Overview structure of algebras

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Representations

$\mathcal{A}$ Q-algebra:

Representation = $\mathcal{H}$ inner product space + $l : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ $*$-algebra homomorphism. Isomorphism = faithful representation.

Def. $(A, \phi) \mapsto A\phi := l(A)\phi$.

- $A(a\phi + b\psi) = aA\phi + bA\psi$
- $(aA + bB)\phi = aA\phi + bB\phi$
- $(AB)\phi = A(B\phi)$
- $1\phi = 1$
- $\langle \phi | A\psi \rangle = \langle A^*\phi | \psi \rangle$.

Representation = inner product space $\mathcal{H}$ + map $(A, \phi) \mapsto A\phi$ with these properties.

Def. homomorphism of representations: linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ s.t.

$$\begin{array}{ll} \langle U\phi | U\psi \rangle_2 = \langle \phi | \psi \rangle_1 & \quad \text{unitary} \\ AU\psi = UA\psi & \quad \forall \psi \in \mathcal{H}_1. \end{array}$$

If U bijection then isomorphism; say $\mathcal{H}_1, \mathcal{H}_2$ equivalent. $l_2(A) = Ul_1(A)U^{-1}$. Equivalent = same after change of orthonormal basis.

Convention: $\mathcal{H}$ representation $\Rightarrow$ $\mathcal{H} \neq \{0\}$.

Direct sum

$\mathcal{V}_1, \ldots, \mathcal{V}_n$ linear spaces: Every $\psi \in \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n$ has unique decomposition

$$\psi = \underset{\in \mathcal{V}_1}{\phi_1} + \cdots + \underset{\in \mathcal{V}_n}{\phi_n}$$

$\mathcal{H}_1, \ldots, \mathcal{H}_n$ inner product spaces:

$$\langle \phi_1 + \cdots + \phi_n | \psi_1 + \cdots + \psi_n \rangle := \langle \phi_1 | \psi_1 \rangle + \cdots + \langle \phi_n | \psi_n \rangle$$

$\mathcal{H}_i \perp \mathcal{H}_j \ \forall i \neq j$.

$\mathcal{A}_1, \ldots, \mathcal{A}_n$ Q-algebras:

$$\begin{aligned}
&(A_1 + \cdots + A_n)(B_1 + \cdots + B_n) : A_1 B_1 + \cdots + A_n B_n \\
&(A_1 + \cdots + A_n)^* := A_1^* + \cdots + A_n^* \\
&1 = 1_1 + \cdots + 1_n
\end{aligned}$$

$\mathcal{H}_1, \ldots, \mathcal{H}_n$ representations:

$$A(\phi_1 + \cdots + \phi_n) := A\phi_1 + \cdots + A\phi_n$$

Decomposition of algebras

Def. ideal $\mathcal{I} \subset \mathcal{A}$ subspace s.t. $A \in \mathcal{A},\ B \in \mathcal{I} \ \Rightarrow\ AB \in \mathcal{I},\ BA \in \mathcal{I}$.

Def. faktor: algebra with no proper ideals.

$l : \mathcal{A} \to \mathcal{B}$ algebra homomorphism $\Rightarrow \mathrm{Ker}(l) = \{A \in \mathcal{A} : l(A) = 0\}$ ideal.

$\mathcal{A}$ Q-algebra $\Rightarrow \mathcal{A} \cong \underset{\uparrow \text{factor}}{\mathcal{A}_1} \oplus \cdots \oplus \mathcal{A}_n$

Proof τ pseudotrace $\Rightarrow \langle A|B \rangle_\tau := \tau(A^* B)$ inner product on $\mathcal{A}$.
$\mathcal{I}$ ideal $\Rightarrow\ \mathcal{I}^\perp$ ideal. $1 = \underset{\in \mathcal{I}}{1_1} + \underset{\in \mathcal{I}^\perp}{1_2}$. $\mathcal{I}$ algebra with identity 1_1. $\mathcal{A} = \mathcal{I} \oplus \mathcal{I}^\perp$.

Decomposition of representations

Def. $\mathcal{F} \subset \mathcal{H}$ invariant subspace if $A \in \mathcal{A}$, $\phi \in \mathcal{F} \Rightarrow A\phi \in \mathcal{F}$.

Def. $\mathcal{H}$ irreducible: no proper invariant subspaces.

$\mathcal{H}$ representation $\Rightarrow \mathcal{H} \cong \underset{\uparrow \text{irreps}}{\mathcal{H}_1} \oplus \cdots \oplus \mathcal{H}_n$

Proof $\mathcal{F}$ invariant $\Rightarrow \mathcal{F}^\perp$ invariant, $\mathcal{H} = \mathcal{F} \oplus \mathcal{F}^\perp$. If $\mathcal{F}$ has invariant subspace then continue.

Main facts

Theorem 1 Equivalent are:

(i) $\mathcal{A}$ is factor

(ii) $\mathcal{A}$ has a faithful irreducible representation

(iii) $\mathcal{A} \cong \mathcal{L}(\mathcal{H})$.

Theorem 2 $\mathcal{A}$ factor, then:

(i) Every representation of $\mathcal{A}$ is faithful.

(ii) All irreducible representations of $\mathcal{A}$ are equivalent.

(iii) $\mathcal{H}$ irreducible representation $\Rightarrow l : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ $*$-algebra isomorphism.

Structure of algebras

$\mathcal{A}$ Q-algebra:

$$\mathcal{A} = \underset{\uparrow\text{factor}}{\mathcal{A}_1} \oplus \cdots \oplus \mathcal{A}_n.$$

$\mathcal{H}_k$ irreducible representation of $\mathcal{A}_k$. Note: $\mathcal{H}_k$ up to equivalence unique, automatically faithful, and $l_k : \mathcal{A}_k \to \mathcal{L}(\mathcal{H}_k)$ $*$-algebra isomorphism. View $\mathcal{H}_k$ as representation of $\mathcal{A}$ by defining:

$$(\underset{\in\mathcal{A}_1}{A_1} + \cdots + A_n)\phi := A_k\phi$$

Theorem Every representation of $\mathcal{A}$ is equivalent to a representation of the form

$$\mathcal{H} = (\underbrace{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_1}_{m_1 \text{ times}}) \oplus \cdots \oplus (\underbrace{\mathcal{H}_n \oplus \cdots \oplus \mathcal{H}_n}_{m_n \text{ times}}),$$

with $m_i \geq 0$ $(i = 1, \ldots, n)$. $\mathcal{H}$ is faithful $\Leftrightarrow$ $m_i \geq 1$ for all $i = 1, \ldots, n$. $\mathcal{A}$ is abelian $\Leftrightarrow$ $\dim(\mathcal{H}_k) = 1$ $\forall k$.

Proof $\mathcal{H}$ representation $\Rightarrow \mathcal{H} \cong \underset{\uparrow\text{irreps}}{\mathcal{H}_1} \oplus \cdots \oplus \mathcal{H}_m$. Def. $\mathcal{I}_k := \mathrm{Ker}(l_k)^\perp$. Then $l_k : \mathcal{I}_k \to \mathcal{L}(\mathcal{H}_k)$ faithful irreducible representation $\Rightarrow \mathcal{I}_k$ factor algebra. $\mathcal{I}_k \cap \mathcal{I}_l$ ideal of $\mathcal{I}_k$ hence $= \{0\}$ or $\mathcal{I}_k$. Hence

$$\forall k \neq l : \quad \mathcal{I}_k = \mathcal{I}_l \text{ or } \mathcal{I}_k \cap \mathcal{I}_l = \{0\}.$$

By Theorem 2 (ii), $\mathcal{I}_k = \mathcal{I}_l \;\Rightarrow \mathcal{H}_1$ and $\mathcal{H}_2$ equivalent.

Note $\psi \in \mathcal{H} \;\Rightarrow$

$$\psi = \underset{\in\mathcal{H}_1}{\phi_{1,1}} + \cdots + \underset{\in\mathcal{H}_1}{\phi_{1,m_1}} + \cdots + \underset{\in\mathcal{H}_n}{\phi_{n,1}} + \cdots + \underset{\in\mathcal{H}_n}{\phi_{n,m_n}}$$

Now

$$\begin{aligned}
&(\underset{\in\mathcal{A}_1}{A_1} + \cdots + \underset{\in\mathcal{A}_n}{A_n})\big(\underset{\in\mathcal{H}_1}{\phi_{1,1}} + \cdots + \underset{\in\mathcal{H}_1}{\phi_{1,m_1}} + \cdots + \underset{\in\mathcal{H}_n}{\phi_{n,1}} + \cdots + \underset{\in\mathcal{H}_n}{\phi_{n,m_n}}\big) \\
&= \big(A_1\phi_{1,1} + \cdots + A_1\phi_{1,m_1} + \cdots + A_n\phi_{n,1} + \cdots + A_1\phi_{n,m_n}\big) \\
&= \begin{pmatrix} A_1 & & & & & \\ & \ddots & & & & \\ & & A_1 & & & \\ & & & \ddots & & \\ & & & & A_n & \\ & & & & & \ddots & \\ & & & & & & A_n \end{pmatrix} \begin{pmatrix} \phi_{1,1} \\ \vdots \\ \phi_{1,m_1} \\ \vdots \\ \phi_{n,1} \\ \vdots \\ \phi_{n,m_n} \end{pmatrix}
\end{aligned}$$

Proofs

$\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ sub-$*$-algebra:
Def. $\mathcal{A}^{\mathrm{c}} := \{B \in \mathcal{L}(\mathcal{H}) : AB = BA \; \forall A \in \mathcal{A}\}$ commutant.

Neumann's bicommutant theorem $(\mathcal{A}^{\mathrm{c}})^{\mathrm{c}} = \mathcal{A}$.

Proof Theorem 2 (i) imagine $\mathcal{H}$ not faithful, then $\mathrm{Ker}(l)$ proper ideal, contradiction.

Proof Theorem 1 (i)⇒(ii) By assumption there exists a faithful representation of $\mathcal{A}$, Decomposing into irreducible representations proofs $\mathcal{A}$ has at least one irreducible representation. By what we just proved faithful.

Proof Theorem 2 (iii) Imagine $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ sub-$*$-algebra, $\mathcal{H}$ has no invariant subspaces. By Neumann suffices to prove that $\mathcal{A}^{\mathrm{c}} = \{a1 : a \in \mathbb{C}\}$. Little argument shows $\mathcal{A}^{\mathrm{c}} = \mathrm{span}\{B \in \mathcal{A}^{\mathrm{c}} : B \text{ hermitian}\}$. Imagine $B \in \mathcal{A}^{\mathrm{c}}$ hermitian, not multiple of 1, then functional calculus for normal operators $\Rightarrow \; \exists$ projektor $P \in \mathcal{A}^{\mathrm{c}}$, $P \neq 0, 1$. Now $P = P_{\mathcal{F}}$ where $\mathcal{F}$ invariant subspace, contradiction.

Proof Theorem 1 (ii)⇒(iii) Immediate from what we have just proved.

Proof Theorem 1 (iii)⇒(i) Imagine $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ not a factor, then $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Now identity $1_1 \in \mathcal{A}_1$ satisfies $1_1 \in \mathcal{A}^{\mathrm{c}} \Rightarrow \mathcal{A}^{\mathrm{c}} \neq \{a1 : a \in \mathbb{C}\}$. But it is easy to see that if $\mathcal{A} = \mathcal{L}(\mathcal{H})$, then $\mathcal{A}^{\mathrm{c}} = \{a1 : a \in \mathbb{C}\}$.

Proof Theorem 2 (ii) See lecture notes.

Proof Neumann's bicommutant theorem See lecture notes.