# Exam Advanced Topics in Markov Chains 

May 24th, 2018

Hints: You can use all results proved in the lecture notes (without proving them yourselves). You can also use claims from exercises in the lecture notes (without solving these exercises). Finally, you can also use a claim you are supposed to prove in one excercise below to solve another excercise (even if you did not prove the claim). Partial solutions also yield points.

Exercise 1 (Trapping probabilities) Let $\left(X_{k}\right)_{k>0}$ be i.i.d. Bernoulli random variables with $\mathbb{P}\left[X_{k}=0\right]=\mathbb{P}\left[X_{k}=1\right]=\frac{1}{2}(k \geq 0)$. Set

$$
\tau_{101}:=\inf \left\{k \geq 0:\left(X_{k}, X_{k+1}, X_{k+2}\right)=(1,0,1)\right\}
$$

and define $\tau_{010}$ similarly. Calculate, for each $\left(x_{0}, x_{1}, x_{2}\right) \in\{0,1\}^{3}$, the conditional probabilities

$$
\mathbb{P}\left[\tau_{101}<\tau_{010} \mid\left(X_{0}, X_{1}, X_{2}\right)=\left(x_{0}, x_{1}, x_{2}\right)\right] .
$$

Hint: The picture in Exercise 1.17 in the lecture notes may be helpful.
Exercise 2 (Quasi-stationary law) Let $\left(X_{k}\right)_{k \geq 0}, \tau_{101}$, and $\tau_{010}$ be as in the previous exercise and define

$$
\tau:=\tau_{101} \wedge \tau_{010} .
$$

(a) Show that there exist constants $c \in(0,1)$ and $r \in(0, \infty)$ such that $\mathbb{P}[\tau>k] \sim r c^{k}$, i.e.,

$$
c^{-k} \mathbb{P}[\tau>k] \underset{k \rightarrow \infty}{\longrightarrow} r
$$

(b) Calculate, for each $\left(x_{0}, x_{1}, x_{2}\right) \in\{0,1\}^{3}$, the limit

$$
\lim _{k \rightarrow \infty} \mathbb{P}\left[\left(X_{k}, X_{k+1}, X_{k+2}\right)=\left(x_{0}, x_{1}, x_{2}\right) \mid \tau>k\right] .
$$

(c) Calculate, for each $\left(x_{0}, x_{1}, x_{2}\right) \in\{0,1\}^{3}$, the limit

$$
\lim _{k \rightarrow \infty} \mathbb{P}\left[\left(X_{k}, X_{k+1}, X_{k+2}\right)=\left(x_{0}, x_{1}, x_{2}\right) \mid \tau>2 k\right] .
$$

(Note that here we condition on $\tau>2 k$ instead of $>k$ ).
Hint: The most important thing is to show which equations you need to solve to calculate these quantities. Explicit solutions are less important, although it is possible to obtain them.

Please turn over.

Exercise 3 (A two-type branching process) Let $\left(X_{k}(1), X_{k}(2)\right)_{k \geq 0}$ be a two-type branching process with offspring distribution

$$
\begin{array}{ll}
\mathbb{P}^{\delta_{1}}\left[\left(X_{1}(1), X_{1}(2)\right)=(3,0)\right]=\frac{1}{2}, & \mathbb{P}^{\delta_{1}}\left[\left(X_{1}(1), X_{1}(2)\right)=(0,1)\right]=\frac{1}{2} \\
\mathbb{P}^{\delta_{2}}\left[\left(X_{1}(1), X_{1}(2)\right)=(0,3)\right]=\frac{1}{2}, & \mathbb{P}^{\delta_{2}}\left[\left(X_{1}(1), X_{1}(2)\right)=(0,0)\right]=\frac{1}{2},
\end{array}
$$

i.e., particles of type 1 either give birth to three particles of type 1 or to one particle of type 2, with equal probabilities, while particles of type 2 either give birth to three particles of type 2 or have no offspring at all, again with equal probabilities.
(a) Is $\left(X_{k}(1)\right)_{k \geq 0}$ an autonomous Markov chain?
(b) Same question as under (a) but for $\left(X_{k}(2)\right)_{k \geq 0}$.
(c) Let $A$ and $B$ denote the events

$$
\begin{aligned}
A & :=\left\{X_{k}(1) \neq 0 \forall k \geq 1\right\} \\
B & :=\left\{\exists n \geq 0 \text { s.t. } X_{k}(2) \neq 0 \forall k \geq n\right\} .
\end{aligned}
$$

Prove that $\mathbb{P}^{x}(B \mid A)=1$ for all $x$ such that $x(1) \geq 1$.

## Solutions

Ex 1
The process $\left(X_{k+1}, X_{k+2}, X_{k+3}\right)_{k \geq 0}$ is a Markov chain with transitions as indicated in the picture in Exercise 1.17 in the lecture notes, where each arrow has probability $1 / 2$. If we stop the process as soon as it enters one of the states 101 and 010, then we get a Markov chain as in the picture on the right. Let $P$ denote the transition kernel of this stopped Markov chain, and let

$$
h(x):=\mathbb{P}\left[\tau_{101}<\tau_{010} \mid\left(X_{0}, X_{1}, X_{2}\right)=x\right] .
$$

By Lemma 1.2 in the lecture notes, this is a harmonic function, i.e., $P h=h$. By Lemma 0.16 in the lecture notes, the stopped Markov chain a.s. ends up in one of the traps 101 and 010, and therefore by Lemma 1.3 in the lecture notes, $h$ is in fact
 the unique solution of

$$
P h=h \quad \text { with } \quad h(101)=1, h(010)=0
$$

It is not hard to see that $h$ is the function indicated in the second picture on the right.


## Ex 2

Let $S=\{000,001,010,011,100,101,110,111\}$ be the state space of the Markov chain from the previous exercise and let $S^{\prime}:=S \backslash\{101,010\}$. Let $P$ be the transition kernel of the Markov chain and let $Q:=\left.P\right|_{S^{\prime}}$ denote its restriction to $S^{\prime}$. We see from the picture that it is possible to get from any state in $S^{\prime}$ to any other state in $S^{\prime}$ without leaving $S^{\prime}$, so $S^{\prime}$ is irreducible. The Perron-Frobenius theorem (Theorem 2.15 in the lecture notes) now tells us that there exists a function $h: S^{\prime} \rightarrow(0, \infty)$, unique up to scalar multiples, and a unique constant $c>0$ such that $Q h=c h$. Applying the Perron-Frobenius theorem to the adjoint $Q^{\dagger}$, we see that there also exists a function $\eta: S^{\prime} \rightarrow(0, \infty)$, unique up to scalar multiples, and a unique constant $c^{\prime}>0$ such that $\eta Q=c^{\prime} \eta$. Here in fact $c=\rho(Q)=\rho\left(Q^{\dagger}\right)=c^{\prime}$ (see Theorem 2.7 and Proposition 2.12 in the lecture notes). Since $S^{\prime \prime}$ is finite, we can normalize $h$ and $\eta$ such that

$$
\sum_{x \in S^{\prime}} \eta(x)=1 \quad \text { and } \quad \sum_{x \in S^{\prime}} \eta(x) h(x)=1 .
$$

Also, the finiteness of $S^{\prime}$ implies that $\inf _{x \in S^{\prime}} h(x)>0$, so Theorem 2.16 from the lecture notes is applicable.
(a) Corollary 2.17 from the lecture notes tells us that

$$
c^{-k} \mathbb{P}\left[\tau>k \mid\left(X_{0}, X_{1}, X_{2}\right)=x\right] \underset{k \rightarrow \infty}{\longrightarrow} h(x)
$$

for each $x \in S^{\prime}$. Since

$$
\mathbb{P}[\tau>k]=\sum_{x \in S} \mathbb{P}\left[\tau>k \mid\left(X_{0}, X_{1}, X_{2}\right)=x\right] \mathbb{P}\left[\left(X_{0}, X_{1}, X_{2}\right)=x\right],
$$

and $\mathbb{P}\left[\left(X_{0}, X_{1}, X_{2}\right)=x\right]=1 / 8$ for each $x \in S^{\prime}$, we conclude that

$$
c^{-k} \mathbb{P}[\tau>k] \underset{k \rightarrow \infty}{\longrightarrow} \frac{1}{8} \sum_{x \in S^{\prime}} h(x) .
$$

(b) We can apply Corollary 2.18 from the lecture notes to conclude that

$$
\lim _{k \rightarrow \infty} \mathbb{P}\left[\left(X_{k}, X_{k+1}, X_{k+2}\right)=x \mid \tau>k,\left(X_{0}, X_{1}, X_{2}\right)=y\right]=\eta(x)
$$

for each $x, y \in S^{\prime}$. Since

$$
\begin{aligned}
& \mathbb{P}\left[\left(X_{k}, X_{k+1}, X_{k+2}\right)=x \mid \tau>k\right] \\
& =\sum_{y \in S^{\prime}} \mathbb{P}\left[\left(X_{k}, X_{k+1}, X_{k+2}\right)=x \mid \tau>k,\left(X_{0}, X_{1}, X_{2}\right)=y\right] \mathbb{P}\left[\left(X_{0}, X_{1}, X_{2}\right)=y \mid \tau>k\right],
\end{aligned}
$$

using the finiteness of $S^{\prime}$, we conclude that

$$
\mathbb{P}\left[\left(X_{k}, X_{k+1}, X_{k+2}\right)=x \mid \tau>k,\left(X_{0}, X_{1}, X_{2}\right)=y\right] \underset{k \rightarrow \infty}{\longrightarrow} \eta(x)
$$

for each $x \in S^{\prime \prime}$.
(c) We can apply Exercise 2.19 from the lecture notes to conclude that

$$
\lim _{k \rightarrow \infty} \mathbb{P}\left[\left(X_{k}, X_{k+1}, X_{k+2}\right)=x \mid \tau>2 k,\left(X_{0}, X_{1}, X_{2}\right)=y\right]=\eta(x) h(x)
$$

for each $x, y \in S^{\prime}$. By the same argument as in part (a), it follows that also

$$
\lim _{k \rightarrow \infty} \mathbb{P}\left[\left(X_{k}, X_{k+1}, X_{k+2}\right)=x \mid \tau>2 k\right]=\eta(x) h(x)
$$

for each $x \in S^{\prime}$.
The constant $c$ and functions $h$ and $\eta$ can actually explicitly be calculated. We start by observing that the problem is symmetric if we replace all 0 's by 1's and vice versa, so $h(000)=h(111), h(001)=h(110)$ etc. Using this, the equation $Q h=c h$ gives

$$
\begin{aligned}
& Q h(100)=\frac{1}{2} h(000)+\frac{1}{2} h(001) \\
&=\operatorname{ch}(100), \\
& Q h(000)=\frac{1}{2} h(000)+\frac{1}{2} h(001)=\operatorname{ch}(000), \\
& Q h(001)=\frac{1}{2} h(011)=\frac{1}{2} h(100)=\operatorname{ch}(001) .
\end{aligned}
$$

We know in advance that the Perron-Frobenius eigenvalue satisfies $c>0$ so the first two equations tell us that $h(100)=h(000)$. It follows that $h$ (up to the normalization, that we will choose later) is given by

$$
\left(\begin{array}{l}
h(100) \\
h(000) \\
h(001)
\end{array}\right)=\left(\begin{array}{c}
1 \\
1 \\
2 c
\end{array}\right) .
$$

To calculate $c$, we use the equation $Q h(100)=c h(100)$ which gives

$$
\frac{1}{2}+\frac{1}{2} c=c^{2} \quad \Leftrightarrow \quad c=\frac{1}{4}(1 \pm \sqrt{5}) .
$$

Since the Perron-Frobenius eigenvalue satisfies $c>0$, we conclude that $c=(1+\sqrt{5}) / 4$. One can check that the spectrum of $Q$ is $\left\{\frac{1}{4}(1-\sqrt{5}), 0, \frac{1}{4}(1+\sqrt{5})\right\}$, so $c$ is the largest eigenvalue, in line with Gelfand's formula (Lemma A. 1 from the lecture notes). The equation $\eta Q=c \eta$ gives the equations

$$
\begin{aligned}
& \eta Q(100)=\frac{1}{2} \eta(110)=\frac{1}{2} \eta(001)=c \eta(100), \\
& \eta Q(000)=\frac{1}{2} \eta(100)+\frac{1}{2} \eta(000)=c \eta(000), \\
& \eta Q(001)=\frac{1}{2} \eta(100)+\frac{1}{2} \eta(000)=c \eta(001),
\end{aligned}
$$

which up to normalization can be solved as

$$
\left(\begin{array}{c}
\eta(100) \\
\eta(000) \\
\eta(001)
\end{array}\right)=\left(\begin{array}{c}
1 \\
2 c \\
2 c
\end{array}\right) .
$$

It is now straightforward (but a bit tedious) to calculate the right normalizations of $h$ and $\eta$ as in Theorem 2.16 from the lecture notes.

## Ex 3

(a) The process $\left(X_{k}(1)\right)_{k \geq 0}$ is a branching process where each particle with equal probabilities either gives birth to three particles or has no offspring at all. In particular, the transition probabilities of $\left(X_{k}(1)\right)_{k \geq 0}$ do not depend on the current state of $\left(X_{k}(2)\right)_{k \geq 0}$, so $\left(X_{k}(1)\right)_{k \geq 0}$ is an autonomous Markov chain.
(b) If $X_{0}(1)=1$, then in the first time step the process $\left(X_{k}(2)\right)_{k \geq 0}$ jumps from 0 to 1 with probability $1 / 2$, but if $X_{0}(1)=0$, then this probability is zero. This shows that the transition probabilities of $\left(X_{k}(2)\right)_{k \geq 0}$ depend on the current state of $\left(X_{k}(1)\right)_{k \geq 0}$, and hence $\left(X_{k}(2)\right)_{k \geq 0}$ is not an autonomous Markov chain. (Nevertheless, in the special case that we start with no particles of type 1 , the process $\left(X_{k}(2)\right)_{k \geq 0}$, on its own, is a Markov chain and even a branching process.)
(c) We claim that if we start with a single particle of type 2 , then

$$
p:=\mathbb{P}^{\delta_{2}}\left[X_{k}(2) \neq 0 \forall k \geq 0\right]>0
$$

Indeed, if initially there are no particles of type 1 , then $\left(X_{k}(2)\right)_{k \geq 0}$ is a branching process where each particle with equal probabilities either gives birth to three particles or has no offspring at all. Since the average number of particles produced by a single particle in each step is $3 / 2$, which is larger than one, such a branching process is supercritical, so by Proposition 4.17 such a process survives with positive probability. By the branching property (Lemma 4.2 in the lecture notes), the same is true if we add more particles at time zero, so we see that more generally

$$
\mathbb{P}^{x}\left[X_{k}(2) \neq 0 \forall k \geq 0\right] \geq p>0 \quad \forall x \text { s.t. } x(2) \geq 1 .
$$

Since particles of type 1 produce with probability $\frac{1}{2}$ a particle of type 2 , it also follows that

$$
\rho(x):=\mathbb{P}^{x}\left[X_{k}(2) \neq 0 \forall k \geq 1\right] \geq \frac{1}{2} p>0 \quad \forall x \text { s.t. } x(1) \geq 1 .
$$

In particular, the event $A$ implies that $\rho\left(X_{k}\right)$ does not tend to zero, so the claim follows from the principle "what can happen must eventually happen" (Proposition 0.14 in the lecture notes).

