The Hausdorff metric

Let (E, d) be a metric space, let $\mathcal{K}(E)$ be the space of all compact subsets of E and set $\mathcal{K}_+(E) := \{K \in \mathcal{K}(E) : K \neq \emptyset\}$. Then the *Hausdorff metric* $d_{\rm H}$ on $\mathcal{K}_+(E)$ is defined as

$$\begin{aligned} d_{\rm H}(K_1, K_2) &:= \sup_{x_1 \in K_1} \inf_{x_2 \in K_2} d(x_1, x_2) \vee \sup_{x_2 \in K_2} \inf_{x_1 \in K_1} d(x_1, x_2) \\ &= \sup_{x_1 \in K_1} d(x_1, K_2) \vee \sup_{x_2 \in K_2} d(x_2, K_1), \end{aligned} \tag{1}$$

where $d(x, A) := \inf_{y \in A} d(x, y)$ denotes the distance between a point $x \in E$ and a set $A \subset E$. The corresponding topology is called the *Hausdorff topology*. We extend this topology to $\mathcal{K}(E)$ by adding $\emptyset$ as an isolated point. The next lemma shows that the Hausdorff topology depends only on the topology on E, and not on the choice of the metric.

Lemma 1 (Convergence criterion) *Let $K_n, K \in \mathcal{K}_+(E)$ $(n \geq 1)$. Then $K_n \to K$ in the Hausdorff topology if and only if there exists a $C \in \mathcal{K}_+(E)$ such that $K_n \subset C$ for all $n \geq 1$ and*

$$\begin{aligned} K &= \{x \in E : \exists x_n \in K_n \text{ s.t. } x_n \to x\} \\ &= \{x \in E : \exists x_n \in K_n \text{ s.t. } x \text{ is a cluster point of } (x_n)_{n \in \mathbb{N}}\}. \end{aligned} \tag{2}$$

The following lemma shows that $\mathcal{K}(E)$ is Polish if E is.

Lemma 2 (Properties of the Hausdorff metric)

(a) *If (E, d) is separable, then so is $(\mathcal{K}_+(E), d_{\rm H})$.*

(b) *If (E, d) is complete, then so is $(\mathcal{K}_+(E), d_{\rm H})$.*

Recall that a subset A of a metric space is *precompact* if its closure is compact. This is equivalent to the statement that each sequence of points $x_n \in A$ has a convergent subsequence.

Lemma 3 (Compactness in the Hausdorff topology) *A set $\mathcal{A} \subset \mathcal{K}(E)$ is precompact if and only if there exists a $C \in \mathcal{K}(E)$ such that $K \subset C$ for each $K \in \mathcal{A}$.*

The following lemma is useful when proving convergence of $\mathcal{K}(E)$-valued random variables.

Lemma 4 (Tightness criterion) *Assume that E is a Polish space and let K_n $(n \geq 1)$ be $\mathcal{K}(E)$-valued random variables. Then the collection of laws $\{\mathbb{P}[K_n \in \cdot\,] : n \geq 1\}$ is tight if and only if for each $\varepsilon > 0$ there exists a compact $C \subset E$ such that $\mathbb{P}[K_n \subset C] \geq 1 - \varepsilon$ uniformly in $n \in \mathbb{N}$.*

If E is compact, then the Hausdorff topology on $\mathcal{K}(E)$ coincides with the Fell topology defined in [Kal02, Thm. A.2.5]. The Hausdorff metric may more generally be defined on the space of nonempty bounded closed subsets of (E, d). In particular, if d is bounded, then $d_{\rm H}(A_1, A_2)$ can be defined for any nonempty closed A_1, A_2. In this more general set-up, Lemma 2 (b) and the 'if' part of Lemma 3 remain true, as well as the 'if' part of Lemma 5 below. This is Excercise 7 (with some hints for a possible solution) in [Mun00, § 45]. A detailed solution of this excercise can be found in [Hen99]. We are not aware of any reference for the other statements in Lemmas 1–4, although they appear to be well-known. For completeness, we provide self-contained proofs of all these lemmas. We start with some preparations.

Recall that for any metric space (E, d), a set $A \subset E$ is *totally bounded* if for every $\varepsilon > 0$ there exists a finite collection of points $x_1, \ldots, x_n \in E$ such that $A \subset \bigcup_{i=1}^n B_\varepsilon(x_i)$, where $B_\varepsilon(x)$ denotes the open ball of radius ε around x. This is equivalent to the statement that every sequence $x_n \in A$ has a Cauchy subsequence. As a consequence, a set $A \subset E$ is compact if and only if it is complete and totally bounded.

Lemma 5 (Totally bounded sets in the Hausdorff metric) *A set $\mathcal{A} \subset \mathcal{K}_+(E)$ is totally bounded in $(\mathcal{K}_+(E), d_{\rm H})$ if and only if the set $A := \{x \in E : \exists K \in \mathcal{A} \text{ s.t. } x \in K\}$ is totally bounded in (E, d).*

Proof. Assume that A is totally bounded. Let $\varepsilon > 0$ and let $\Delta \subset E$ be a finite set such that $A = \bigcup_{x\in\Delta} B_\varepsilon(x)$. Let $K \in \mathcal{A}$ and set $\Delta' := \{x \in \Delta : B_\varepsilon(x) \cap K \neq \emptyset\}$. Then for all $y \in K$ there is an $x \in \Delta'$ such that $d(x, y) < \varepsilon$ and for all $x \in \Delta'$ there is a $y \in K$ such that $d(x, y) < \varepsilon$ proving that $d_{\rm H}(\Delta', K) < \varepsilon$. This shows that $\mathcal{A}$ is covered, in the Hausdorff metric, by the collection of open balls of radius ε centered around finite subsets of Δ. Since ε is general, we conclude that $\mathcal{A}$ is totally bounded.

Conversely, if $\mathcal{A}$ is totally bounded, then for each $\varepsilon > 0$ we can find $K_1, \ldots, K_n \in \mathcal{K}_+(E)$ such that $\mathcal{A} \subset \bigcup_{k=1}^n \mathcal{B}_{\varepsilon/2}(K_n)$, where $\mathcal{B}_\varepsilon(K)$ denotes the open ball in the Hausdorff metric of radius ε centered around a compact set K. Since each K_k is compact, there exist $x_{k,1}, \ldots, x_{k,m_k}$ such that $K_k \subset \bigcup_{j=1}^{m_k} B_{\varepsilon/2}(x_{k,j})$, hence $A \subset \bigcup_{k=1}^n \bigcup_{j=1}^{m_k} B_\varepsilon(x_{k,j})$. ∎

Lemma 6 (Cauchy sequences in the Hausdorff metric) *Let $K_n \in \mathcal{K}_+(E)$ be a Cauchy sequence in $(\mathcal{K}_+(E), d_{\rm H})$. Then there exists a closed set K such that (2) holds.*

Proof. The sets on the first and second line of the right-hand side of (2) are, respectively,

$$A = \{x \in E : \lim_{n\to\infty} d(x, K_n) = 0\} \quad \text{and} \quad B = \{x \in E : \liminf_{n\to\infty} d(x, K_n) = 0\}. \tag{3}$$

If $x \in B\backslash A$, then there is some $\varepsilon > 0$ such that for each $k \geq 1$ we can find $n, m \geq k$ such that $d(x, K_n) \leq \varepsilon$ and $d(x, K_m) \geq 2\varepsilon$, hence $d_{\rm H}(K_n, K_m) \geq \varepsilon$, contradicting the assumption that the K_n form a Cauchy sequence.

To complete the proof, it suffices to show that if $A = B$, then $K := A = B$ is closed. We will show that if $x_k \in A$ satisfy $x_k \to x$ for some $x \in E$, then $x \in B$. Since $x_k \in A$ we can find $x_{k,n} \in K_n$ such that $x_{k,n} \to x_k$ as $n \to \infty$. For each k, we can choose $n(k) \geq k$ such that $d(x_{k,n(k)}, x_k) \leq d(x_k, x)$. Then $n(k) \to \infty$ and $d(x, K_{n(k)}) \leq d(x_{k,n(k)}, x) \leq 2d(x_k, x) \to 0$ as $k \to \infty$ and hence $x \in B$. ∎

Lemma 7 (Sufficient conditions for convergence) *The conditions for convergence in the Hausdorff topology given in Lemma 1 are sufficient.*

Proof. Our assumptions imply that $d(x, K_n) \to 0$ for each $x \in K$. We wish to show that in fact $\sup_{x\in K} d(x, K_n) \to 0$. If this is not the case, then by going to a subsequence if necessary we may assume that there exist $x_n \in K$ and $\varepsilon > 0$ such that $\liminf_{n\to\infty} d(x_n, K_n) \geq \varepsilon$. Since K is compact, by going to a further subsequence if necessary, we may assume that $x_n \to x \in K$. But then $\liminf_{n\to\infty} d(x, K_n) \geq \liminf_{n\to\infty}(d(x_n, K_n) - d(x, x_n)) \geq \varepsilon$ for this subsequence, contradicting the fact that for the original sequence, $d(x, K_n) \to 0$ for each $x \in K$.

The proof that $\sup_{x\in K_n} d(x, K) \to 0$ is similar. If this is not true, then we can go to a subsequence of the K_n and then find $x_n \in K_n$ such that $d(x_n, K) \geq \varepsilon$ for all n, for some $\varepsilon > 0$. Using the compactness of C, we can select a further subsequence such that $x_n \to x \in C$. Now x is a cluster point of some $x_n \in K_n$ but $d(x, K) \geq \varepsilon$, contradicting the fact that the two sets on the right-hand side of (2) are equal. ∎

Proof of Lemma 2. To prove part (a), it suffices to show that if $\mathcal{D}$ is a countable dense subset of (E, d), then the collection of finite subsets of $\mathcal{D}$ is a countable dense subset of $(\mathcal{K}_+(E), d_{\rm H})$. Since a compact set $K \subset E$ is totally bounded, for each $\varepsilon > 0$, we can find a finitely many points $x_1, \ldots, x_n \in E$ such that $K \subset \bigcup_{i=1}^n B_{\varepsilon/2}(x_i)$. Since $\mathcal{D}$ is dense, we can choose $x'_i \in \mathcal{D} \cap B_{\varepsilon/2}(x_i)$. Then $d_{\rm H}(K, \{x'_1, \ldots, x'_n\}) \leq \varepsilon$, proving our claim.

To prove part (b), let $K_n \in \mathcal{K}_+(E)$ be a Cauchy sequence. Then, by Lemma 6, there exists a closed set K such that (2) holds. Since each sequence in the set $\{K_n : n \geq 1\}$ contains a Cauchy subsequence, the set $\{K_n : n \geq 1\}$ is totally bounded, hence by Lemma 5, there exists some totally bounded set containing all of the K_n. Let C denote its closure. Then C is compact since E is complete, hence also $K \subset C$ is compact and Lemma 7 implies that $K_n \to K$. ∎

Proof of Lemma 3. It suffices to prove the statement for $\mathcal{A} \subset \mathcal{K}_+(E)$. If there exists a compact $C \subset E$ such that $K \subset C$ for all $K \in \mathcal{A}$, then C is totally bounded and complete, so by Lemmas 5 and 2 (b), the same is true for $\{K \in \mathcal{K}_+(E) : K \subset E\}$, implying the latter is compact and hence its subset $\mathcal{A}$ is precompact. To complete the proof, it suffices to show that if the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ is compact, then the set $C := \{x \in E : \exists K \in \overline{\mathcal{A}} \text{ s.t. } x \in K\}$ is compact. Since $\overline{\mathcal{A}}$ is totally bounded, Lemma 5 implies that C is totally bounded too. It therefore suffices to show that C is complete. For this, it suffices to show that each sequence $x_n \in C$ has a cluster point $x \in C$. Choose $K_n \in \overline{\mathcal{A}}$ such that $x_n \in K_n$. Since $\overline{\mathcal{A}}$ is compact, by going to a subsequence if necessary, we may assume that $K_n \to K$ for some $K \in \overline{\mathcal{A}}$. Choose $x'_n \in K$ such that $d(x_n, x'_n) \to 0$. Since K is compact, by going to a further subsequence if necessary, we may assume that $x'_n \to x$ for some $x \in K$. Since $d(x_n, x) \leq d(x_n, x'_n) + d(x'_n, x) \to 0$ this proves that the sequence x_n has a cluster point $x \in K \subset C$. ∎

Proof of Lemma 4. Immediate from Lemma 3 and the definition of tightness. ∎

Proof of Lemma 1. By Lemma 7, we only need to prove that if $K_n \in \mathcal{K}_+(E)$ converge to a limit K, then there exists a $C \in \mathcal{K}_+(E)$ such that $K_n \subset C$ for all n and (2) holds. If $K_n \to K$ then the set $\{K_n : n \geq 1\}$ is precompact, hence by Lemma 3 there exists a $C \in \mathcal{K}_+(E)$ such that $K_n \subset C$ for all n. Formula (2) follows from the facts that if $x \in K$, then $d(x, K_n) \to 0$ hence there exist $K_n \ni x_n \to x$, while if $x \notin K$, then $B_\varepsilon(x) \cap K_n = \emptyset$ for all n large enough such that $\sup_{x' \in K} d(x', K_n) < \varepsilon$, hence x is not a cluster point of some $x_n \in K_n$. ∎

References

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