mathrm{z}} S_{\mathrm{z}}.$$

It follows from the proof of Lemma 4.2.2 in the lecture notes that S_θ is a hermitian operator with spectrum $\sigma(S_\theta) = \{-1, +1\}$. Conversely, each operator (on our two-dimenional space) with these properties is of the form $S = S_\theta$ for some $\theta \in \mathbb{R}^3$ with $\|\theta\| = 1$.

In Exercises 2 and 3, we will be interested in the state vectors

$$\phi := \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big) \quad \text{and} \quad \psi := \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big).$$

Please turn over.

Exercise 2 (Entangled pure states)
(a) Prove that
$$\langle\phi|S_\theta \otimes S_{\theta'}|\phi\rangle = -\theta_{\rm x}\theta'_{\rm x} - \theta_{\rm y}\theta'_{\rm y} - \theta_{\rm z}\theta'_{\rm z}.$$
(b) Prove that
$$\langle\psi|S_\theta \otimes S_{\theta'}|\psi\rangle = \theta_{\rm x}\theta'_{\rm x} + \theta_{\rm y}\theta'_{\rm y} - \theta_{\rm z}\theta'_{\rm z}.$$
(c) Let P be a projection operator that projects on a one-dimensional subspace. Prove that
$$\rho_\phi(P \otimes P) = 0.$$
(d) Give an example of a projection operator P that projects on a one-dimensional subspace such that
$$\rho_\psi(P \otimes P) \neq 0.$$

Exercise 3 (Entangled mixed states) Let ϕ and ψ be as in Exercise 1. In this exercise we are interested in the mixed state
$$\rho_{(p)} := p\rho_\phi + (1-p)\rho_\psi.$$
(a) Prove that
$$\rho_{(1/2)} = \tfrac{1}{2}\rho_{[01\rangle} + \tfrac{1}{2}\rho_{[10\rangle}.$$
(b) Is the state $\rho_{(1/2)}$ entangled?

(c) Prove that $\rho_{(p)}$ is entangled when p is sufficiently close to zero or one. *Hint:* Bell's inequality says that
$$\big|\rho(S_{\theta^1} \otimes S_{\theta^2}) + \rho(S_{\theta^3} \otimes S_{\theta^2}) + \rho(S_{\theta^1} \otimes S_{\theta^4}) - \rho(S_{\theta^3} \otimes S_{\theta^4})\big| \leq 2.$$
If $\theta^1_{\rm z} = \theta^2_{\rm z} = 0$, then you can use parts (a) and (b) of the previous exercise to get an expression for $\rho(S_{\theta^1} \otimes S_{\theta^2})$. Now you need to choose $\theta^1, \ldots, \theta^4$ in a clever way, similar to what we did in the lecture notes.

Solutions

Ex 1

(a) Since A is normal, there exists an orthonormal basis $\{e(1), \ldots, e(n)\}$ such that

$$A = \sum_{k=1}^{n} \lambda_k |e(k)\rangle\langle e(k)|.$$

Now $A^2 + 1 = 0$ is equivalent to

$$\sum_{k=1}^{n} (\lambda_k^2 + 1)|e(k)\rangle\langle e(k)| = 0,$$

which is in turn equivalent to $\lambda_k = \pm i$ for all $k = 1, \ldots, n$. Since $\lambda_k^{-1} = \lambda_k^*$ for each k, it follows that $A^{-1} = A^*$ which shows that A is unitary.

(b) Let us set $\alpha_k := \pi/2$ if $\lambda_k = i$ and $\alpha_k := -\pi/2$ if $\lambda_k = -i$. Then $\lambda_k = e^{i\alpha_k}$ for all k. Now if we set

$$B := \sum_{k=1}^{n} e^{i\alpha_k/2} |e(k)\rangle\langle e(k)|,$$

then B is unitary since $(e^{i\alpha_k/2})^{-1} = (e^{i\alpha_k/2})^*$ and $B^2 = A$ since $(e^{i\alpha_k/2})^2 = e^{i\alpha_k} = \lambda_k$.

Ex 2

(a) We start by writing

$$\langle\phi|S_\theta \otimes S_{\theta'}|\phi\rangle = \sum_{\mathrm{v}\in\{\mathrm{x,y,z}\}} \sum_{\mathrm{w}\in\{\mathrm{x,y,z}\}} \theta_{\mathrm{v}}\theta'_{\mathrm{w}} \langle\phi|S_{\mathrm{v}} \otimes S_{\mathrm{w}}|\phi\rangle.$$

We observe that

$$\begin{aligned} S_{\mathrm{x}} \otimes S_{\mathrm{z}}|\phi\rangle &= \frac{1}{\sqrt{2}}\big(- |11\rangle - |00\rangle\big) \\ S_{\mathrm{y}} \otimes S_{\mathrm{z}}|\phi\rangle &= \frac{1}{\sqrt{2}}\big(- i|11\rangle + i|00\rangle\big), \end{aligned}$$

which are orthogonal to ϕ. For the same reason, $S_{\mathrm{z}} \otimes S_{\mathrm{x}}$ and $S_{\mathrm{z}} \otimes S_{\mathrm{y}}$ give a zero contribution. Since moreover

$$\begin{aligned} S_{\mathrm{x}} \otimes S_{\mathrm{y}}|\phi\rangle &= \frac{1}{\sqrt{2}}\big(- i|10\rangle - i|01\rangle\big) = -i\psi \\ S_{\mathrm{y}} \otimes S_{\mathrm{y}}|\phi\rangle &= \frac{1}{\sqrt{2}}\big(i|10\rangle + i|01\rangle\big) = i\psi, \end{aligned}$$

are orthogonal to ϕ, only the diagonal terms contribute. Using moreover that

$$\begin{aligned} S_{\mathrm{x}} \otimes S_{\mathrm{x}}|\phi\rangle &= \frac{1}{\sqrt{2}}\big(|10\rangle - |01\rangle\big) = -\phi \\ S_{\mathrm{y}} \otimes S_{\mathrm{y}}|\phi\rangle &= \frac{1}{\sqrt{2}}\big(i \cdot (-i)|10\rangle - (-i) \cdot i|01\rangle\big) = -\phi \\ S_{\mathrm{z}} \otimes S_{\mathrm{z}}|\phi\rangle &= \frac{1}{\sqrt{2}}\big(- |01\rangle + |10\rangle\big) = -\phi, \end{aligned}$$

we obtain that

$$\langle\phi|S_\theta\otimes S_{\theta'}|\phi\rangle = -\theta_{\rm x}\theta'_{\rm x} - \theta_{\rm y}\theta'_{\rm y} - \theta_{\rm z}\theta'_{\rm z}.$$

(b) The contribution of the off-diagonal terms is zero for the same reason as under (a). For the diagonal terms, we obtain

$$\begin{aligned}
S_{\rm x}\otimes S_{\rm x}|\psi\rangle &= \frac{1}{\sqrt{2}}\big(|10\rangle + |01\rangle\big) = \psi\\
S_{\rm y}\otimes S_{\rm y}|\psi\rangle &= \frac{1}{\sqrt{2}}\big(i\cdot(-i)|10\rangle + (-i)\cdot i|01\rangle\big) = \psi\\
S_{\rm z}\otimes S_{\rm z}|\psi\rangle &= \frac{1}{\sqrt{2}}\big(-|01\rangle + |10\rangle\big) = -\psi,
\end{aligned}$$

which yields

$$\langle\psi|S_\theta\otimes S_{\theta'}|\psi\rangle = \theta_{\rm x}\theta'_{\rm x} + \theta_{\rm y}\theta'_{\rm y} - \theta_{\rm z}\theta'_{\rm z}.$$

(c) By Lemma 4.2.2 in the lecture notes, every projection operator P that projects on a one-dimensional subspace is of the form $P = P_\theta := \frac{1}{2}1 + \frac{1}{2}S_\theta$ for some $\theta\in\mathbb{R}^3$ with $\|\theta\| = 1$. We can write

$$S_\theta = 2P_\theta - 1 = P_\theta - (1 - P_\theta) = P_\theta - P_{-\theta},$$

where $\{P_\theta, P_{-\theta}\}$ is a partition of the identity. Now part (a) tells us that

$$\begin{aligned}
-1 &= \langle\phi|S_\theta\otimes S_\theta|\phi\rangle = \rho_\phi(S_\theta\otimes S_\theta)\\
&= 1\cdot 1\,\rho_\phi(P_\theta\otimes P_\theta) + 1\cdot(-1)\,\rho_\phi(P_\theta\otimes P_{-\theta})\\
&\quad + (-1)\cdot 1\,\rho_\phi(P_{-\theta}\otimes P_\theta) + (-1)\cdot(-1)\,\rho_\phi(P_{-\theta}\otimes P_{-\theta}),
\end{aligned}$$

which implies that the probabilities $\rho_\phi(P_\theta\otimes P_\theta)$ and $\rho_\phi(P_{-\theta}\otimes P_{-\theta})$ are zero. Alternatively, we can calculate

$$\rho_\phi(P_\theta\otimes P_\theta) = \tfrac{1}{4}\big\{\rho_\phi(S_\theta\otimes S_\theta) + \rho_\phi(S_\theta\otimes 1) + \rho_\phi(1\otimes S_\theta) + \rho_\phi(1\otimes 1)\big\}.$$

Here $\rho_\phi(S_\theta\otimes S_\theta) = -1$ by part (a) and $\rho_\phi(1\otimes 1) = 1$. To calculate the other terms, we write

$$\rho_\phi(S_\theta\otimes 1) = \langle\phi|S_\theta\otimes 1|\phi\rangle = \sum_{\rm v\in\{x,y,z\}}\theta_{\rm v}\langle\phi|S_{\rm v}|\phi\rangle.$$

Here

$$\begin{aligned}
S_{\rm x}\phi &= \tfrac{1}{\sqrt{2}}\big(|11\rangle - |00\rangle\big),\\
S_{\rm y}\phi &= \tfrac{1}{\sqrt{2}}\big(i|11\rangle + i|00\rangle\big),\\
S_{\rm z}\phi &= \tfrac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big),
\end{aligned}$$

are all orthogonal to ϕ, so $\rho_\phi(S_\theta\otimes 1) = 0$. In the same way, we see that $\rho_\phi(1\otimes S_\theta) = 0$. It follows that

$$\rho_\phi(P_\theta\otimes P_\theta) = \tfrac{1}{4}\{-1 + 0 + 0 + 1\} = 0.$$

(d) If $\theta_{\mathrm{z}} < 1$, then part (b) tells us that

$$\begin{aligned} -1 &< \langle\psi|S_\theta \otimes S_\theta|\psi\rangle = \rho_\psi(S_\theta \otimes S_\theta) \\ &= \rho_\psi(P_\theta \otimes P_\theta) - \rho_\psi(P_\theta \otimes P_{-\theta}) - \rho_\psi(P_{-\theta} \otimes P_\theta) + \rho_\psi(P_{-\theta} \otimes P_{-\theta}), \end{aligned}$$

which implies that at least one of the probabilities $\rho_\psi(P_\theta \otimes P_\theta)$ and $\rho_\psi(P_{-\theta} \otimes P_{-\theta})$ must be nonzero. More explicitly, we can calculate

$$\begin{aligned} \rho_\psi(P_\theta \otimes P_\theta) &= \tfrac{1}{4}\big\{\rho_\psi(S_\theta \otimes S_\theta) + \rho_\psi(S_\theta \otimes 1) + \rho_\psi(1 \otimes S_\theta) + \rho_\psi(1 \otimes 1)\big\} \\ &= \tfrac{1}{4}\big\{\theta_{\mathrm{x}}^2 + \theta_{\mathrm{y}}^2 - \theta_{\mathrm{z}}^2 + 0 + 0 + 1\big\}, \end{aligned}$$

which is in fact nonzero always except when $(\theta_{\mathrm{x}}, \theta_{\mathrm{y}}, \theta_{\mathrm{z}}) = (0, 0, 1)$.

Ex 3

(a) Let $\mathcal{F}$ be the subspace of $\mathcal{H} \otimes \mathcal{H}$ spanned by the orthonormal vectors ϕ and ψ. Then, for any $A \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$,

$$\begin{aligned} \rho_{(1/2)}(A) &= \tfrac{1}{2}\rho_\phi(A) + \tfrac{1}{2}\rho_\psi(A) = \tfrac{1}{2}\big(\langle\phi|A|\phi\rangle + \langle\psi|A|\psi\rangle\big) = \tfrac{1}{2}\mathrm{tr}(P_{\mathcal{F}}A) \\ &= \tfrac{1}{2}\big(\langle 01|A|01\rangle + \langle 10|A|10\rangle\big) = \tfrac{1}{2}\rho_{[01\rangle} + \tfrac{1}{2}\rho_{[10\rangle}, \end{aligned}$$

since $\{|01\rangle, |10\rangle\}$ is also an orthonormal basis for $\mathcal{F}$.

(b) The state $\rho_{(1/2)}$ is not entangled since $\rho_{[01\rangle} = \rho_{[0\rangle} \otimes \rho_{[1\rangle}$ and $\rho_{[10\rangle} = \rho_{[1\rangle} \otimes \rho_{[0\rangle}$ are product states.

(c) We choose $\theta_{\mathrm{z}}^1 = \theta_{\mathrm{z}}^2 = \theta_{\mathrm{z}}^3 = \theta_{\mathrm{z}}^4 = 0$ and

$$\begin{aligned} (\theta_{\mathrm{x}}^1, \theta_{\mathrm{y}}^1) &= \big(\cos(0), \sin(0)\big), & (\theta_{\mathrm{x}}^2, \theta_{\mathrm{y}}^2) &= \big(\cos(\gamma), \sin(\gamma)\big), \\ (\theta_{\mathrm{x}}^3, \theta_{\mathrm{y}}^3) &= \big(\cos(2\gamma), \sin(2\gamma)\big), & (\theta_{\mathrm{x}}^4, \theta_{\mathrm{y}}^4) &= \big(\cos(-\gamma), \sin(-\gamma)\big), \end{aligned}$$

which by parts (a) and (b) of Exercise 2 yields

$$\begin{aligned} \rho_\phi(S_{\theta^1} \otimes S_{\theta^2}) + \rho_\phi(S_{\theta^3} \otimes S_{\theta^2}) + \rho_\phi(S_{\theta^1} \otimes S_{\theta^4}) - \rho_\phi(S_{\theta^3} \otimes S_{\theta^4}) &= -3\cos(\gamma) - \cos(3\gamma), \\ \rho_\psi(S_{\theta^1} \otimes S_{\theta^2}) + \rho_\psi(S_{\theta^3} \otimes S_{\theta^2}) + \rho_\psi(S_{\theta^1} \otimes S_{\theta^4}) - \rho_\psi(S_{\theta^3} \otimes S_{\theta^4}) &= 3\cos(\gamma) + \cos(3\gamma). \end{aligned}$$

The calculations on pages 102 and 103 in the lecture notes show that the optimal choice is $\gamma = \pi/4$, for which $3\cos(\gamma) + \cos(3\gamma) = 2\sqrt{2}$. It follows that

$$\rho_{(p)}(S_{\theta^1} \otimes S_{\theta^2}) + \rho_{(p)}(S_{\theta^3} \otimes S_{\theta^2}) + \rho_{(p)}(S_{\theta^1} \otimes S_{\theta^4}) - \rho_{(p)}(S_{\theta^3} \otimes S_{\theta^4}) = (1 - 2p)2\sqrt{2},$$

so as long as $|p - \frac{1}{2}| > 1/(2\sqrt{2})$, this expression is in absolute value larger than 2 and we conclude that $\rho_{(p)}$ is entangled.