Random graphs and networks

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The Erdős-Rényi random graph

Def $[n] := \{1, \ldots, n\}$.
Def K_n complete graph: vertex set $[n]$, edge set $\big\{\{s,t\} : s,t \in [n],\ s \neq t\big\}$.
Notation $st = ts = \{s,t\}$.
Def $\mathrm{ER}_n(p)$ Erdős-Rényi random graph with vertex set $[n]$, random edge set: edges i.i.d. present with edge probability p.
Def $s \leftrightarrow t$ if s,t connected in $\mathrm{ER}_n(p)$.

$$\mathcal{C}(v) := \big\{x \in [n] : v \leftrightarrow x\big\} \quad \text{connected component containing } v.$$

Def $\mathcal{C}_{\max}$ largest connected component
Set $\lambda := pn$. Let $n \to \infty$ for fixed λ.
Theorems 4.4 & 4.5 say that for $\lambda < 1$

$$\mathbb{P}\big[(\kappa_\lambda - \varepsilon)\log(n) \leq |\mathcal{C}_{\max}| \leq (\kappa_\lambda + \varepsilon)\log(n)\big] \underset{n\to\infty}{\longrightarrow} 1 \qquad \forall \varepsilon > 0,$$

with $\kappa_\lambda := (\lambda - 1 - \log(\lambda))^{-1}$.
Theorem 4.8 says that for $\lambda > 1$

$$\mathbb{P}\big[\zeta_\lambda n - n^\nu \leq |\mathcal{C}_{\max}| \leq \zeta_\lambda n + n^\nu\big] \underset{n\to\infty}{\longrightarrow} 1 \qquad \forall \nu \in (\tfrac{1}{2}, 1),$$

for some $0 < \zeta_\lambda < 1$.

Generalized Random Graphs

Edges independent but not identically distributed.
Vertex i has weight $w_i > 0$.
Graph $\mathrm{GRG}_n(w)$. Edge ij present with probab. $p_{ij} := \frac{w_i w_j}{w_i w_j + \sum_k w_k}$.
Def W_n weight of unif chosen random vertex.
Assume $\exists W$ s.t.

- $\mathbb{P}[W_n \in \cdot] \underset{n\to\infty}{\Longrightarrow} \mathbb{P}[W \in \cdot]$,
- $\mathbb{E}[W_n] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[W]$,
- $\mathbb{E}[W_n^2] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[W^2]$.

Typical choice

$$\mathbb{P}\big[W > x\big] \sim \mathrm{cst} \cdot x^{1-\tau}$$

Degree D_n of unif chosen random vertex has mixed Poisson distribution.
(Compared to $\mathrm{Pois}(\lambda)$ for Erdős-Rényi.)

The Configuration Model

In $\mathrm{CM}_n(d)$, the degrees $d_1, \ldots, d_n$ are given.
Assume $\sum_{i\in[n]} d_i$ even.

- $\forall i$, draw d_i half-edges out of i.
- Enumerate the half edges.
- Pair the first half edge to a unif chosen free partner.
- Continue till no half-edges left.

Result: multigraph: may contain multiple edges and loops.
In the Erased Configuration Model, all loops are erased, all multiple edges reduced to a single edge.
Assume $\exists D$ s.t.

- $\mathbb{P}[D_n \in \cdot] \underset{n\to\infty}{\Longrightarrow} \mathbb{P}[D \in \cdot]$,
- $\mathbb{E}[D_n] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[D]$,
- $\mathbb{E}[D_n^2] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[D^2]$.

Typical choice

$$\mathbb{P}\big[D = k\big] \sim \mathrm{cst} \cdot k^{-\tau}.$$

Recall $\tau = 2.2, = 2.1$ observed in real networks.
If we choose $\tau \in (1, 2)$, then after erasing multiple edges and loops $\tau = 2$ (Theorem 7.24).
If we choose $\tau > 2$, then only few multiple edges and loops.

Preferential Attachment Models

Grow a multigraph with n vertices as follows.
Fix $\delta \geq -1$.
Let $D_i(t) :=$ degree of $i = 1, 2, \dots$ at time $t = 0, 1, 2, \dots$.
$D_1(0) := 1$ (half-edge) and $D_i(0) := 0$ $\forall i \geq 2$.
Inductively for $t = 0, 1, \dots$:

- Connect the half edge at $t+1$ to random $i \in [t+1]$ chosen with probab. proportional to $D_i(t) + \delta$ (weight of vertex).
- Add a half-edge to $t + 2$.

Sum of degrees $\sum_{i=1}^{\infty} D_i(t) = 2t + 1$.
Sum of weights $\sum_{i=1}^{\infty} \big(D_i(t) + \delta\big) = 2t + 1 + \delta(t + 1)$.
Probability to attach to $i \in [t + 1]$

$$\frac{D_i(t) + \delta}{2t + 1 + \delta(t + 1)}.$$

More generally, fix $m = 1, 2, \dots$ and $\delta \geq -m$.
For $t = 0, 1, \dots$
 For $k = 0, 1, \dots, m - 1$

- Connect the half edge at $t+1$ to random $i \in [t+1]$ chosen with probab. proportional to $D_i(tm + k) + \delta/m$ (weight of vertex).
- If $k < m - 1$, add another half-edge to $t + 1$.
- If $k = m - 1$, add a half-edge to $t + 2$.

 end
end
Observation: model with $\delta \geq -m$ and $m \geq 2$ can be obtained from model with $\delta' := \delta/m$ and $m' := 1$ by merging vertices in groups of m.
<u>Claim</u> Choose I unif from $[t + 1]$. Then

$$\mathbb{P}\big[D_I \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \mathbb{P}\big[D \in \cdot\,\big]$$

for a r.v. D s.t.

$$\mathbb{P}\big[D = k\big] \sim \mathrm{cst} \cdot k^{-\tau}$$

with $\tau = 3 + \delta/m$.

Levels of randomness

Let $G_n(w)$ be a deterministic graph with vertex set $[n]$ and vertex weights $(w_i)_{i\in[n]}$.
Then the empirical weight distribution and empirical distribution function

$$\mu_n := \frac{1}{n}\sum_{i=1}^{n}\delta_{w_i} \quad F_n(x) := \frac{1}{n}\sum_{i=1}^{n}1_{\{w_i \le x\}}$$

are deterministic objects.
Let $I(n)$ be uniformly distributed in $[n]$. Then

$$F_n(x) = \mathbb{P}[W_{I(n)} \le x].$$

Our assumption that $\mathbb{P}[W_{I(n)} \in \cdot\,] \underset{n\to\infty}{\Longrightarrow} \mathbb{P}[W \in \cdot\,]$ for some W is equivalent to

$$F_n(x) \underset{n\to\infty}{\longrightarrow} F(x) \quad \text{in all continuity points of } F,$$

with $F(x) := \mathbb{P}[W \le x]$.
Let $(w_i)_{i\in[n]}$ be i.i.d. with law μ.
These define a random weighted graph with

$$F_n(x) \xrightarrow[n\to\infty]{\mathrm{p}} F(x) \quad \text{in all continuity points of } F,$$

where $\xrightarrow{\mathrm{p}}$ denotes convergence in probability.
Let $I(n)$ be uniformly distributed in $[n]$ independent of $(w_i)_{i\in[n]}$. Then

$$\mathbb{P}\Big[\big|\mathbb{P}[W_{I_n} \le x \,|\, (w_i)_{i\in[n]}] - F(x)\big| \ge \varepsilon\Big] \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in all continuity points of } F.$$

Note two levels of randomness.
Conditional on $(w_i)_{i\in[n]}$, construct $\mathrm{GRG}_n(w)$ with:
edge ij present with probab. $p_{ij} := \frac{w_i w_j}{w_i w_j + \sum_k w_k}$.
Now three levels of randomness: $(w_i)_{i\in[n]}$, the edges, and I_n.
Similarly, construct configuration model with random degrees

1. Choose $(d_i)_{i\in[n]}$ i.i.d. with law $\mathbb{P}[D \in \cdot\,]$.
2. Conditional on $(d_i)_{i\in[n]}$, randomly pair up half-edges.
3. Independently of $(d_i)_{i\in[n]}$ AND the edges, choose uniform random vertex $I_n \in [n]$.

Local limits

Let G_n be deterministic graphs with vertex set $[n]$.
Let I_n uniformly distributed on $[n]$.
Look at:

- I_n
- all neighbors of I_n.
- all neighbors of neighbors of I_n.
- etc.

Example The configuration model with $d_i = 3$ for all i (and n even).

- I_n has 3 neighbors.
- each neighbor of I_n has 3 neighbors.
- etc.

Moreover, for large n,

- $\mathbb{P}[I_n \text{ part of a triangle}] \underset{n\to\infty}{\longrightarrow} 0$.
- $\mathbb{P}[I_n \text{ part of a cycle of length } 4] \underset{n\to\infty}{\longrightarrow} 0$.
- $\mathbb{P}[I_n \text{ part of a cycle of length } k] \underset{n\to\infty}{\longrightarrow} 0 \quad \forall k$.

The $\mathrm{CM}_n(d)$ locally looks like a 3-regular tree.

Example In the Erdős-Rényi random graph $\mathrm{ER}_n(\lambda/n)$,

- I_n has $\mathrm{Pois}(\lambda)$ neighbors.
- each neighbor of I_n has $\mathrm{Pois}(\lambda)$ further neighbors.

$\mathrm{ER}_n(\lambda/n)$ locally looks like a branching process
with $\mathrm{Pois}(\lambda)$ offspring distribution.