Skorohod's topologies on path space

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December 15, 2022

Abstract

We introduce the path space over a general metrisable space. Elements of this space are paths, which are pairs consisting of a closed subset of the real line and a cadlag function that is defined on that subset and takes values in the metrisable space. We equip the space of all paths with topologies that generalise Skorohod's J1 and M1 topologies, prove that these topologies are Polish, and derive compactness criteria.

The central idea is that the closed graph (in case of the J1 topology) and the filled-in graph (in case of the M1 topology) of a path can naturally be viewed as totally ordered compact sets. We define a variant of the Hausdorff metric that measures the distance between two compact sets, each of which is equipped with a total order. We show that the topology generated by this metric is Polish and derive a compactness criterion. Specialising to closed or filled-in graphs then yields Skorohod's J1 and M1 topologies, generalised to functions that need not all be defined on the same domain.

MSC 2020. To be filled in.
Primary: 26A15; Secondary: 06A05, 54E35, 60G07
Keywords. Skorohod topology, J1 topology, M1 topology, path space, Hausdorff metric.
Acknowledgments. Work sponsored by GAČR grant 22-12790S.

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Contents

1 Introduction

A function is *cadlag* (from the French "continue à droit, limite à gauche") if it is right-continuous with left limits. In his classical paper [Sko56], Skorohod introduced four topologies on the space of real cadlag functions on a compact interval, which he called J1, J2, M1, and M2. Of these, the J1 topology has proved to be the most natural in many situations, in particular, when discussing convergence of Markov processes [EK86]. For this reason, Skorohod's J1 topology is now normally known as the "Skorohod topology". Classical textbook discussions of the Skorohod topology can be found in [EK86, Section 3.5] and [Bil99, Section 12]. All four topologies introduced by Skorohod are discussed in [Whi02, Section 11.5].

Motivated by the theory of the Brownian web and net [FINR04, SSS17], we study *paths*, which, roughly speaking, are cadlag functions that are defined on an arbitrary closed subset of the real line and take values in a metrisable topological space $\mathcal{X}$. We generalise Skorohod's topologies to the space of all paths over a fixed space $\mathcal{X}$. Roughly, a sequence of paths converges when the domains on which they are defined converge and moreover the paths themselves converge in the sense of Skorohod's J1 or M1 topologies. We show that the path space, equipped with such a topology, is a Polish space, and we derive compactness criteria in terms of a suitable modulus of continuity. Polish spaces play a crucial role in probability theorem, as they are required for Prohorov's theorem [Bil99, Thm 5.2], and sufficient for many other theorems requiring some regularity of a measurable space.

In Section 2, we first develop the necessary topological material, which in Section 3 we then apply to path space. The remaining Sections 4 and 5 contain proofs.

In Subsections 2.1 and 2.2, we develop an observation, due to Kolmogorov [Kol56], that a cadlag function defined on an interval, together with its left-continuous modification, can be viewed as a continuous function on a somewhat peculiar topological space introduced by Alexandroff and Urysohn [AU29]. This makes it possible to allow cadlag functions to jump at their initial times, which later simplifies the compactness criteria, and also provides the right set-up to define paths whose domains may not be intervals.

In Subsection 2.3 we recall the Hausdorff metric, which measures the distance between two compact subsets of a metric space. In Subsection 2.3, we introduce a variant of this metric that measures the distance between two compact subsets that are moreover each equipped with a total order. We will later apply this to the closed graph (in case of the J1 topology) or the filled-in graph (in case of the M1 topology), which can naturally be viewed as totally ordered compact sets.

The classical M1 topology is defined only for paths taking values in the real line. For paths taking values in more general spaces, there are several possible ways to generalise the M1 topology. In Subsection 2.5, we introduce the concept of a "betweenness", which will allow us to treat the J1 topology and various possible variants of the M1 topology in a unified framework.

Skorohod and Kolmogorov [Sko56, Kol56] only considered cadlag functions defined on compact time intervals. The extension to unbounded time intervals is important in applications, but not completely trivial. For continuous paths, one defines locally uniform convergence of a sequence of functions by requiring that their restrictions to any compact time interval converge uniformly. For the J1 and M1 topologies, this approach is not feasible, since the map that restricts a function to a smaller time interval is not continuous. To solve this, in Subsection 2.6, we use an idea first developed in the theory of the Brownian web [FINR04], which is to introduce a topology on space-time that cares less about the spatial distance between two space-time points if their time coordinates are large. This will allow us to view graphs of cadlag functions as compact sets, even when they are defined on unbounded time intervals.

With all the right ingredients in place, in Section 3 we introduce and study the J1 and M1 topologies on path space. In Subsection 3.1, we show that the closed and filled-in graphs of a path can be viewed as totally ordered compact sets. In Subsection 3.2, we then define metrics for the J1 and M1 topologies by measuring the distance between closed and filled-in graphs using the ordered Hausdorff metric from Subsection 2.3. In Subsection 3.3, we derive compactness criteria in terms of a suitable modulus of continuity, generalising results from [Sko56, Kol56]. In Section 3.4, we specialise to cadlag functions defined on a fixed time interval and show how our results relate to the classical textbook definitions of the J1 and M1 topologies, and also briefly discuss the less commonly used J2 and M2 topologies.

2 Preliminaries

2.1 The split real line

Let $\mathbb{R}_{\mathfrak{s}}$ be the space that consists of all words of the form $t\star$ where $t \in \mathbb{R}$ is a real number and $\star \in \{-,+\}$ is a sign. We think of $\mathbb{R}_{\mathfrak{s}}$ as obtained by cutting each point of the real line into two. Consequently, we call $\mathbb{R}_{\mathfrak{s}}$ the *split real line* and call elements of $\mathbb{R}_{\mathfrak{s}}$ *split real numbers*. We denote split real numbers either by words $t\star$ consisting of a Roman letter and a sign, or by a single Greek letter. In this case, if $\tau = t\star$, then we call $\underline{\tau} := t$ the *real part* of τ and we call $\mathfrak{s}(\tau) := \star$ its *sign*.

We equip $\mathbb{R}_{\mathfrak{s}}$ with the lexicographic order, i.e., we set $\sigma \leq \tau$ if and only if either $\underline{\sigma} < \underline{\tau}$ or $\underline{\sigma} = \underline{\tau}$ and $\mathfrak{s}(\sigma) \leq \mathfrak{s}(\tau)$, where $\{-,+\}$ is equipped with the natural total order in which $- \leq +$. We write $\sigma < \tau$ if $\sigma \leq \tau$ and $\sigma \neq \tau$. We use notation for intervals in $\mathbb{R}_{\mathfrak{s}}$ similar to the usual notation for the real line, i.e.,

$$\begin{aligned} (\sigma,\rho) &:= \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma < \tau < \rho\}, \qquad & [\sigma,\rho) &:= \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma \leq \tau < \rho\}, \\ (\sigma,\rho] &:= \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma < \tau \leq \rho\}, \qquad & [\sigma,\rho] &:= \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma \leq \tau \leq \rho\}. \end{aligned} \tag{2.1}$$

Note that there is some redundancy in this notation: for example, $(s-,t+) = [s+,t-]$. We equip $\mathbb{R}_{\mathfrak{s}}$ with the *order topology*, which means that by definition, a basis for the topology on $\mathbb{R}_{\mathfrak{s}}$ is formed by all intervals of the form (σ,ρ) with $\sigma,\rho \in \mathbb{R}_{\mathfrak{s}}$. We defer the (simple) proof of the following lemma, and all further results stated in this section, to Section 4.

Lemma 2.1 (Convergence criterion) *For $\tau_n \in \mathbb{R}_{\mathfrak{s}}$ and $t \in \mathbb{R}$, one has*

(i) $\tau_n \to t+$ *if and only if* $\underline{\tau}_n \to t$ *and* $\tau_n \geq t+$ *for all n sufficiently large,*

(ii) $\tau_n \to t-$ *if and only if* $\underline{\tau}_n \to t$ *and* $\tau_n \leq t-$ *for all n sufficiently large.*

Intervals of the form (σ,ρ) (resp. $[\sigma,\rho]$) are open (resp. closed) in the topology on $\overline{\mathbb{R}}_{\mathfrak{s}}$. In particular, $(s-,t+) = [s+,t-]$ is both open and closed. The following lemma lists some elementary properties of $\mathbb{R}_{\mathfrak{s}}$.

Lemma 2.2 (The split real line) *The space $\mathbb{R}_{\mathfrak{s}}$ is first countable, Hausdorff, and separable, but not second countable and not metrisable. Moreover, $\mathbb{R}_{\mathfrak{s}}$ is totally disconnected, meaning that its only connected subsets are singletons.*

We equip the product space $\mathbb{R}_{\mathfrak{s}}^d$ with the product topology. By definition, a subset $A \subset \mathbb{R}_{\mathfrak{s}}^d$ is *bounded* if $A \subset [\sigma,\tau]^d$ for some $\sigma,\tau \in \mathbb{R}_{\mathfrak{s}}$. The following proposition gives a characterisation of the compact subsets of $\mathbb{R}_{\mathfrak{s}}^d$, similar to the well-known characterisation of compact subsets of $\mathbb{R}^d$.

Proposition 2.3 (Compact sets) *For a subset $C \subset \mathbb{R}_{\mathfrak{s}}^d$, the following three claims are equivalent: (i) C is compact, (ii) C is sequentially compact, and (iii) C is closed and bounded.*

We can compactify $\mathbb{R}_{\mathfrak{s}}$ by adding two points $\pm\boldsymbol{\infty}$, in such a way that $-\boldsymbol{\infty} < \tau < +\boldsymbol{\infty}$ for all $\tau \in \mathbb{R}_{\mathfrak{s}}$, and then equipping $\overline{\mathbb{R}}_{\mathfrak{s}} := \mathbb{R}_{\mathfrak{s}} \cup \{-\boldsymbol{\infty}, +\boldsymbol{\infty}\}$ with the topology generated by intervals of the form (σ, τ), $[-\boldsymbol{\infty}, \tau)$, or $(\sigma, +\boldsymbol{\infty}]$. Instead of $+\boldsymbol{\infty}$ we also write $\boldsymbol{\infty}$. We call $\overline{\mathbb{R}}_{\mathfrak{s}}$ the *extended split real line*. Note that we notationally distinguish the points at infinity $\pm\boldsymbol{\infty}$ of the extended split real line from the points $\pm\infty$ of the extended real line. We extend the functions $\tau \mapsto \underline{\tau}$ and $\tau \mapsto \mathfrak{s}(\tau)$ that assign to each split real number τ its real part $\underline{\tau}$ and sign $\mathfrak{s}(\tau)$ to the extended split real line $\overline{\mathbb{R}}_{\mathfrak{s}}$ by setting $\underline{\pm\boldsymbol{\infty}} := \pm\infty$ and $\mathfrak{s}(-\boldsymbol{\infty}) := +$, $\mathfrak{s}(\boldsymbol{\infty}) = -$. Note that with these definitions $\overline{\mathbb{R}}_{\mathfrak{s}}$ is naturally isomorphic to $[0+, 1-]$. The extended split real line provides us with a natural way to denote half infinite intervals in $\mathbb{R}_{\mathfrak{s}}$; for example, $[\sigma, \boldsymbol{\infty}) = \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma \leq \tau\}$.

2.2 Cadlag functions

Let $I \subset \mathbb{R}$ be an interval, let $\mathcal{X}$ be a Hausdorff topological space, and let $f : I \to \mathcal{X}$ be a function. By definition, we say that f is *cadlag* if it is right-continuous with left limits, i.e.:

(i) $f(t) = \lim_{n\to\infty} f(t_n)$ whenever $t_n, t \in I$ satisfy $t_n \underset{n\to\infty}{\longrightarrow} t$ and $t_n > t$ for all n,

(ii) $f(t-) := \lim_{n\to\infty} f(t_n)$ exists whenever $t_n, t \in I$ satisfy $t_n \underset{n\to\infty}{\longrightarrow} t$ and $t_n < t$ for all n.

Similarly, a *caglad* function (from the French "continue à gauche, limite à droit") is left-continuous with right limits. For any closed interval $I \subset \mathbb{R}$ of nonzero length, we let $\mathcal{D}_I(\mathcal{X})$ denote the space of all functions $f : I \to \mathcal{X}$ such that:

$$\text{(i) } f \text{ is cadlag,} \quad \text{(ii) if } t := \sup I < \infty, \text{ then } f(t-) = f(t). \tag{2.2}$$

We impose condition (ii) in order to have a more symmetric definition, since cadlag functions can by construction not have a jump at the left boundary of their domain. We let $\mathcal{D}_I^-(\mathcal{X})$ denote the space of all functions $f : I \to \mathcal{X}$ such that:

$$\text{(i) } f \text{ is caglad,} \quad \text{(ii) if } t := \inf I > -\infty, \text{ then } f(t) = f(t+), \tag{2.3}$$

where $f(t+)$ denotes the right limit of f at t. The *left-continuous modification* of a function $f \in \mathcal{D}_I(\mathcal{X})$ is the function $f^- \in \mathcal{D}_I^-(\mathcal{X})$ uniquely defined by the requirement that $f^-(t) := f(t-)$ for all $t \in I$ where the left limit is defined. Right-continuous modifications of functions in $\mathcal{D}_I^-(\mathcal{X})$ are defined similarly. A cadlag function $f \in \mathcal{D}_I(\mathcal{X})$ and its left-continuous modification f^- uniquely determine each other. Indeed, f is the right-continuous modification of f^-.

If $\mathcal{I} \subset \mathbb{R}_{\mathfrak{s}}$ is a closed subinterval of the split real line and $\mathcal{X}$ is a Hausdorff topological space, then by Lemma 2.1, a function $f : \mathcal{I} \to \mathcal{X}$ is continuous if and only if

(i) $f(\tau_n) \to f(t+)$ for all $t+ \in \mathcal{I}$ and $\tau_n \in \mathcal{I}$ such that $\tau_n \geq t+$ for all n and $\underline{\tau}_n \to t$,

(ii) $f(\tau_n) \to f(t-)$ for all $t- \in \mathcal{I}$ and $\tau_n \in \mathcal{I}$ such that $\tau_n \leq t-$ for all n and $\underline{\tau}_n \to t$.

We let $\mathcal{C}_{\mathcal{I}}(\mathcal{X})$ denote the space of continuous functions $f : \mathcal{I} \to \mathcal{X}$. Continuous functions on a closed subinterval of the split real line correspond more or less to cadlag functions on a closed subinterval of the real line. To make this connection precise, for any closed interval $I \subset \mathbb{R}$ of nonzero length, we define $I_{\rm in} \subset \mathbb{R}_{\mathfrak{s}}$ by

$$I_{\rm in} := \big\{t- : (t-\varepsilon, t] \subset I \text{ for some } \varepsilon > 0\big\} \cup \big\{t+ : [t, t+\varepsilon) \subset I \text{ for some } \varepsilon > 0\big\}. \tag{2.4}$$

In particular, if $I = [s, u]$, then $I_{\rm in} = [s+, u-]$. Then we have the following lemma.

Lemma 2.4 (Cadlag functions as continuous functions) *Let I be a closed real interval of nonzero length and let $\mathcal{X}$ be a Hausdorff topological space. Let $f^+ \in \mathcal{D}_I(\mathcal{X})$ and let $f^- \in \mathcal{D}_I^-(\mathcal{X})$ be its left-continuous modification. Then setting*

$$f(t\pm) := f^\pm(t) \qquad (t\pm \in I_{\rm in}) \tag{2.5}$$

defines a function $f \in \mathcal{C}_{I_{\rm in}}(\mathcal{X})$, and each function $f \in \mathcal{C}_{I_{\rm in}}(\mathcal{X})$ is of this form.

In particular, if $[s,u]$ is a compact real interval, then Lemma 2.4 says that there is a natural isomorphism between the space of cadlag functions $\mathcal{D}_{[s,u]}(\mathcal{X})$ and the space of continuous functions $\mathcal{C}_{[s+,u-]}(\mathcal{X})$. An advantage of working with the split real line is that we can also easily allow for functions that jump at the endpoints of their domain. Indeed, if we replace $\mathcal{C}_{[s+,u-]}(\mathcal{X})$ by the slightly larger space $\mathcal{C}_{[s-,u+]}(\mathcal{X})$, then we obtain a space of functions that can also jump at the endpoints s and u of the real interval $[s,u]$.

Although we will not need this in the present paper, we note that the split real line also leads to a natural definition of cadlag functions of several variables, since we can simply define them as continuous functions defined on (a subset of) the product space $\mathbb{R}_{\mathfrak{s}}^n$. This seems much simpler than the approach used by other authors such as [Neu71].

2.3 The Hausdorff metric

For any metric space $(\mathcal{X},d)$, we let $\mathcal{K}_+(\mathcal{X})$ denote the space of all nonempty compact subsets of $\mathcal{X}$. The *Hausdorff metric* $d_{\rm H}$ on $\mathcal{K}_+(\mathcal{X})$ is defined as

$$d_{\rm H}(K_1,K_2) := \sup_{x_1\in K_1} d(x_1,K_2) \vee \sup_{x_2\in K_2} d(x_2,K_1), \tag{2.6}$$

where $d(x,A) := \inf_{y\in A} d(x,y)$ denotes the distance between a point $x \in \mathcal{X}$ and a set $A \subset \mathcal{X}$. We can alternatively define $d_{\rm H}$ in terms of correspondences. A *correspondence* between two sets A_1, A_2 is a set $R \subset A_1 \times A_2$ such that

$$\forall x_1 \in A_1\ \exists x_2 \in A_2 \text{ s.t. } (x_1,x_2) \in R \quad\text{and}\quad \forall x_2 \in A_2\ \exists x_1 \in A_1 \text{ s.t. } (x_1,x_2) \in R. \tag{2.7}$$

We let $\mathrm{Cor}(A_1,A_2)$ denote the set of all correspondences between A_1 and A_2.

Lemma 2.5 (Hausdorff metric and correspondences) *Let $(\mathcal{X},d)$ be a metric space. Then*

$$d_{\rm H}(K_1,K_2) = \inf_{R\in\mathrm{Cor}(K_1,K_2)} \sup_{(x_1,x_2)\in R} d(x_1,x_2). \tag{2.8}$$

Moreover, there exists an $R \in \mathrm{Cor}(K_1,K_2)$ such that $d_{\rm H}(K_1,K_2) = \max_{(x_1,x_2)\in R} d(x_1,x_2)$.

We cite the following lemma from [SSS14, Lemma B.1].

Lemma 2.6 (Convergence criterion) *Let $K_n, K \in \mathcal{K}_+(\mathcal{X})$ $(n \geq 1)$. Then $K_n \to K$ in the Hausdorff topology if and only if there exists a $C \in \mathcal{K}_+(\mathcal{X})$ such that $K_n \subset C$ for all $n \geq 1$ and*

$$\begin{aligned} K &= \{x \in \mathcal{X} : \exists x_n \in K_n \textit{ s.t. } x_n \to x\} \\ &= \{x \in \mathcal{X} : \exists x_n \in K_n \textit{ s.t. } x \textit{ is a cluster point of } (x_n)_{n\in\mathbb{N}}\}. \end{aligned} \tag{2.9}$$

Lemma 2.6 shows that if d and d' are two metrics generating the same topology on $\mathcal{X}$, then the corresponding Hausdorff metrics $d_{\rm H}$ and $d'_{\rm H}$ generate the same topology on $\mathcal{K}_+(\mathcal{X})$. We call this topology the *Hausdorff topology*. Note the subtle difference between "the Hausdorff topology" (the topology generated by the Hausdorff metric) and "a Hausdorff topology" (any topology satisfying Hausdorff's separation axiom).

The following lemma is [SSS14, Lemma B.2]. In particular, it shows that $\mathcal{K}_+(\mathcal{X})$ is Polish if $\mathcal{X}$ is.

Lemma 2.7 (Properties of the Hausdorff metric)

(a) *If* $(\mathcal{X}, d)$ *is separable, then so is* $(\mathcal{K}_+(\mathcal{X}), d_{\rm H})$.

(b) *If* $(\mathcal{X}, d)$ *is complete, then so is* $(\mathcal{K}_+(\mathcal{X}), d_{\rm H})$.

Recall that a set is called precompact if its closure is compact. The following lemma is [SSS14, Lemma B.3]. In particular, it shows that $\mathcal{K}_+(\mathcal{X})$ is compact if $\mathcal{X}$ is.

Lemma 2.8 (Compactness in the Hausdorff topology) *A set* $\mathcal{A} \subset \mathcal{K}_+(\mathcal{X})$ *is precompact if and only if there exists a* $C \in \mathcal{K}_+(\mathcal{X})$ *such that* $K \subset C$ *for each* $K \in \mathcal{A}$.

The following lemma says connectedness is a property of compact sets that is preserved under limits.

Lemma 2.9 (Preservation of connectedness) *The set* $\mathcal{K}_{\rm c}(\mathcal{X})$ *of all connected nonempty compact subsets of* $\mathcal{X}$ *is a closed subset of* $\mathcal{K}_+(\mathcal{X})$.

2.4 The ordered Hausdorff metric

We will need a variant of the Hausdorff metric that measures the distance between two compact sets, each of which is equipped with a total order. For any metric space $(\mathcal{X}, d)$, we let $\mathcal{K}_{\rm part}(\mathcal{X})$ denote the space of all pairs $(K, \preceq)$ where K is a nonempty compact subset of $\mathcal{X}$ and $\preceq$ is a partial order on K that is *compatible with the topology* in the sense that the set

$$K^{\langle 2\rangle} := \big\{(x, y) \in K^2 : x \preceq y\big\} \tag{2.10}$$

is a closed subset of K^2, equipped with the product topology. Note that we do not assume that $\mathcal{X}$ is equipped with a partial order; in particular, the partial order on K does not have to come from an order on $\mathcal{X}$, although we always assume that the topology on K is the induced topology from $\mathcal{X}$. We will sometimes be sloppy and denote elements of $\mathcal{K}_{\rm part}(\mathcal{X})$ simply as K, where it is implicitly understood that K is equipped with a partial order that is compatible with the topology. We equip the space $\mathcal{X}^2$ with the metric

$$d^2\big((x_1, y_1), (x_2, y_2)\big) := d(x_1, x_2) \vee d(y_1, y_2), \tag{2.11}$$

which generates the product topology, and we equip the space $\mathcal{K}_+(\mathcal{X}^2)$ of compact nonempty subsets of $\mathcal{X}^2$ with the associated Hausdorff metric

$$d^2_{\rm H}(A_1, A_2) := \sup_{(x_1,y_1)\in A_1} d^2\big((x_1, y_1), A_2\big) \vee \sup_{(x_2,y_2)\in A_2} d^2\big((x_2, y_2), A_1\big) \qquad \big(A_1, A_2 \in \mathcal{K}_+(\mathcal{X}^2)\big). \tag{2.12}$$

An element $(K, \preceq)$ of $\mathcal{K}_{\rm part}(\mathcal{X})$ is clearly uniquely determined by the compact set $K^{\langle 2\rangle} \subset \mathcal{X}^2$ defined in (2.10), so setting

$$d_{\rm part}(K_1, K_2) := d^2_{\rm H}(K_1^{\langle 2\rangle}, K_2^{\langle 2\rangle}) \qquad \big(K_1, K_2 \in \mathcal{K}_{\rm part}(\mathcal{X})\big) \tag{2.13}$$

defines a metric $d_{\rm part}$ on $\mathcal{K}_{\rm part}(\mathcal{X})$.

We let $\mathcal{K}_{\rm tot}(\mathcal{X})$ denote the space of all pairs $(K, \preceq) \in \mathcal{K}_{\rm part}(\mathcal{X})$ such that $\preceq$ is a total order on K. There is a natural way to define a metric on $\mathcal{K}_{\rm tot}(\mathcal{X})$ that is at first sight very different from the definition in (2.13). Recall the definition of a correspondence from Subsection 2.3. By definition, a correspondence R between two totally ordered spaces $(K_1, \preceq_1)$ and $(K_2, \preceq_2)$ is *monotone* if

$$\text{there are no } (x_1, x_2), (y_1, y_2) \in R \text{ such that } x_1 \prec_1 y_1 \text{ and } y_2 \prec_2 x_2, \tag{2.14}$$

where $x \prec y$ means that $x \preceq y$ and $x \neq y$. We let $\mathrm{Cor}_+(K_1, K_2)$ denote the space of all monotone correspondences between two totally ordered spaces K_1 and K_2, and define a metric $d_{\rm tot}$ on $\mathcal{K}_{\rm tot}(\mathcal{X})$ by

$$d_{\rm tot}(K_1, K_2) := \inf_{R \in \mathrm{Cor}_+(K_1,K_2)} \sup_{(x_1,x_2)\in R} d(x_1, x_2) \qquad \big(K_1, K_2 \in \mathcal{K}_{\rm tot}(\mathcal{X})\big). \tag{2.15}$$

The following theorem says that $d_{\rm part}$ and $d_{\rm tot}$ generate the same topology on $\mathcal{K}_{\rm tot}(\mathcal{X})$ and satisfy $d_{\rm part} \leq d_{\rm tot}$, but they do not satisfy an opposite inequality of the form $d_{\rm tot} \leq C d_{\rm part}$ for any $C < \infty$.

Theorem 2.10 (The ordered Hausdorff topology) *Let $(\mathcal{X}, d)$ be a metric space. Then the metrics $d_{\rm part}$ and $d_{\rm tot}$ defined in (2.13) and (2.15) generate the same topology on $\mathcal{K}_{\rm tot}(\mathcal{X})$. Also, if d and d' generate the same topology on $\mathcal{X}$ and $d_{\rm part}$ and $d'_{\rm part}$ are defined in terms of d and d' as in (2.13), then $d_{\rm part}$ and $d'_{\rm part}$ generate the same topology on $\mathcal{K}_{\rm tot}(\mathcal{X})$. One has*

$$d_{\rm H}(K_1, K_2) \leq d_{\rm part}(K_1, K_2) \leq d_{\rm tot}(K_1, K_2) \qquad \big(K_1, K_2 \in \mathcal{K}_{\rm tot}(\mathcal{X})\big). \tag{2.16}$$

If $\mathcal{X} = [0, 1]$, then for each $\varepsilon > 0$, there exist $K_1, K_2 \in \mathcal{K}_{\rm tot}(\mathcal{X})$ such that $d_{\rm part}(K_1, K_2) \leq \varepsilon d_{\rm tot}(K_1, K_2)$.

We call the topology on $\mathcal{K}_{\rm tot}(\mathcal{X})$ generated by $d_{\rm part}$ or $d_{\rm tot}$ the *ordered Hausdorff topology.* The second claim of Theorem 2.10 says that this topology depends only on the topology on $\mathcal{X}$ and not on the choice of metric on $\mathcal{X}$. We recall that a topological space $\mathcal{X}$ is *Polish* if $\mathcal{X}$ is separable and there exists a complete metric generating the topology on $\mathcal{X}$. Note that being Polish is a property of the topology and not a property of the metric. In fact, on each non-compact Polish space $\mathcal{X}$, there also exist non-complete metrics that generate the topology on $\mathcal{X}$.[1] If $(\mathcal{X}, d)$ is complete, then as we will show in Lemma 4.13 below, $\mathcal{K}_{\rm tot}(\mathcal{X})$ is not in general complete in the metrics $d_{\rm part}$ or $d_{\rm tot}$. Nevertheless, we have the following result.

Proposition 2.11 (Preservation of Polishness) *If $\mathcal{X}$ is a Polish space, then so is $\mathcal{K}_{\rm tot}(\mathcal{X})$, equipped with the ordered Hausdorff topology.*

Our next result characterises the compact subsets of $\mathcal{K}_{\rm tot}(\mathcal{X})$. For $K \in \mathcal{K}_{\rm tot}(\mathcal{X})$ and $\varepsilon > 0$, we define the *mismatch modulus* $m_\varepsilon(K)$ as

$$\begin{aligned} m_\varepsilon(K) := \sup \big\{ & d(x_1, y_1) \vee d(x_2, y_2) : x_1, y_1, x_2, y_2 \in K \\ & d(x_1, x_2) \vee d(y_1, y_2) \leq \varepsilon,\ x_1 \preceq y_1,\ y_2 \preceq x_2 \big\}. \end{aligned} \tag{2.17}$$

Theorem 2.12 (Compact subsets) *Let $(\mathcal{X}, d)$ be a metric space and let $\mathcal{K}_{\rm tot}(\mathcal{X})$ be equipped with the ordered Hausdorff topology. Then a set $\mathcal{A} \subset \mathcal{K}_{\rm tot}(\mathcal{X})$ is precompact if and only if*

$$\text{(i) } \exists \text{ compact } C \subset \mathcal{X} \text{ s.t. } K \subset C\ \forall K \in \mathcal{A} \quad \text{and} \quad \text{(ii) } \lim_{\varepsilon \to 0} \sup_{K \in \mathcal{A}} m_\varepsilon(K) = 0. \tag{2.18}$$

Recall the definition of the space $\mathcal{D}_{[0,1]}(\mathcal{X})$ of cadlag functions $f : [0, 1] \to \mathcal{X}$ in (2.2). A *cadlag parametrisation* of an element $K \in \mathcal{K}_{\rm tot}(\mathcal{X})$ is a function $\gamma \in \mathcal{D}_{[0,1]}(\mathcal{X})$ such that

$$K = \big\{\gamma(t), \gamma^-(t) : t \in [0, 1]\big\} \quad \text{and} \quad \gamma(s) \prec \gamma(t)\ \forall 0 \leq s < t \leq 1, \tag{2.19}$$

[1] Indeed, it is well-known that each separable metric space $\mathcal{X}$ is homeomorphic to a subset of a compact metric space $\mathcal{Y}$. The completion of $\mathcal{X}$ in the metric from $\mathcal{Y}$ is equal to the closure of $\mathcal{X}$ in $\mathcal{Y}$, so unless $\mathcal{X}$ is compact, it is not complete in the metric from $\mathcal{Y}$.

where γ^- denotes the caglad modification of γ. Clearly, not every element of $\mathcal{K}_{\rm tot}(\mathcal{X})$ has a cadlag representation. For those that do, the following proposition gives an expression for the metrics $d_{\rm H}$ and $d_{\rm tot}$ that will later help us make the connection between our definitions and the classical definitions of the J1 and M1 topologies. Let Λ be the space of all bijections $\lambda:[0,1]\to[0,1]$ and let Λ_+ be the subset consisting of all bijections λ that are monotone in the sense that $s\leq t$ implies $\lambda(s)\leq\lambda(t)$. Note that each $\lambda\in\Lambda_+$ is continuous and strictly increasing with $\lambda(0)=0$ and $\lambda(1)=1$.

Proposition 2.13 (Distance between cadlag curves) *Let (X,d) be a metric space, and assume that $K_1,K_2\in\mathcal{K}_{\rm tot}(\mathcal{X})$ have cadlag parametrisations γ_1,γ_2, respectively. Then*

$$\begin{aligned} d_{\rm H}(K_1,K_2) &= \inf_{\lambda\in\Lambda}\sup_{t\in[0,1]} d\big(\gamma_1(t),\gamma_2\big(\lambda(t)\big)\big),\\ d_{\rm tot}(K_1,K_2) &= \inf_{\lambda\in\Lambda_+}\sup_{t\in[0,1]} d\big(\gamma_1(t),\gamma_2\big(\lambda(t)\big)\big). \end{aligned} \tag{2.20}$$

2.5 Betweenness

There are great similarities between Skorohod's J1 and M1 topologies. In fact, it turns out to be possible to treat them in a unified framework. To this aim, we introduce a natural concept that we will call "betweenness" and that seems to be new in this context. It seems quite conceivable it may have been invented in other contexts before, but we have been unable to find a reference. If $\mathcal{X}$ is a set, then we define a *betweenness* on $\mathcal{X}$ to be a function that assigns to each pair x,z of elements of $\mathcal{X}$ a subset $\langle x,z\rangle$ of $\mathcal{X}$, such that the following axioms hold for all $x,y,z\in\mathcal{X}$:

(i) $\langle x,z\rangle=\langle z,x\rangle$,

(ii) $x\in\langle x,z\rangle$,

(iii) $y\in\langle x,z\rangle \ \Rightarrow\ \langle x,y\rangle\cap\langle y,z\rangle=\{y\}$,

(iv) $y\in\langle x,z\rangle \ \Rightarrow\ \langle x,y\rangle\cup\langle y,z\rangle=\langle x,z\rangle$.

If $y\in\langle x,z\rangle$, then we say that y lies *between* x and z. We call $\langle x,z\rangle$ the *segment* with *endpoints* x and z. The following lemma lists some elementary consequences of the axioms (i)–(iv).

Lemma 2.14 (Elementary properties) *Each betweenness satisfies, for each $x,y,y',z\in\mathcal{X}$:*

(v) $\langle x,x\rangle=\{x\}$,

(vi) $y\in\langle x,z\rangle \ \Rightarrow\ \langle x,y\rangle\subset\langle x,z\rangle$,

(vii) $x\in\langle y,z\rangle$ *and* $y\in\langle x,z\rangle \ \Rightarrow\ x=y$.

(viii) $y,y'\in\langle x,z\rangle$ *and* $y'\in\langle x,y\rangle \ \Rightarrow\ y\in\langle y',z\rangle$.

For $x,z\in\mathcal{X}$ and $y,y'\in\langle x,z\rangle$, one has

$$\langle x,y\rangle\subset\langle x,y'\rangle \ \Leftrightarrow\ y\in\langle x,y'\rangle \ \Leftrightarrow\ y'\in\langle y,z\rangle \ \Leftrightarrow\ \langle y,z\rangle\supset\langle y',z\rangle. \tag{2.21}$$

Setting $y\leq_{x,z}y'$ if any of these equivalent conditions holds defines a total order on $\langle x,z\rangle$.

We next give some examples of betweennesses. For any set $\mathcal{X}$, it is straightforward to check that setting $\langle x,z\rangle := \{x,z\}$ defines a betweenness. We call this the *trivial betweenness.* If $\mathcal{X}$ is a linear space, then it is easy to see that

$$\langle x,z\rangle := \big\{(1-p)x+pz : p\in[0,1]\big\} \qquad (x,z\in\mathcal{X}) \tag{2.22}$$

defines a betweenness on $\mathcal{X}$. We call this the *linear betweenness.* If $(\mathcal{X},\leq)$ is a totally ordered space, then one can check that setting

$$\langle x,z\rangle := \big\{y\in\mathcal{X} : x\leq y\leq z \text{ or } z\leq y\leq x\big\} \qquad (x,z\in\mathcal{X}) \tag{2.23}$$

defines a betweenness on $\mathcal{X}$. We call this the *order betweenness.* If $(\mathcal{X},d)$ is a metric space, then we recall that a *geodesic* in $(\mathcal{X},d)$ is a subset Γ of $\mathcal{X}$ that is isometric to a compact real interval, i.e., there exists an isometry $\gamma:[s,u]\to\mathcal{X}$ (with $s,u\in\mathbb{R},\ s\leq u$) such that Γ is the image of $[s,u]$ under γ. Clearly, γ is uniquely determined by Γ up to translations and mirror images of the interval $[s,u]$. The points $\gamma(s),\gamma(u)$ are called the *endpoints* of the geodesic Γ. We say that a metric space has *unique geodesics* if for each $x,z\in\mathcal{X}$, there exists a unique geodesic Γ with endpoints x,z. We call the betweenness defined in the following lemma the *geodesic betweennesss.*

Lemma 2.15 (Geodesic betweenness) *Let $(\mathcal{X},d)$ be a metric space with unique geodesics. Then letting $\langle x,z\rangle$ denote the unique geodesic with endpoints x,z defines a betweenness on $\mathcal{X}$.*

As an example of spaces without a linear structure where Lemma 2.15 is applicable we mention real-trees [DT96]. We say that a betweenness on a metrisable space $\mathcal{X}$ is *generated by an interpolation function* if there exists a continuous function $\varphi:\mathcal{X}^2\times[0,1]\to\mathcal{X}$ that satisfies $\varphi(x,z,0)=x$, $\varphi(x,z,1)=z$, and

$$\langle x,z\rangle = \big\{\varphi(x,z,p) : p\in[0,1]\big\} \qquad (x,z\in\mathcal{X}). \tag{2.24}$$

A metric space is called *proper* if for each $x\in\mathcal{X}$ and $r\geq 0$, the closed ball $\{y\in\mathcal{X} : d(x,y)\leq r\}$ is a compact subset of $\mathcal{X}$. We do not know if the properness assumption in the following lemma is needed, but it is certainly sufficient.

Lemma 2.16 (Interpolation functions) *If $\mathcal{X}$ is a normed linear space, then the linear betweenness is generated by an interpolation function. If $\mathcal{X}$ is a proper metric space with unique geodesics, then the same is true for the geodesic betweenness.*

If $\mathcal{X}$ is a metrisable space, then we say that a betweenness on $\mathcal{X}$ is *compatible with the topology* if $\langle x,z\rangle$ is compact for each $x,z\in\mathcal{X}$, and the map $\mathcal{X}^2\ni(x,z)\mapsto\langle x,z\rangle\in\mathcal{K}_+(\mathcal{X})$ is continuous with respect to the product topology on $\mathcal{X}^2$ and the Hausdorff topology on $\mathcal{K}_+(\mathcal{X})$.

Lemma 2.17 (Compatible betweennesses) *If $\mathcal{X}$ is a metrisable space, then the trivial betweenness is compatible with the topology. The same is true for any betweenness that is generated by an interpolation function. If $\mathcal{X}$ is a closed subset of $\mathbb{R}$, then the order betweenness on $\mathcal{X}$ is compatible with the topology.*

2.6 Squeezed space

We will need to view graphs of functions as compact sets. This will require us to compactify the real time axis by adding points at $\pm\infty$. At the same time, we want to equip space-time $\mathcal{X}\times\mathbb{R}$ with a metric that cares less about the spatial distance between two points if their time coordinates are very large. In the special case when $\mathcal{X} = \mathbb{R}$, such a topology has been introduced in [FINR04, formula (3.4)]. Here, we generalise this to $\mathcal{X}$ being any metrisable space.

Let $(\mathcal{X}, d)$ be a metric space and let $*$ be a point not contained in $\mathcal{X}$. Then we call

$$\mathcal{R}(\mathcal{X}) := (\mathcal{X} \times \mathbb{R}) \cup \big\{(*, -\infty), (*, \infty)\big\} \tag{2.25}$$

the *squeezed space* associated with $\mathcal{X}$. Let $d_{\overline{\mathbb{R}}}$ be a metric that generates the topology on the extended real line $\overline{\mathbb{R}}$ and let $\phi : \overline{\mathbb{R}} \to [0, \infty)$ be a continuous function such that $\phi(\pm\infty) = 0$ and $\phi(t) > 0$ for all $t \in \mathbb{R}$. We define $d_{\rm sqz} : \mathcal{R}(\mathcal{X})^2 \to [0, \infty)$ by

$$d_{\rm sqz}\big((x, s), (y, t)\big) := \big(\phi(s) \wedge \phi(t)\big)\big(d(x, y) \wedge 1\big) + \big|\phi(s) - \phi(t)\big| + d_{\overline{\mathbb{R}}}(s, t), \tag{2.26}$$

where naturally the first term is zero if (x, s) or (y, t) are elements of $\big\{(*, -\infty), (*, \infty)\big\}$ (even though $d(x, y)$ is not defined in this case).

Lemma 2.18 (Squeezed space) *Let $(\mathcal{X}, d)$ be a metric space. Then $d_{\rm sqz}$ is a metric on $\mathcal{R}(\mathcal{X})$. One has $d_{\rm sqz}\big((x_n, t_n), (x, t)\big) \to 0$ if and only if:*

(i) *$t_n \to t$,*

(ii) *if $t \in \mathbb{R}$, then $x_n \to x$.*

Usually, we will only be interested in $\mathcal{R}(\mathcal{X})$ as a topological space. The conditions (i) and (ii) show that the topology on $\mathcal{R}(\mathcal{X})$ depends only on the topology on $\mathcal{X}$ and not on the choice of the metric d on $\mathcal{X}$, the metric $d_{\overline{\mathbb{R}}}$ on $\overline{\mathbb{R}}$, and the function ϕ. Condition (ii) is trivially satisfied if $t_n \to -\infty$ or $\to +\infty$, i.e., we have that $(x_n, t_n) \to (*, \pm\infty)$ if and only if $t_n \to \pm\infty$, with no conditions on the sequence x_n. The squeezed space $\mathcal{R}(\overline{\mathbb{R}})$ plays an important role in the theory of the Brownian web, see [SSS17, Figure 6.2].

We need some elementary properties of squeezed space. The following lemma shows that $\mathcal{R}(\mathcal{X})$ is Polish if $\mathcal{X}$ is.

Lemma 2.19 (Preservation of Polishness)

(a) *If $(\mathcal{X}, d)$ is separable, then so is $(\mathcal{R}(\mathcal{X}), d_{\rm sqz})$.*

(b) *If $(\mathcal{X}, d)$ is complete, then so is $(\mathcal{R}(\mathcal{X}), d_{\rm sqz})$.*

The following lemma identifies the compact subsets of $\mathcal{R}(\mathcal{X})$. In particular, the lemma shows that $\mathcal{R}(\mathcal{X})$ is compact if $\mathcal{X}$ is compact.

Lemma 2.20 (Compactness criterion) *A set $A \subset \mathcal{R}(\mathcal{X})$ is precompact if and only if for each $T < \infty$, there exists a compact set $K \subset \mathcal{X}$ such that $\{x \in \mathcal{X} : \exists t \in [-T, T]$ s.t. $(x, t) \in A\} \subset K$.*

3 Topologies on path space

3.1 Path space

For any set $I \subset \mathbb{R}$, we let $I_\mathfrak{s}$ denote the subset of the split real line defined as $I_\mathfrak{s} := \big\{t-, t+ : t \in I\big\}$. Let $\mathcal{X}$ be a metrisable space. By definition, a *path* in $\mathcal{X}$ is an object that consists of two parts: a closed subset $I(\pi) \subset \mathbb{R}$ (possibly empty) and a continuous function $\pi : I_\mathfrak{s}(\pi) \to \mathcal{X}$. The path with $I(\pi) = \emptyset$ is called the *trivial path*. We usually denote a path simply by π, which includes both the function and its domain. We let $\Pi(\mathcal{X})$ denote the space of all paths in $\mathcal{X}$ and let

$$\Pi_{\rm c}(\mathcal{X}) := \big\{\pi \in \Pi(\mathcal{X}) : \pi(t-) = \pi(t+)\ \forall t \in I(\pi)\big\} \tag{3.1}$$

denote the subspace consisting of paths without jumps. For $\pi \in \Pi_c(\mathcal{X})$ and $t \in I(\pi)$ we simply write $\pi(t) := \pi(t-) = \pi(t+)$. Then $\pi : I(\pi) \to \mathcal{X}$ is a continuous function. For this reason, we call $\Pi_c(\mathcal{X})$ the space of *continuous paths*, even though using the split real line, we can also view paths with jumps as continuous functions. We call

$$\sigma_\pi := \inf I(\pi) \quad \text{and} \quad \tau_\pi := \sup I(\pi) \tag{3.2}$$

the *starting time* and *final time* of a path π, respectively. By convention, $\sigma_\pi = \infty$ and $\tau_\pi = -\infty$ for the trivial path π. We let

$$\begin{aligned} \Pi^{|}(\mathcal{X}) &:= \big\{\pi \in \Pi(\mathcal{X}) : t \in I(\pi)\ \forall s, u \in I(\pi) \text{ and } t \in \mathbb{R} \text{ s.t. } s < t < u\big\}, \\ \Pi^{\uparrow}(\mathcal{X}) &:= \big\{\pi \in \Pi(\mathcal{X}) : t \in I(\pi)\ \forall s \in I(\pi) \text{ and } t \in \mathbb{R} \text{ s.t. } s < t\big\}, \\ \Pi^{\downarrow}(\mathcal{X}) &:= \big\{\pi \in \Pi(\mathcal{X}) : t \in I(\pi)\ \forall u \in I(\pi) \text{ and } t \in \mathbb{R} \text{ s.t. } t < u\big\} \end{aligned} \tag{3.3}$$

denote the sets of paths π for which $I(\pi)$ is an interval, an interval that is unbounded from above, and an interval that is unbounded from below, respectively. Note that all these sets contain the trivial path. We also set

$$\Pi^{\updownarrow}(\mathcal{X}) := \big\{\pi \in \Pi(\mathcal{X}) : I(\pi) = \mathbb{R}\big\}, \tag{3.4}$$

which is $\Pi^{\uparrow}(\mathcal{X}) \cap \Pi^{\downarrow}(\mathcal{X})$ minus the trivial path. We write $\Pi^{|}_c(\mathcal{X}) := \Pi^{|}(\mathcal{X}) \cap \Pi_c$ etc.

By definition, the *closed graph* of a path $\pi \in \Pi(\mathcal{X})$ is the set $\mathcal{G}(\pi) \subset \mathcal{R}(\mathcal{X})$ defined as

$$\mathcal{G}(\pi) := \big\{(x,t) : t \in I(\pi),\ x \in \{\pi(t-), \pi(t+)\}\big\} \cup \big\{(*, -\infty), (*, +\infty)\big\}. \tag{3.5}$$

Note that $\mathcal{G}(\pi)$ is nonempty, since we always add the points $(*, \pm\infty)$, even for the trivial path. If $\mathcal{X}$ is equipped with a betweenness (see Subsection 2.5), then we define the *filled-in graph*[2] of a path $\pi \in \Pi(\mathcal{X})$ as

$$\mathcal{G}_f(\pi) := \big\{(x,t) : t \in I(\pi),\ x \in \langle \pi(t-), \pi(t+)\rangle\big\} \cup \big\{(*, -\infty), (*, +\infty)\big\}. \tag{3.6}$$

Note that for the trivial betweenness, the filled-in and closed graphs coincide. This will allow us to treat Skorohod's J1 and M1 topologies in a unified framework. The filled-in graph $\mathcal{G}_f(\pi)$ is naturally equipped with a total order, which is defined by setting $(x,s) \preceq (y,t)$ if either $s < t$ and x, y are arbitrary, or $s = t$ and $x \leq_{\pi(t-),\pi(t+)} y$, where $\leq_{\pi(t-),\pi(t+)}$ is the total order on the segment $\langle \pi(t-), \pi(t+)\rangle$ defined in Lemma 2.14. Informally, the total order $\preceq$ corresponds to the direction of time. Recall the definitions of $\mathcal{K}_{\rm tot}$ and $\mathcal{R}(\mathcal{X})$ from Subsections 2.4 and 2.6. The following lemma says that we can view $\mathcal{G}_f(\pi)$ as an element of the space $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$.

Lemma 3.1 (Filled-in graphs) *Assume that $\mathcal{X}$ is a metrisable space that is equipped with a betweenness that is compatible with the topology. Then for any path π, the filled-in graph $\mathcal{G}_f(\pi)$ is a compact subset of the squeezed space $\mathcal{R}(\mathcal{X})$, and the total order $\preceq$ is compatible with the induced topology on $\mathcal{G}_f(\pi)$.*

A path π is uniquely determined by the totally ordered compact set $\big(\mathcal{G}_f(\pi), \preceq\big)$. If $\pi \in \Pi_c(\mathcal{X})$ or if $I(\pi)$ does not contain any isolated points, then π is uniquely determined by $\mathcal{G}_f(\pi)$ *as a set*, but if t is an isolated point of $I(\pi)$ and $\pi(t-) \neq \pi(t+)$, then one needs the order $\preceq$ to find out which of the two endpoints of the segment $\langle \pi(t-), \pi(t+)\rangle$ is $\pi(t-)$ and which is $\pi(t+)$. The following lemma gives another indication why it may be useful to view filled-in graphs as totally ordered sets.

[2]Except for the points at infinity, this is what Whitt [Whi02] calls the *completed graph*.

Lemma 3.2 (Characterisation of graphs) *Let $\mathcal{X}$ be a metrisable space that is equipped with a betweennesss that is compatible with the topology. Assume that $(G, \preceq) \in \mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ and $(*, \pm\infty) \in G$. Then $(G, \preceq)$ is the filled-in graph of a path $\pi \in \Pi(\mathcal{X})$ if and only if the following conditions are satisfied.*

(i) *For each $t \in \mathbb{R}$ and $(x_1, t), (x_2, t), (x_3, t) \in G$ with $(x_1, t) \preceq (x_2, t) \preceq (x_3, t)$, one has $x_2 \in \langle x_1, x_2 \rangle$.*

(ii) *$(x, s) \preceq (y, t)$ for all $(x, s), (y, t) \in G$ such that $s < t$.*

Applying Lemma 3.2 to the trivial betweenness, it is easy to see that a nonempty compact set $G \subset \mathcal{R}(\mathcal{X})$ with $(*, \pm\infty) \in G$ is the closed graph of a path if and only if it is possible to equip G with a total order that is compatible with the topology, such that it satisfies (ii) above and

(i)' For each $t \in \mathbb{R}$, the set $\{x \in \mathcal{X} : (x, t) \in G\}$ has at most two elements.

The total order $\preceq$ is essential here. To see this, assume that $x, y \in \mathcal{X}$ satisfy $x \neq y$, and let $G := \{x\} \times [-1, 1] \cup \{(y, 0)\}$. Then G is not the closed graph of a path $\pi \in \Pi(\mathcal{X})$, while it satisfies condition (i)'. However, it is not possible to equip G with a total order $\preceq$ as in (ii) so that $\preceq$ is compatible with the topology.

3.2 Metrics on path space

Let $(\mathcal{X}, d)$ be a metric space that is equipped with a betweennesss that is compatible with the topology and let $\Pi(\mathcal{X})$ be the path space defined in Section 3.1. We view the filled-in graph $\mathcal{G}_{\rm f}(\pi)$ of a path π as an element of the space $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ of totally ordered compact subsets of the squeezed space $\mathcal{R}(\mathcal{X})$ defined in Section 2.6. Let $d_{\rm sqz}$ be any metric that generates the topology on $\mathcal{R}(\mathcal{X})$, and let $d_{\rm part}$ and $d_{\rm tot}$ be the metrics on $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ defined in terms of $d_{\rm sqz}$ as in (2.13) and (2.15). Since a path is uniquely characterised by its filled-in graph (viewed as an element of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$), setting

$$d^{\rm S}_{\rm part}(\pi_1, \pi_2) := d_{\rm part}\big(\mathcal{G}_{\rm f}(\pi_1), \mathcal{G}_{\rm f}(\pi_2)\big) \quad \text{and} \quad d^{\rm S}_{\rm tot}(\pi_1, \pi_2) := d_{\rm tot}\big(\mathcal{G}_{\rm f}(\pi_1), \mathcal{G}_{\rm f}(\pi_2)\big) \tag{3.7}$$

$\big(\pi_1, \pi_2 \in \Pi(\mathcal{X})\big)$ defines two metrics on the path space $\Pi(\mathcal{X})$, that in view of Theorem 2.10 generate the same topology. Letting $d_{\rm H}$ denote the Hausdorff metric on $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ associated with the metric $d_{\rm sqz}$ on $\mathcal{R}(\mathcal{X})$, we moreover define a pseudometric on $\Pi(\mathcal{X})$ by setting

$$d^{\rm H}(\pi_1, \pi_2) := d_{\rm H}\big(\mathcal{G}_{\rm f}(\pi_1), \mathcal{G}_{\rm f}(\pi_2)\big). \tag{3.8}$$

Restricted to the space $\Pi_{\rm c}(\mathcal{X})$ of continuous paths, this is a metric.

We call the topology on the path space $\Pi(\mathcal{X})$ generated by the metrics $d^{\rm S}_{\rm part}$ and $d^{\rm S}_{\rm tot}$ the *Skorohod topology* associated with the given betweenness. In particular, we define the *J1 topology* to be the Skorohod topology associated with the trivial betweenness. In the special case when $\mathcal{X} = \mathbb{R}$, we define the *M1 topology* to be the Skorohod topology associated with the linear betweenness. More generally, Skorohod topologies associated with a betweenness that is generated by an interpolation function may naturally be viewed as generalisations of the classical M1 topology. In view of Theorem 2.10 and Lemma 2.18, the definition of a Skorohod topology only depends on the topology on $\mathcal{X}$ and on the choice of the betweenness, and not on the precise choice of the metrics on $\mathcal{X}$ and $\mathcal{R}(\mathcal{X})$. The following proposition says that Skorohod topologies are Polish.

Proposition 3.3 (Skorohod topologies are Polish) *If $\mathcal{X}$ is a Polish space, then so is $\Pi(\mathcal{X})$, equipped with the Skorohod topology, for any choice of the betweennesss that is compatible with the topology.*

Recall from (3.1) that $\Pi_{\rm c}(\mathcal{X})$ denotes the space of continuous paths and that $d^{\rm H}$ is a metric on $\Pi_{\rm c}(\mathcal{X})$. Since paths in $\Pi_{\rm c}(\mathcal{X})$ make no jumps, the definitions of $d^{\rm S}_{\rm part}$, $d^{\rm S}_{\rm tot}$, and $d^{\rm H}$ restricted to $\Pi_{\rm c}(\mathcal{X})$ do not depend on the choice of the betweenness. As a result of the following proposition, all these metrics generate the same topology on $\Pi_{\rm c}(\mathcal{X})$, so we simply call the resulting topology *the topology on* $\Pi_{\rm c}(\mathcal{X})$. The space $\Pi^{\uparrow}_{\rm c}(\overline{\mathbb{R}})$ of half-infinite paths with values in $\overline{\mathbb{R}}$, equipped with the topology we have just defined, plays an important role in the theory of the Brownian web and net [SSS14, Subsection 6.2.1].

Proposition 3.4 (Space of continuous paths) *For paths $\pi_n \in \Pi(\mathcal{X})$ and $\pi \in \Pi_{\rm c}(\mathcal{X})$, the following statements are equivalent:*

$$\text{(i) } d^{\rm S}_{\rm part}(\pi_n,\pi) \underset{n\to\infty}{\longrightarrow} 0, \quad \text{(ii) } d^{\rm S}_{\rm tot}(\pi_n,\pi) \underset{n\to\infty}{\longrightarrow} 0, \quad \text{(iii) } d^{\rm H}(\pi_n,\pi) \underset{n\to\infty}{\longrightarrow} 0.$$

In particular, these metrics all generate the same topology on $\Pi_{\rm c}(\mathcal{X})$. If $\mathcal{X}$ is a Polish space, then so is $\Pi_{\rm c}(\mathcal{X})$, equipped with this topology.

The final statement of the following lemma reveals a special property of the J1 topology that does not hold for general Skorohod topologies. We note that it is not hard to check that $\Pi_{\rm c}(\mathcal{X})$, contrary to $\Pi^{|}_{\rm c}(\mathcal{X})$, is in general not closed in the J1 topology.

Lemma 3.5 (Closed subspaces) *Let $\mathcal{X}$ be a metrisable space that is equipped with a betweenness that is compatible with the topology. Then $\Pi^{|}(\mathcal{X}), \Pi^{\uparrow}(\mathcal{X})$, and $\Pi^{\downarrow}(\mathcal{X})$ are closed subsets of $\Pi(\mathcal{X})$, equipped with the Skorohod topology. If the betweenness is the trivial betweenness, then also $\Pi^{|}_{\rm c}(\mathcal{X})$ is a closed subset of $\Pi(\mathcal{X})$.*

3.3 Compactness criteria

In this subsection, we give criteria for compactness in the spaces $\Pi(\mathcal{X})$ and $\Pi_{\rm c}(\mathcal{X})$. These criteria are similar to well-known results for spaces of functions defined on a fixed domain. Let $(\mathcal{X}, d)$ be a metric space. We say that a set $\mathcal{A} \subset \Pi(\mathcal{X})$ satisfies the *compact containment condition* if

$$\forall T < \infty\ \exists \text{ compact } C \subset \mathcal{X} \text{ s.t. } \pi(t\pm) \in C\ \forall \pi \in \mathcal{A} \text{ and } t \in I(\pi) \cap [-T, T]. \tag{3.9}$$

For each $0 < T < \infty$ and $\delta > 0$, we define the (traditional) *modulus of continuity* of a path $\pi \in \Pi_{\rm c}(\mathcal{X})$ as

$$m_{T,\delta}(\pi) := \sup\big\{ d\big(\pi(t_1), \pi(t_2)\big) : t_1, t_2 \in I(\pi),\ -T \le t_1 < t_2 \le T,\ t_2 - t_1 \le \delta \big\}. \tag{3.10}$$

We say that a set $\mathcal{A} \subset \Pi_{\rm c}(\mathcal{X})$ is *equicontinuous* if

$$\lim_{\delta\to 0} \sup_{\pi\in\mathcal{A}} m_{T,\delta}(\pi) = 0 \qquad \forall T < \infty. \tag{3.11}$$

The following theorem generalises the classical Arzela-Ascoli theorem to sets of functions that are not necessarily all defined on the same domain, which moreover does not need to be an interval.

Theorem 3.6 (Arzela-Ascoli) *Let $\mathcal{X}$ be a metric space. Then a set $\mathcal{A} \subset \Pi_{\rm c}(\mathcal{X})$ is precompact if and only if it is equicontinuous and satisfies the compact containment condition.*

For paths with jumps, it is possible to give a very similar compactness criterion. Assume that $\mathcal{X}$ is equipped with a betweenness that is compatible with the topology. For each $0 < T < \infty$ and $\delta > 0$, we define the *Skorohod modulus of continuity* as

$$\begin{aligned} m^{\rm S}_{T,\delta}(\pi) := \sup\big\{d\big(\pi(\tau_2),\langle\pi(\tau_1),\pi(\tau_3)\rangle\big) : \tau_1,\tau_2,\tau_3 \in I_{\mathfrak{s}}(\pi),\ \tau_1 \leq \tau_2 \leq \tau_2, \\ -T \leq \underline{\tau}_1,\ \underline{\tau}_3 \leq T,\ \underline{\tau}_3 - \underline{\tau}_1 \leq \delta\big\}, \end{aligned} \tag{3.12}$$

where as before $d(x,A)$ denotes the distance of a point x to a set A and $\langle x,y\rangle$ is the segment with endpoints x and y. We say that a set $\mathcal{A} \subset \Pi(\mathcal{X})$ is *Skorohod-equicontinuous* if

$$\lim_{\delta\to 0}\sup_{\pi\in\mathcal{A}} m^{\rm S}_{T,\delta}(\pi) = 0 \qquad \forall T < \infty. \tag{3.13}$$

Specialising these definitions to the trivial betweenness, for which $\langle\pi(\tau_1),\pi(\tau_3)\rangle = \{\pi(\tau_1),\pi(\tau_3)\}$, yields the definitions of the *J1-modulus of continuity* and *J1-equicontinuity*. If $\mathcal{X} = \mathbb{R}$, equipped with the linear betweenness, then we speak of the *M1-modulus of continuity* and *M1-equicontinuity*.

Theorem 3.7 (Compactness criterion) *Let $(\mathcal{X},d)$ be a metric space that is equipped with a betweenness that is compatible with the topology. Then a set $\mathcal{A} \subset \Pi(\mathcal{X})$ is precompact in the Skorohod topology if and only if it is Skorohod-equicontinuous and satisfies the compact containment condition.*

3.4 Paths on fixed domains

Let $\mathcal{X}$ be a metrisable space that is equipped with a betweenness that is compatible with the topology. Let I be a closed real interval of positive length, let $\mathrm{int}(I)$ denote its interior and let $\partial I := I\backslash\mathrm{int}(I)$ denote its boundary. Let $\mathcal{D}_I(\mathcal{X})$ be the set of cadlag functions $f : I \to \mathcal{X}$ defined in (2.2). We have seen in Lemma 2.4 that we may identify $\mathcal{D}_I(\mathcal{X})$ with the set of paths

$$\big\{\pi \in \Pi(\mathcal{X}) : I(\pi) = I,\ \pi(t-) = \pi(t+) \text{ if } t \in \partial I\big\}. \tag{3.14}$$

In this identification, $d^{\rm S}_{\rm part}, d^{\rm S}_{\rm tot}$, and $d^{\rm H}$ are metrics[3] on $\mathcal{D}_I(\mathcal{X})$. We have already seen that $d^{\rm S}_{\rm part}$ and $d^{\rm S}_{\rm tot}$ generate the same topology on the larger space $\Pi(\mathcal{X})$ and hence the same is true on $\mathcal{D}_I(\mathcal{X})$. If $\mathcal{X}$ is equipped with the trivial betweenness, then we call the topology on $\mathcal{D}_I(\mathcal{X})$ generated by the metrics $d^{\rm S}_{\rm part}$ and $d^{\rm S}_{\rm tot}$ the *J1 topology*, and we call the topology generated by $d^{\rm H}$ the *J2 topology*. If $\mathcal{X} = \mathbb{R}$, equipped with the linear betweenness, then we call these the *M1 topology* and *M2 topology*, respectively.

Skorohod [Sko56] only considered compact time intervals. It is easy to see that his definition of the M2 topology [Sko56, Def 2.2.6] coincides with our definition. For the J1, J2, and M1 topologies, the equivalence of [Sko56, Defs 2.2.2, 2.2.3, and 2.2.4] with our definitions follows from Proposition 2.13. It is interesting to note that all previous treatments of the Skorohod topology seem to have been based on variants of the metric $d^{\rm S}_{\rm tot}$, while te fact that the metric $d^{\rm S}_{\rm part}$ generates the same topology seems to have been overlooked.

Skorohod [Sko56] did not consider unbounded time intervals but other authors such as [EK86, Whi02] have done so. To see that our definitions agree with their definitions, one can use the following simple lemma. We note that for the Skorohod topologies, the restriction map that restricts a function to a smaller time interval is in general not a continuous map, which is why in (3.15) we have to restrict ourselves to continuity points of the limit function.

[3]For the metric $d^{\rm H}$, the assumption that I has positive length is essential, since otherwise this would only be a pseudometric.

Lemma 3.8 (Convergence of restricted functions) *Let $(\mathcal{X},d)$ be a metric space that is equipped with a betweenness that is compatible with the topology. Let $g\big|_{[0,t]}$ denote the restriction of a function g to the interval $[0,t]$. Then for all $f_n, f \in \mathcal{D}_{[0,\infty)}(\mathcal{X})$, one has*

$$
\begin{aligned}
d^{\mathrm{H}}(f_n,f) \underset{n\to\infty}{\longrightarrow} 0 \quad &\Leftrightarrow \quad d^{\mathrm{H}}\big(f_n\big|_{[0,t]}, f\big|_{[0,t]}\big) \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall t>0 \text{ s.t. } f(t-)=f(t),\\
d^{\mathrm{S}}_{\mathrm{tot}}(f_n,f) \underset{n\to\infty}{\longrightarrow} 0 \quad &\Leftrightarrow \quad d^{\mathrm{S}}_{\mathrm{tot}}\big(f_n\big|_{[0,t]}, f\big|_{[0,t]}\big) \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall t>0 \text{ s.t. } f(t-)=f(t).
\end{aligned}
\tag{3.15}
$$

It is not hard to see that for $f \in \mathcal{D}_I(\mathcal{X})$, the Skorohod modulus of continuity defined in (3.12) can alternatively be written as

$$
\begin{aligned}
m^{\mathrm{S}}_{T,\delta}(f) = \sup\big\{ d\big(f(t_2), \langle f(t_1), f(t_3)\rangle\big) : t_1,t_2,t_3 \in I \\
-T \le t_1 < t_2 < t_3 \le T,\ t_3 - t_1 \le \delta \big\}.
\end{aligned}
\tag{3.16}
$$

Moreover, a set $\mathcal{F} \subset \mathcal{D}_I(\mathcal{X})$ satisfies the compact containment condition if and only if

$$
\forall T<\infty\ \exists \text{ compact } C \subset \mathcal{X} \text{ s.t. } f(t) \in C\ \forall f \in \mathcal{F} \text{ and } t \in I. \tag{3.17}
$$

In other words, these last two formulas say that in (3.9) and (3.12), it suffices to consider $\pi(t+)$ only. As a straightforward application of Theorem 3.7, we obtain the following.

Theorem 3.9 (Compactness criterion) *Let $(\mathcal{X},d)$ be a metric space that is equipped with a betweenness that is compatible with the topology and let I be a closed real interval of positive length. Then a set $\mathcal{F} \subset \mathcal{D}_I(\mathcal{X})$ is precompact in the Skorohod topology if and only if:*

(i) *the compact containment condition holds,*

(ii) $\lim_{\delta\to 0} \sup_{f\in\mathcal{F}} m^{\mathrm{S}}_{T,\delta}(f) = 0$ *for all $T<\infty$,*

(iii) $\lim_{\delta\to 0} \sup_{f\in\mathcal{F}} \sup\big\{ d\big(f(s), f(t)\big) : s \in I,\ |s-t| \le \delta \big\} = 0$ *for all $t \in \partial I$.*

Note that compared to Theorem 3.7, we need the extra condition (iii) to guarantee that a sequence of functions in $\mathcal{F}$ cannot converge to a function with a discontinuity at a time $t \in \partial I$. If in (3.14) we drop the condition that $\pi(t-) = \pi(t+)$ for $t \in \partial I$, then condition (iii) of Theorem 3.9 can be dropped. For the J1 topology on $\mathcal{D}_{[0,1]}$, Theorem 3.9 was first proved by Kolmogorov in [Kol56, Thm IV]. The analogue statement for the M1 topology was proved by Skorohod in [Sko56, 2.7.3].

3.5 Interpolation

It often happens that a sequence of functions that are defined on a countable subset of $\mathbb{R}$ converge to a limit that is defined on a subinterval of $\mathbb{R}$. In such situations, to formulate what convergence means, it is common practise to interpolate the approximating functions, so that all functions are defined on the same domain. With the use of path space, one can compare functions that are defined on different domains. In the present subsection, we show that in such situations, for the J1 topology, there is no need to interpolate. For the M1 topology, on the other hand, it still makes sense to interpolate.

Let $\hat{I}$ denote the convex hull of a closed set $I \subset \mathbb{R}$. Fix $\pi \in \Pi_{\mathrm{c}}(\mathcal{X})$ and for each $t \in \hat{I}(\pi)\backslash I(\pi)$, let

$$
t_{\mathrm{l}} := \sup\{s \in I : s < t\} \quad \text{and} \quad t_{\mathrm{r}} := \inf\{s \in I : s > t\}, \tag{3.18}
$$

where the subscripts l and r stand for "left" and "right". Then we can uniquely define *interpolated paths* $\pi^{\rm l} \in \mathcal{D}_{\hat{I}(\pi)}(\mathcal{X})$ and $\pi^{\rm r} \in \mathcal{D}^-_{\hat{I}(\pi)}(\mathcal{X})$ by

$$\pi^{\rm l}(t+) := \begin{cases} \pi(t) & \text{if } t \in I(\pi), \\ \pi(t_{\rm l}) & \text{if } t \in \hat{I}(\pi)\backslash I(\pi), \end{cases} \qquad \pi^{\rm r}(t-) := \begin{cases} \pi(t) & \text{if } t \in I(\pi), \\ \pi(t_{\rm r}) & \text{if } t \in \hat{I}(\pi)\backslash I(\pi). \end{cases} \tag{3.19}$$

As in (3.14), we can identify $\mathcal{D}_{\hat{I}(\pi)}(\mathcal{X})$ and $\mathcal{D}^-_{\hat{I}(\pi)}(\mathcal{X})$ with subsets of $\Pi(\mathcal{X})$ and hence view $\pi^{\rm l}$ and $\pi^{\rm r}$ as paths. Let the metric on the squeezed space $\mathcal{R}(\mathcal{X})$ be defined as in (2.26) in terms of a metric $d_{\overline{\mathbb{R}}}$ generating the topology on $\overline{\mathbb{R}}$ and a function ϕ. By setting up a monotone correspondence, it is easy to see that for the J1 topology

$$d^{\rm S}_{\rm tot}(\pi, \pi^{\rm l}) \leq \varepsilon_{\rm l}(\pi) := \sup_{t \in \hat{I}(\pi)\backslash I(\pi)} \big[d_{\overline{\mathbb{R}}}(t, t_{\rm l}) + |\phi(t) - \phi(t_{\rm l})|\big], \tag{3.20}$$

and similarly for $\pi^{\rm r}$ (with a similar $\varepsilon_{\rm r}(\pi)$). In particular, when $\pi_n \in \Pi_{\rm c}(\mathcal{X})$ and $\pi \in \Pi^|(\mathcal{X})$ satisfy $\varepsilon_{\rm l}(\pi_n) \to 0$, then with respect to the J1 topology one has $\pi_n \to \pi$ if and only if $\pi^{\rm l}_n \to \pi$. In other words, no interpolation is needed. Indeed, convergence of the uninterpolated paths $\pi_n \to \pi$ gives more information since this also implies $\varepsilon_{\rm l}(\pi_n) \to 0$.

We next consider Skorohod topologies associated with a betweenness that is generated by an interpolation function φ. In this case, for any $\pi \in \Pi_{\rm c}(\mathcal{X})$, we can define a continuously interpolated path π^φ by $I(\pi^\phi) := \hat{I}(\pi)$ and

$$\pi^\varphi(t+) := \begin{cases} \pi(t) & \text{if } t \in I(\pi), \\ \varphi\big(\pi(t_{\rm l}), \pi(t_{\rm r}), p(t)\big) & \text{if } t \in \hat{I}(\pi)\backslash I(\pi), \end{cases} \qquad \text{where} \quad p(t) := \frac{t - t_{\rm l}}{t_{\rm r} - t_{\rm l}}. \tag{3.21}$$

By setting up a monotone correspondence, it is easy to see that

$$d^{\rm S}_{\rm tot}(\pi^\varphi, \pi^{\rm l}), d^{\rm S}_{\rm tot}(\pi^\varphi, \pi^{\rm r}) \leq \varepsilon(\pi) := \varepsilon_{\rm l}(\pi) \vee \varepsilon_{\rm r}(\pi). \tag{3.22}$$

Thus, when $\pi_n \in \Pi_{\rm c}(\mathcal{X})$ and $\pi \in \Pi^|(\mathcal{X})$ satisfy $\varepsilon(\pi_n) \to 0$, then with respect to the M1 topology, the conditions $\pi^\varphi_n \to \pi$, $\pi^{\rm l}_n \to \pi$, and $\pi^{\rm r}_n \to \pi$ are all equivalent. In other words, for the M1 topology, it makes sense to interpolate, but it does not matter if we interpolate in a continuous way or in a piecewise constant manner.

3.6 Open problems

By definition, a *laglad* function (from the French "limite à gauche, limite à droit") is a function that has both left- and right- limits in each point, but whose value in a point does not need to be equal to either the left or right limit at that point. Whitt [Whi02, Chapter 15] introduces a topology on spaces of laglad functions. It seems it should be possible to develop the theory of laglad functions very much in parallel to the theory of cadlag functions, except that instead of splitting each point of the real line into two points, as we did in the split real line, one now needs a topological space where each point of the real line is replaced by three points. A slight complication is that the closed graph of a laglad function cannot always be equipped with a total order that is compatible with the topology, as pointed out below Lemma 3.2. However, it seems likely this can be overcome and it should be possible to prove a compactnes criterion similar to Theorem 3.7, this time involving a modulus of continuity that compares the function values at four consecutive times, rather than three as for the Skorohod modulus of continuity.

Let $\mathcal{X}$ be a metrisable space. Recall from Subsection 2.4 that $\mathcal{K}_{\rm part}(\mathcal{X})$ denotes the space of all compact subsets $K \subset \mathcal{X}$ that are equipped with a partial order $\preceq$ that is compatible with the (induced) topology on K. For each finite partially ordered set $(S, \leq)$, let K^S denote the space of all monotone functions $f : S \to K$, i.e., functions such that $i \leq j$ implies $f(i) \preceq f(j)$. Then $K^{\langle 2 \rangle}$, defined in (2.10), is the same as K^S where S is the totally ordered space $\{1, 2\}$. In Subsection 4.3 below, we more generally define $K^{\langle m \rangle} := K^S$ with S the totally ordeed set $\{1, \ldots, m\}$. For each partially ordered set $(S, \leq)$, similar to (2.13), we can define a pseudometric d^S by

$$d^S(K_1, K_2) := d_{\mathrm{H}}(K_1^S, K_2^S), \tag{3.23}$$

where d_{H} is the Hausdorff metric on the product space $\mathcal{X}^S$, equipped with a product metric as in (2.11). In Subsection 2.4, we show that the metrics d^S with S a finite totally ordered set with at least two elements all generate the same topology on the space $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ of totally ordered compact sets. It seems this result does not generalise to partially ordered sets. A natural idea is therefore to equip the larger space $\mathcal{K}_{\mathrm{part}}(\mathcal{X})$ with a topology such that $K_n \to K$ in the topology on $\mathcal{K}_{\mathrm{part}}(\mathcal{X})$ if and only if $d^S(K_n, K) \to 0$ for every finite partially ordered set. Such a topology can be generated by a metric, for example by setting $d(K_1, K_2) := \sum_S r_S d^S(K_1, K_2)$ where r_S are positive weights such that the sum over all partially ordered finite sets $\sum_S r_S$ is finite. It then seems interesting to study the associated "ordered" Gromov-Hausdorff distance between two partially ordered sets K_1, K_2, which is the infimum of $d(K_1, K_2)$ over all isometric embeddings of K_1 and K_2 into a common metric space $\mathcal{X}$.

As a final, minor open problem, we ask whether the properness assumption in Lemma 2.16 can be removed. This may be known; we have just not managed to find this in the literature.

3.7 Outline of the proofs

The results from Section 2 are proved in Section 4 and the from Section 3 are proved in Section 5. More precisely, Lemmas 2.1 and 2.2, Proposition 2.3, and Lemma 2.4 are proved in Subsection 4.1. Lemmas 2.5 and 2.9 are proved in Subsection 4.2. We cited Lemmas 2.6–2.8 from [SSS14, Appendix B], so these don't need proofs. Theorem 2.10 is proved in Subsection 4.4. Proposition 2.11 is proved in Subsection 4.5. Theorem 2.12 is proved in Subsection 4.6. Proposition 2.13 is proved in Subsection 4.7. Lemmas 2.14, 2.15, 2.16, and 2.17 are proved in Subsection 4.8.Lemmas 2.18, 2.19, and 2.20 are proved in Subsection 4.9.

Lemmas 3.1 and 3.2 are proved in Subsection 5.1. Propositions 3.3 and 3.4 are proved in Subsection 5.2, which also contains the proof of Lemma 3.5. Theorems 3.6 and 3.7 are proved in Subsection 5.3. Lemma 3.8 and Theorem 3.9 are proved in Subsection 5.4.

4 Proofs of the preliminary results

4.1 The split real line

In this subsection, we prove Lemmas 2.1 and 2.2, Proposition 2.3, and Lemma 2.4, as well as one more lemma that will be needed in what follows. We recall some basic definitions from topology. A *topology* on a set $\mathcal{X}$ is a collection $\mathcal{O}$ of subsets of $\mathcal{X}$ that are called *open* and that have the properties that $\emptyset, \mathcal{X} \in \mathcal{O}$ and $\mathcal{O}$ is closed under finite intersections and arbitrary unions. If $\mathcal{Y}$ is a subset of $\mathcal{X}$, then the *induced topology* is defined as $\{O \cap \mathcal{Y} : O \in \mathcal{O}\}$. A *basis* for the topology on $\mathcal{X}$ is a subset $\mathcal{O}' \subset \mathcal{O}$ such that each element of $\mathcal{O}$ can be written as the union of elements of $\mathcal{O}'$. The set $\mathcal{V}_x$ of *neighbourhoods* of a point $x \in \mathcal{X}$ is $\mathcal{V}_x := \{V \subset \mathcal{X} : x \in O \subset V$ for some $O \in \mathcal{O}\}$. A *fundamental system of neighbourhoods* is a set $\mathcal{V}'_x \subset \mathcal{V}_x$ such that $\forall V \in \mathcal{V}_x \ \exists V' \in \mathcal{V}'_x$ s.t. $V' \subset V$.

A *Hausdorff* topology is a topology that has the Hausdorff property, i.e., for all $x_1, x_2 \in \mathcal{X}$ with $x_1 \neq x_2$ there exist disjoint $O_1, O_2 \in \mathcal{O}$ such that $x_1 \in O_1$, $x_2 \in O_2$. A topology is *first countable* if each point has a countable fundamental system of neighbourhoods and *second countable* if there exists a countable basis for the topology. A sequence converges to a limit, denoted $x_n \to x$, if for each $V \in \mathcal{V}_x$, there exists an m such that $x_n \in V$ for all $n \geq m$. It suffices to check this for a fundamental system of neighbourhoods. In a Hausdorff space, a sequence can have at most one limit.

A set is *closed* if its complement is open and *sequentially closed* if it contains the limits of all convergent sequences that lie inside it; in first countable spaces, the concepts are equivalent. The *closure* of a set is the smallest closed set that contains it and a *dense* set is a set whose closure is the whole space. A topological space $\mathcal{X}$ is *separable* if it contains a countable dense set and *connected* if $\emptyset, \mathcal{X}$ are the only sets that are both open and closed. A set $C \subset \mathcal{X}$ is *compact* if each covering with open sets has a finite subcover and *sequentially compact* if each sequence in C has a subsequence that converges to a limit in C; in second countable spaces, the concepts are equivalent. A metric defines a topology in the usual way; a topology that is generated by a metric is called *metrisable*.

Proof of Lemma 2.1 By symmetry, it suffices to prove (i). By definition, a basis for the topology is formed by all intervals of the form (σ, ρ) with $\sigma, \rho \in \mathbb{R}_\mathfrak{s}$. If $t+ \in (\sigma, \rho)$, then $\sigma < t+ < \rho$ and hence $(t-, u+) \subset (\sigma, \rho)$ for some $u > t$. It follows that the sets of the form $(t-, u+) = [t+, u-]$ with $u \in \{t + n^{-1} : n \geq 1\}$ form a fundamental system of neighbourhoods of t, which is easily seen to imply the claim. ∎

Proof of Lemma 2.2 It is easy to see that $\mathbb{R}_\mathfrak{s}$ has the Hausdorff property. In the proof of Lemma 2.1, we have already seen that each point has a countable fundamental system of neighbourhoods, so $\mathbb{R}_\mathfrak{s}$ is first countable. On the other hand, each basis of the topology must for each $t \in \mathbb{R}$ contain an open set O such that $t \in O \subset (t-, (t+1)+) = [t+, (t+1)-]$. These open sets are all distinct, so $\mathbb{R}_\mathfrak{s}$ is not second countable. By Lemma 2.1, the set $\{t+ : t \in \mathbb{Q}\}$ is dense so $\mathbb{R}_\mathfrak{s}$ is separable. Since in metric spaces, separability implies second countability, we conclude that $\mathbb{R}_\mathfrak{s}$ is not metrisable. Since for each $t \in \mathbb{R}$, we can write $\mathbb{R}_\mathfrak{s}$ as the union of two disjoint closed sets as $\mathbb{R}_\mathfrak{s} = (-\infty, t-] \cup [t+, \infty)$, we see that $\mathbb{R}_\mathfrak{s}$ is totally disconnected. ∎

The next lemma prepares for the proof of Proposition 2.3. Even though $\mathbb{R}_\mathfrak{s}$ is not second countable, it has a property that is almost as good.

Lemma 4.1 (Strong Lindelöf property) *Every open cover of a subset of* $\mathbb{R}_\mathfrak{s}$ *has a countable subcover.*

Proof This is proved in [AU29], but since the latter reference is not readily available to everyone, including the present authors, we provide our own proof. The *Sorgenfrey line* is the set of real numbers equipped with the *lower limit topology* that is generated by intervals of the form $[a, b)$ [SS95]. Similarly, the *upper limit topology* on $\mathbb{R}$ is generated by intervals of the form $(a, b]$. One can check that the topology on $\mathbb{R}_\mathfrak{s}$ induces the lower limit topology on its subspace $\{t+ : t \in \mathbb{R}\}$ and the upper limit topology on its subspace $\{t- : t \in \mathbb{R}\}$. In view of this, it suffices to prove that the Sorgenfrey line has the strong Lindelöf property. This is well-known [Sor47] but for completeness we provide a proof. Since sets of the form $[a, b)$ form a basis for the topology, it suffices to prove that if $A \subset \mathbb{R}$ and $\mathcal{I}$ is a collection of intervals of the form $[a, b)$ that covers A, then a countable subset of $\mathcal{I}$ covers A.

Fix $n \geq 1$, let $\mathcal{I}'$ denote the subset of $\mathcal{I}$ consisting of all intervals $[a, b) \in \mathcal{I}$ with $b - a \geq 2/n$, and let A' be the union of all elements of $\mathcal{I}'$. It suffices to show that for each $n \geq 1$, a countable subset of $\mathcal{I}'$ already covers A'. Fix $k \in \mathbb{Z}$, let $A'' := A' \cap [k/n, (k+1)/n]$, and let $\mathcal{I}''$ denote the subset of $\mathcal{I}'$ consisting of all intervals $[a, b) \in \mathcal{I}'$ with $[a, b) \cap [k/n, (k+1)/n] \neq \emptyset$. It suffices to show that for each $k \in \mathbb{Z}$, a countable subset of $\mathcal{I}''$ already covers A''. Since $b - a \geq 2/n$

for all $[b,a) \in \mathcal{I}''$, each element of $\mathcal{I}''$ must contain either k/n or $(k+1)/n$, or both. If $\mathcal{I}''$ contains an element $[a,b)$ that contains both k/n and $(k+1)/n$ we are done. Otherwise, we can select countable subsets of $\{[a,b) \in \mathcal{I}'' : k/n \in [a,b)\}$ and $\{[a,b) \in \mathcal{I}'' : (k+1)/n \in [a,b)\}$ that cover A''. ∎

We note that the product space $\mathbb{R}_\mathfrak{s} \times \mathbb{R}_\mathfrak{s}$, equipped with the product topology, does *not* have the strong Lindelöf property. Indeed, the collection of open sets

$$\big\{[t+, \infty-) \times [-t+, \infty-) : t \in \mathbb{R}\big\} \tag{4.1}$$

covers the set $\{(t+, -t+) : t \in \mathbb{R}\}$, but no countable subset of (4.1) has this property. The set $\{(s+, t+) : s, t \in \mathbb{R}\}$ with the induced topology from $\mathbb{R}_\mathfrak{s}^2$ is a well-known counterexample in topology, known as the *Sorgenfrey plane*.

Proof of Proposition 2.3 We first prove the statement for $\mathbb{R}_\mathfrak{s}$. In any first countable space, being sequentially compact is equivalent to being countably compact, which means that every countable open covering has a finite subcovering. By the strong Lindelöf property, a subset of $\mathbb{R}_\mathfrak{s}$ is compact if and only if it is countably compact, proving that (i) and (ii) are equivalent.

Since the map $\tau \mapsto \underline{\tau}$, that assigns to a split real number τ its real part, is continuous, and since the continuous image of a compact set is compact, (i) implies that $\underline{C} := \{\underline{\tau} : \tau \in C\}$ is closed and bounded. Since moreover a compact subset of a Hausdorff space is closed, (i) implies (iii).

Property (iii) implies that each sequence $\tau_n \in C$ has a subsequence τ'_n such that $\underline{\tau}'_n$ converges to a limit $t \in \underline{C}$. The τ'_n must then contain a further subsequence τ''_n such that one of the following three cases occurs: 1. $\underline{\tau}''_n < t$ for all n, 2. $\underline{\tau}''_n > t$ for all n, or 3. $\underline{\tau}''_n$ is constant. In either case, the fact that C is closed implies that τ''_n converges to a limit in C, proving the implication (iii)⇒(ii). This completes the proof for $\mathbb{R}_\mathfrak{s}$.

We saw before that the strong Lindelöf property does not hold for $\mathbb{R}_\mathfrak{s}^d$ in dimensions $d \geq 2$, so to prove the statement for these spaces we have to proceed differently. Property (i) implies countable compactness which by the fact that $\mathbb{R}_\mathfrak{s}$ and hence also $\mathbb{R}_\mathfrak{s}^d$ are first countable is equivalent to (ii). The continuous image of a countably compact set is countably compact. Applying this to the coordinate projections and using what we already know for $\mathbb{R}_\mathfrak{s}$, we see that (ii) implies that C is bounded. Since, moreover, in any first countable Hausdorff space, being sequentially compact implies being closed, we see that (ii) implies (iii). By Tychonoff's theorem and what we already know for $\mathbb{R}_\mathfrak{s}$, the set $[s-, t+]^d$ is compact for each $-\infty < s < t < \infty$. Since a closed subset of a compact set is compact, (iii) implies (i). ∎

Proof of Lemma 2.4 A function $f : I_{\rm in} \to \mathcal{X}$ is continuous if and only if

(i) $f(\tau_n) \to f(t+)$ for all $t+ \in I_{\rm in}$ and $\tau_n \in I_{\rm in}$ such that $\underline{\tau}_n \to t$ and $\tau_n \geq t+$,

(ii) $f(\tau_n) \to f(t-)$ for all $t- \in I_{\rm in}$ and $\tau_n \in I_{\rm in}$ such that $\underline{\tau}_n \to t$ and $\tau_n \leq t-$.

We see from this that for a given $f \in \mathcal{C}_{I_{\rm in}}(\mathcal{X})$, setting

$$\begin{aligned} f^{\pm}(t) &:= f(t\pm) \qquad && \text{if } t\pm \in I_{\rm in}, \\ f^{+}(t) &:= f(t-) && \text{if } t = \sup I < \infty, \\ f^{-}(t) &:= f(t+) && \text{if } t = \inf I > -\infty \end{aligned} \tag{4.2}$$

defines $f^+ \in \mathcal{D}_I(\mathcal{X})$ and $f^- \in \mathcal{D}_I^-(\mathcal{X})$ such that f^- is the left-continuous modification of f^+ and f is given by (2.5).

Conversely, if such $f^{\pm}$ are given, then to see that f defined in (2.5) satisfies $f \in \mathcal{C}_{I_{\rm in}}(\mathcal{X})$, by symmetry, it suffices to check only condition (i) above. To show that $f(\tau_n) \to f(t+)$, it suffices to prove that each subsequence τ'_n contains a further subsequence τ''_n such that $f(\tau''_n) \to f(t+)$. In view of this, without loss of generality, we may assume that either $\mathfrak{s}(\tau_n) = +$ for all n or $\mathfrak{s}(\tau_n) = -$ for all n. In the first case, we have $f(\tau_n) \to f(t+)$ by the right-continuity of f^+, while in the second case we have $f(\tau_n) \to f(t+)$ by the fact that f^+ is the right-continuous modification of f^-. ∎

The following fact is well-known, but since we need this in what follows, for completeness we include the proof.

Lemma 4.2 (Countably many discontinuities) *Let $(\mathcal{X}, d)$ be a metric space, let I be a closed real interval, and let $f \in \mathcal{D}_I(\mathcal{X})$. Then the set $\{t \in I : f(t-) \neq f(t)\}$ is countable.*

Proof For each $\varepsilon > 0$ and $T < \infty$, the set $J := \{t \in I \cap [-T, T] : d\big(f(t-), f(t)\big) \geq \varepsilon\}$ must be finite, since otherwise there exist a strictly increasing or decreasing sequence $t_n \in J$, which is easily seen to contradict the cadlag property. ∎

4.2 The Hausdorff metric

In this subsection, we prove Lemmas 2.5 and 2.9, which are the only results from Subsection 2.3 for which we did not give a reference, as well as Lemma 4.3 below that will be needed in what follows.

Proof of Lemma 2.5 Let $R \in \mathrm{Cor}(K_1, K_2)$ and let $D := \sup_{(x_1,x_2)\in R} d(x_1, x_2)$. Then $d(x_1, K_2) \leq D$ and $d(x_2, K_1) \leq D$ for each $x_1 \in K_1$, $x_2 \in K_2$, and hence $d_{\rm H}(K_1, K_2) \leq D$. On the other hand, by the compactness of K_2 and the continuity of the function $d(x_1, \cdot\,)$, for each $x_1 \in K_1$, there exists an $x_2 \in K_2$ such that $d(x_1, K_2) = d(x_1, x_2)$. The same statement holds with the roles of K_1 and K_2 interchanged, so setting

$$R := \big\{(x_1, x_2) \in K_1 \times K_2 : d(x_1, x_2) \in \{d(x_1, K_2), d(x_2, K_1)\}\big\} \tag{4.3}$$

defines a correspondence between K_1 and K_2. By the compactness of K_1 and the continuity of the map $d(\,\cdot\,, K_2)$, there exists an $x'_1 \in K_1$ such that $d(x'_1, K_2) = \max_{x_1 \in K_1} d(x_1, K_2)$, and similarly there exists an $x''_2 \in K_2$ such that $d(x''_2, K_1) = \max_{x_2 \in K_2} d(x_2, K_1)$. By our earlier arguments, there exist $x'_2 \in K_2$ and $x''_1 \in K_1$ such that $d(x'_1, K_2) = d(x'_1, x'_2)$ and $d(x''_2, K_1) = d(x''_1, x''_2)$. Then

$$d_{\rm H}(K_1, K_2) = d(x'_1, x'_2) \vee d(x''_1, x''_2) = \max_{(x_1,x_2)\in R} d(x_1, x_2). \tag{4.4}$$

∎

Proof of Lemma 2.9 Imagine that $K_n, K \in \mathcal{K}_+(\mathcal{X})$ satisfy $K_n \to K$. If K is not connected, then there exist disjoint nonempty compact sets C_1, C_2 such that $K = C_1 \cup C_2$. Let $\varepsilon := d(C_1, C_2) = \inf\{d(x_1, x_2) : x_1 \in C_1,\ x_2 \in C_2\}$. By the compactness of C_1 and C_2, the infimum is attained and $\varepsilon > 0$. Let $U_i := \{x \in \mathcal{X} : d(x, C_i) \leq \varepsilon/3\}$ $(i = 1, 2)$. Then U_1, U_2 are disjoint closed sets. For all n large enough such that $d_{\rm H}(K_n, K) \leq \varepsilon/3$, one has $K_n \subset U_1 \cup U_2$ while $K_n \cap U_1$ and $K_n \cap U_2$ are both nonempty, which proves that K_n is not connected. ∎

We recall that the image of a compact set under a continuous map is compact. In what follows, we will need the following simple observation.

Lemma 4.3 (Continuous image) *Let $\mathcal{X}, \mathcal{Y}$ be metrisable spaces and let $\psi : \mathcal{X} \to \mathcal{Y}$ be continuous. If $K_n, K \in \mathcal{K}_+(\mathcal{X})$ satisfy $K_n \to K$, then their images under ψ satisfy $\psi(K_n) \to \psi(K)$ in $\mathcal{K}_+(\mathcal{X})$.*

Proof This follows easily from Lemma 2.6. Since $K_n \to K$, there exists a compact set $C \subset \mathcal{X}$ such that $K_n \subset C$ for all n, and now $\psi(C) \subset \mathcal{Y}$ is a compact set such that $\psi(K_n) \subset \psi(C)$ for all n. By (2.9), it now suffices to check that:

$$\begin{aligned} &\text{(i)} \quad \psi(K) \subset \{y \in \mathcal{Y} : \exists x_n \in K_n \text{ s.t. } \psi(x_n) \to y\}, \\ &\text{(ii)} \quad \{y \in \mathcal{Y} : \exists x_n \in K_n \text{ s.t. } y \text{ is a subsequential limit of } (\psi(x_n))_{n\in\mathbb{N}}\} \subset \psi(K). \end{aligned} \tag{4.5}$$

Here (i) follows from (2.9) and the continuity of ψ. To prove (ii), if $\psi(x'_n) \to y$ for some subsequence x'_n, then since $K_n \subset C$ for all n there exists a further subsequence x''_n such that $x''_n \to x$ for some $x \in \mathcal{X}$. Then $x \in K$ by (2.9) and hence $\psi(x) = y$ by the continuity of ψ. ∎

4.3 The ordered Hausdorff metric

In this subsection, we study the metrics $d_{\rm part}$ and $d_{\rm tot}$ defined in (2.13) and (2.15), preparing for the proofs of Theorem 2.10, Proposition 2.11, and Theorem 2.12, which will be given in the next subsection. Let $(\mathcal{X}, d)$ be a metric space. Generalising the definition in (2.10), for each $m \geq 1$ and $K \in \mathcal{K}_{\rm part}(\mathcal{X})$, we set

$$K^{\langle m\rangle} := \big\{(x_1, \ldots, x_m) \in K^m : x_1 \preceq \cdots \preceq x_m\big\}. \tag{4.6}$$

It is straightforward to check that $K^{\langle m\rangle}$ is a closed subset of K^m and hence a compact subset of $\mathcal{X}^m$. Generalising the definitions in (2.11) and (2.12), we equip $\mathcal{X}^m$ with the metric

$$d^m\big((x_1, \ldots, x_m), (y_1, \ldots, y_m)\big) := \bigvee_{k=1}^{m} d(x_k, y_k), \tag{4.7}$$

and we equip $\mathcal{K}_+(\mathcal{X}^m)$ with the associated Hausdorff metric $d^m_{\rm H}$. Generalising the definition in (2.13), for each $m \geq 1$, we define a function $d^{\langle m\rangle}$ on $\mathcal{K}_{\rm part}(\mathcal{X})^2$ by

$$d^{\langle m\rangle}(K_1, K_2) := d^m_{\rm H}(K_1^{\langle m\rangle}, K_2^{\langle m\rangle}) \qquad \big(K_1, K_2 \in \mathcal{K}_{\rm part}(\mathcal{X})\big). \tag{4.8}$$

In particular, when $m \geq 2$, this is a metric on $\mathcal{K}_{\rm part}(\mathcal{X})$ since $(K, \preceq)$ is uniquely characterised by $K^{\langle m\rangle}$ for $m \geq 2$. On the other hand, $d^{\langle 1\rangle}(K_1, K_2)$ is simply the Hausdorff distance between K_1 and K_2 as sets, which gives no information about the partial order. The following lemma describes a simple property of the metric $d^{\langle 2\rangle}$.

Lemma 4.4 (Ordered limit) *Let $\mathcal{X}$ be a metrisable space. Assume that $K_n, K \in \mathcal{K}_{\rm part}(\mathcal{X})$ satisfy $d^{\langle 2\rangle}(K_n, K) \to 0$ and that $x_n, y_n \in K_n$ satisfy $x_n \to x$, $y_n \to y$, and $x_n \preceq y_n$ for all n. Then $x, y \in K$ satisfy $x \preceq y$.*

Proof Since $d^{\langle 2\rangle}(K_n, K) \to 0$, we have $K_n^{\langle 2\rangle} \to K^{\langle 2\rangle}$ and hence by Lemma 2.6 $x, y \in K^{\langle 2\rangle}$, which proves that $x \preceq y$. ∎

The folling lemma gives a one-sided bound between metrics of the form $d^{\langle m\rangle}$ for different values of m.

Lemma 4.5 (One-sided bound) *One has*

$$d^{\langle m\rangle}(K_1, K_2) \leq d^{\langle m+1\rangle}(K_1, K_2) \qquad \big(m \geq 1,\ K_1, K_2 \in \mathcal{K}_{\rm part}(\mathcal{X})\big). \tag{4.9}$$

Proof By Lemma 2.5, there exists a correspondence R between $K_1^{\langle m+1\rangle}$ and $K_2^{\langle m+1\rangle}$ such that $d^{m+1}(x,y)\le d_{\rm H}^{m+1}(K_1^{\langle m+1\rangle},K_2^{\langle m+1\rangle})$ for all $(x,y)\in R$. Let $\psi:\mathcal{X}^{m+1}\to\mathcal{X}$ denote the projection $\psi(x_1,\ldots,x_{m+1}):=(x_1,\ldots,x_m)$. Then (4.7) implies that

$$d^m\big(\psi(x),\psi(y)\big)\le d^{m+1}(x,y)\qquad(x,y\in\mathcal{X}^{m+1}).\tag{4.10}$$

Since $\psi(K_i^{\langle m+1\rangle})=K_i^{\langle m\rangle}$ $(i=1,2)$, it follows that $R':=\big\{\big(\psi(x),\psi(y)\big):(x,y)\in R\big\}$ is a correspondence between $K_1^{\langle m\rangle}$ and $K_2^{\langle m\rangle}$ such that $d^m(x',y')\le d_{\rm H}^{m+1}(K_1^{\langle m+1\rangle},K_2^{\langle m+1\rangle})$ for all $(x',y')\in R'$. By Lemma 2.5, this proves that

$$d_{\rm H}^m(K_1^{\langle m\rangle},K_2^{\langle m\rangle})\le d_{\rm H}^{m+1}(K_1^{\langle m+1\rangle},K_2^{\langle m+1\rangle}),\tag{4.11}$$

which in view of (4.8) implies the claim. ∎

The following lemmas show that in general, the metrics $d^{\langle m\rangle}$ for different values of m are not comparable. More precisely, the one-sided bound in Lemma 4.5 is not matched by an opposite inequality of the form $d^{\langle m+1\rangle}(K_1,K_2)\le Cd^{\langle m\rangle}(K_1,K_2)$ for any finite constant C, and convergence in $d^{\langle m\rangle}$ does not imply convergence in $d^{\langle m+1\rangle}$.

Lemma 4.6 (No opposite inequality) *Let $\mathcal{X}=[0,1]$, equipped with the usual distance. Then for each $m\ge1$ and $0<\varepsilon\le1/4$, there exist $K_1,K_2\in\mathcal{K}_{\rm tot}(\mathcal{X})$ such that $d^{\langle m\rangle}(K_1,K_2)\le\varepsilon$ and $d^{\langle m+1\rangle}(K_1,K_2)\ge1/2$.*

Proof We choose $K_1=\{x_1,\ldots,x_{m+1}\}$ with $x_k\in[0,\varepsilon]$ when k is even and $x_k\in[1-\varepsilon,1]$ if k is odd, and we choose $K_2=\{y_1,\ldots,y_{m+1}\}$ with $y_k\in[0,\varepsilon]$ when k is odd and $y_k\in[1-\varepsilon,1]$ if k is even. We equip K_1 and K_2 with total orders such that $x_1\prec\cdots\prec x_{m+1}$ and $y_1\prec\cdots\prec y_{m+1}$. It is easy to see that

$$d\big((x_1,\ldots,x_{m+1}),K_2^{\langle m+1\rangle}\big)\ge1/2,\tag{4.12}$$

and hence $d^{\langle m+1\rangle}(K_1,K_2)\ge1/2$. On the other hand, it is easy to see that for each $(z_1,\ldots,z_m)\in K_1^{\langle m\rangle}$, there exists a $(z'_1,\ldots,z'_m)\in K_2^{\langle m\rangle}$ such that $|z_k-z'_k|\le\varepsilon$ for all k, and vice versa, so $d^{\langle m\rangle}(K_1,K_2)\le\varepsilon$. ∎

Lemma 4.7 (Different topologies) *Let $\mathcal{X}=[0,1]$, equipped with the usual distance. Then for each $m\ge1$, there exist $K_n\in\mathcal{K}_{\rm part}(\mathcal{X})$ and $K\in\mathcal{K}_{\rm tot}(\mathcal{X})$ such that $d^{\langle m\rangle}(K_n,K)\to0$ as $n\to\infty$ but $d^{\langle m+1\rangle}(K_n,K)\ge1/2$ for all n.*

Proof It will be convenient to use the notation $[m]:=\{1,\ldots,m\}$ $(m\ge1)$. We choose $x_1,\ldots,x_{m+1}$, all different, with $x_k\in[0,1/4]$ when k is even and $x_k\in[3/4,1]$ if k is odd. We set $K=\{x_1,\ldots,x_{m+1}\}$ and we equip K with a total order by setting $x_1\prec\cdots\prec x_{m+1}$. We choose points

$$\big(x_k^l(n)\big)_{k\in[m+1]}^{l\in[m+1]}\tag{4.13}$$

in $[0,1]$, all different, such that $x_k^l(n)\to x_k$ as $n\to\infty$ for all $k,l\in[m+1]$, and we set

$$K_n:=\big\{x_k^l:k,l\in[m+1],\ k\neq l\big\}.\tag{4.14}$$

We equip K_n with a partial order such that

$$x_k^l\preceq x_{k'}^{l'}\quad\Leftrightarrow\quad k\preceq k'\text{ and }l=l'.\tag{4.15}$$

Then it is easy to check that $d^{\langle m\rangle}(K_n,K)\to 0$ as $n\to\infty$ but $d^{\langle m+1\rangle}(K_n,K)\geq 1/2$ for all n. ∎

We note that $d^{\langle m\rangle}(K_1,K_2)\leq\sup_{(x_1,x_2)\in K_1\times K_2}d(x_1,x_2)$, which is finite by the continuity of d and the compactness of $K_1\times K_2$. We use this and Lemma 4.5 to define $d^{\langle\infty\rangle}$ as the increasing limit

$$d^{\langle\infty\rangle}(K_1,K_2):=\lim_{m\to\infty}d^{\langle m\rangle}(K_1,K_2)\qquad\big(K_1,K_2\in\mathcal{K}_{\rm part}(\mathcal{X})\big).\tag{4.16}$$

It is straightforward to check that $d^{\langle\infty\rangle}$ is a metric on $\mathcal{K}_{\rm part}(\mathcal{X})$: symmetry and the triangle inequality follow by taking the limit in the corresponding properties of the metrics $d^{\langle m\rangle}$, and $d^{\langle\infty\rangle}(K_1,K_2)=0$ clearly implies $d^{\langle m\rangle}(K_1,K_2)=0$ for all $m\geq 1$ and hence equality of K_1 and K_2 as partially ordered spaces. In the special case that K_1 and K_2 are totally ordered, the following proposition identifies $d^{\langle\infty\rangle}(K_1,K_2)$ as the metric $d_{\rm tot}$ defined in (2.15).

Proposition 4.8 (Monotone correspondences) *Let $(\mathcal{X},d)$ be a metric space. Then one has $d^{\langle\infty\rangle}(K_1,K_2)=d_{\rm tot}(K_1,K_2)$ for all $K_1,K_2\in\mathcal{K}_{\rm tot}(\mathcal{X})$.*

The proof of Proposition 4.8 uses the following simple lemma.

Lemma 4.9 (Eventually ordered sequences) *Let K be a compact metrisable set that is equipped with a total order that is compatible with the topology. Assume that $x\prec y$ and that $x_n,y_n\in K$ satisfy $x_n\to x$ and $y_n\to y$. Then $x_n\prec y_n$ for all n sufficiently large.*

Proof Since $\preceq$ is a total order, if the statement is not true, then $y_n\preceq x_n$ for infinitely many n, so we can select a subsequence such that $y_n\preceq x_n$ for all n. Taking the limit, using the fact that the total order that is compatible with the topology, we find that $y\preceq x$, which contradicts $x\prec y$. ∎

Proof of Proposition 4.8 We first prove the inequality $d^{\langle\infty\rangle}(K_1,K_2)\leq d_{\rm tot}(K_1,K_2)$. Let R be a monotone correspondence between K_1 and K_2. Let $x_1,\ldots,x_m\in K_1$ satisfy $x_1\preceq\cdots\preceq x_m$. Then we can choose $x'_1,\ldots,x'_m\in K_2$ such that $(x_k,x'_k)\in R$ for all $1\leq k\leq m$, and moreover $x'_k=x'_{k+1}$ whenever $x_k=x_{k+1}$ $(1\leq k<m)$. Since R is monotone and K_2 is totally ordered, we must have $x'_1\preceq\cdots\preceq x'_m$. This shows that

$$d^m\big((x_1,\ldots,x_m),K_2^{\langle m\rangle}\big)\leq\sup_{(x,x')\in R}d(x,x')\qquad\big((x_1,\ldots,x_m)\in K_1^{\langle m\rangle}\big).\tag{4.17}$$

The same is true with the roles of K_1 and K_2 interchanged, so we conclude that

$$d^{\langle m\rangle}(K_1,K_2)=d^m_{\rm H}(K_1^{\langle m\rangle},K_2^{\langle m\rangle})\leq\sup_{(x,x')\in R}d(x,x').\tag{4.18}$$

Taking the infimum over all monotone correspondences between K_1 and K_2 and letting $m\to\infty$ we see that $d^{\langle\infty\rangle}(K_1,K_2)\leq d_{\rm tot}(K_1,K_2)$.

To prove the opposite inequality, let ε_n be positive constants, tending to zero. Since K_1 is totally bounded, for each n, we can find an $m(n)\geq 1$ and $x^n_1,\ldots,x^n_{m(n)}\in K_1$ such that

$$d\big(x,\{x^n_1,\ldots,x^n_{m(n)}\}\big)\leq\varepsilon_n\quad\forall x\in K_1.\tag{4.19}$$

Since K_1 is totally ordered, we can assume without loss of generality that $x^n_1\preceq\cdots\preceq x^n_{m(n)}$. Since

$$d^m_{\rm H}(K_1^{\langle m\rangle},K_2^{\langle m\rangle})=d^{\langle m\rangle}(K_1,K_2)\leq d^{\langle\infty\rangle}(K_1,K_2),\tag{4.20}$$

we can find $y^n_1,\ldots,y^n_{m(n)}\in K_2$ with $y^n_1\preceq\cdots\preceq y^n_{m(n)}$ such that

$$d(x^n_k,y^n_k)\leq d^{\langle\infty\rangle}(K_1,K_2)\qquad(1\leq k\leq m(n)).\tag{4.21}$$

Using the fact that K_2 is totally bounded, adding points to $\{y_1^n, \ldots, y_{m(n)}^n\}$ and making $m(n)$ larger if necessary, we can arrange things so that also

$$d\big(y, \{y_1^n, \ldots, y_{m(n)}^n\}\big) \le \varepsilon \quad \forall y \in K_2. \tag{4.22}$$

Now using again (4.20) we can add corresponding points in K_1 for the new points we have added to K_2 so that (4.21) remains true. Adding points will not spoil (4.19) so we can arrange things such that (4.19), (4.20), and (4.22) are satisfied simultaneously.

Let $R_n \subset K_1 \times K_2$ be the set

$$R_n := \big\{(x_k^n, y_k^n) : 1 \le k \le m(n)\big\}. \tag{4.23}$$

We claim that R_n is monotone in the sense that

$$\text{there are no } (x_k^n, y_k^n), (x_l^n, x_l^n) \in R_n \text{ such that } x_k^n \prec x_l^n \text{ and } y_l^n \prec y_k^n. \tag{4.24}$$

Indeed, $x_k^n \prec x_l^n$ implies $k < l$ and $y_l^n \prec y_k^n$ implies $l < k$, which is a contradiction.

Since $K_1 \times K_2$ is compact, by Lemma 2.8, we can select a subsequence such that $R_n \to R$ in the Hausdorff topology on $\mathcal{K}_+(K_1 \times K_2)$, for some compact set $R \subset K_1 \times K_2$. We claim that R is a correspondence between K_1 and K_2. Indeed, by (4.19), for each $x \in K_1$, we can choose $k(n)$ such that $x_{k(n)}^n \to x$. By the compactness of $K_1 \times K_2$, the sequence $(x_{k(n)}^n, y_{k(n)}^n)$ has at least one cluster point (x, y), and by Lemma 2.6 $(x, y) \in R$. Similarly, for each $y \in K_2$ there exists an $x \in K_1$ such that $(x, y) \in R$.

We next claim that R is monotone. Assume that conversely, there exist $(x, y), (x', y') \in R$ such that $x \prec x'$ and $y' \prec y$. Then by Lemma 2.6, there exist $k(n), k'(n)$ such that $(x_{k(n)}^n, y_{k(n)}^n) \to (x, y)$ and $(x_{k'(n)}^n, y_{k'(n)}^n) \to (x', y')$. By Lemma 4.9, $x_{k(n)}^n \prec y_{k(n)}^n$ and $y_{k'(n)}^n \prec x_{k'(n)}^n$ for all n large enough, which contradicts (4.24).

Taking the limit in (4.21), using Lemma 2.6, we see that

$$d(x, y) \le d^{\langle\infty\rangle}(K_1, K_2) \quad \forall (x, y) \in R, \tag{4.25}$$

and hence by (2.15) $d_{\rm tot}(K_1, K_2) \le d^{\langle\infty\rangle}(K_1, K_2)$. ∎

4.4 The mismatch modulus

In this subsection, we prove Theorem 2.10. Generalising the definition in (2.17) for any $K_1, K_2 \in \mathcal{K}_{\rm tot}(\mathcal{X})$ and $\varepsilon > 0$, we define the *mismatch modulus* $m_\varepsilon(K_1, K_2)$ by

$$\begin{aligned} m_\varepsilon(K_1, K_2) := \sup\big\{\, & d(x_1, y_1) \vee d(x_2, y_2) : x_1, y_1 \in K_1,\ x_2, y_2 \in K_2, \\ & d(x_1, x_2) \vee d(y_1, y_2) \le \varepsilon,\ x_1 \preceq y_1,\ y_2 \preceq x_2\big\}. \end{aligned} \tag{4.26}$$

Lemma 4.10 (Convergence of the mismatch modulus) *Let $\mathcal{X}$ be a metrisable space. Assume that $K_n, K \in \mathcal{K}_{\rm part}(\mathcal{X})$ satisfy $d^{\langle 2\rangle}(K_n, K) \to 0$. Then*

$$m_{\varepsilon_n}(K_n, K) \underset{n\to\infty}{\longrightarrow} 0 \quad \textit{with} \quad \varepsilon_n := d^{\langle 1\rangle}(K_n, K). \tag{4.27}$$

Proof If (4.27) does not hold, then, by going to a subsequence, we can assume that there exists a $\delta > 0$ such that $m_{\varepsilon_n}(K_n, K) \ge \delta$ for all n. It follows that for each n, we can find $x_1(n), y_1(n) \in K_n$ and $x_2(n), y_2(n) \in K$ with $d(x_1(n), y_1(n)) \vee d(x_2(n), y_2(n)) \ge \delta$ and $d(x_1(n), x_2(n)) \vee d(y_1(n), y_2(n)) \le \varepsilon_n$ such that $x_1(n) \preceq y_1(n)$ and $y_2(n) \preceq x_2(n)$. By Lemma 4.5, our assumption $d^{\langle 2\rangle}(K_n, K) \to 0$ implies $\varepsilon_n = d^{\langle 1\rangle}(K_n, K) \to 0$ and hence $K_n \to K$. Therefore, by Lemma 2.6, there exists a compact set $C \subset \mathcal{X}$ such that $K_n \subset C$ for all n, so by going to a subsequence, we can assume that $x_1(n), x_2(n), y_1(n), y_2(n)$ converge to limits x_1, x_2, y_1, y_2 in $\mathcal{X}$. Since $d(x_1(n), x_2(n)) \vee d(y_1(n), y_2(n)) \leq \varepsilon_n \to 0$, we have $x := x_1 = x_2$ and $y := y_1 = y_2$. On the other hand, our assumption that $d(x_1(n), y_1(n)) \vee d(y_2(n), x_2(n)) \geq \delta$ implies that $d(x, y) \geq \delta$. This leads to a contradiction, since by the assumption that $d^{\langle 2\rangle}(K_n, K) \to 0$ and Lemma 4.4, $x_1(n) \preceq y_1(n)$ and $y_2(n) \preceq x_2(n)$ imply $x \preceq y$ and $y \preceq x$ and hence $x = y$. ∎

The following estimate essentially uses that the spaces are totally ordered.

Lemma 4.11 (Estimate in terms of mismatch modulus) *Let $\mathcal{X}$ be a metrisable space and let $K_1, K_2 \in \mathcal{K}_{\rm tot}(\mathcal{X})$ satisfy $d^{\langle 1\rangle}(K_1, K_2) \leq \varepsilon$. Then*

$$d^{\langle m\rangle}(K_1, K_2) \leq m_\varepsilon(K_1, K_2) + \varepsilon \qquad (m \geq 1). \tag{4.28}$$

Proof By symmetry, it suffices to show that for each $x_1 = (x_1^1, \ldots, x_1^m) \in K_1^{\langle m\rangle}$, there exists an $x_2 = (x_2^1, \ldots, x_2^m) \in K_2^{\langle m\rangle}$ such that $d^m(x_1, x_2) \leq m_\varepsilon(K_1, K_2) + \varepsilon$. In view of (4.7), the latter means that $d(x_1^k, x_2^k) \leq m_\varepsilon(K_1, K_2) + \varepsilon$ for all $1 \leq k \leq m$. Since $d^{\langle 1\rangle}(K_1, K_2) \leq \varepsilon$, there exists a $z(1) \in K_2$ such that $d(x_1^1, z(1)) \leq \varepsilon$. We set $x_2^i = z(1)$ for all $1 \leq i < I(1)$, where $I(1) := \inf\{i > 1 : d(x_1^i, z(1)) > m_\varepsilon(K_1, K_2) + \varepsilon\}$. Using again that $d^{\langle 1\rangle}(K_1, K_2) \leq \varepsilon$, there exists a $z(2) \in K_2$ such that $d(x_1^{I(1)}, z(2)) \leq \varepsilon$. Then

$$d\big(z(1), z(2)\big) > d\big(x_1^{I(1)}, z(2)\big) - \varepsilon \geq m_\varepsilon(K_1, K_2) \tag{4.29}$$

and hence by the definition of $m_\varepsilon(K_1, K_2)$ and the fact that $x_1^1 \preceq x_1^{I(1)}$ we cannot have $z(2) \preceq z(1)$. Since K_2 is totally ordered, we conclude that $z(1) \prec z(2)$. This allows us to set $x_2^i = z(2)$ for all $I(1) \leq i < I(2)$, where $I(2) := \inf\{i > I(1) : d(x_1^i, z(2)) > m_\varepsilon(K_1, K_2) + \varepsilon\}$. Continuing inductively, we obtain $(x_2^1, \ldots, x_2^m) \in K_2^{\langle m\rangle}$ such that $d(x_1^k, x_2^k) \leq m_\varepsilon(K_1, K_2) + \varepsilon$ for all $1 \leq k \leq m$. ∎

As a consequence of Lemmas 4.10 and 4.11, we can prove that the metrics $d^{\langle m\rangle}$ with $2 \leq m \leq \infty$ all generate the same topology on $\mathcal{K}_{\rm tot}(\mathcal{X})$. This may be a bit surprising in view of Lemmas 4.6 and 4.7. As the latter shows, the restriction to totally ordered sets is essential in the following lemma.

Lemma 4.12 (Convergence of totally ordered sets) *Let $\mathcal{X}$ be a metrisable space. Then $K_n, K \in \mathcal{K}_{\rm tot}(\mathcal{X})$ satisfy $d^{\langle 2\rangle}(K_n, K) \to 0$ if and only if $d^{\langle\infty\rangle}(K_n, K) \to 0$.*

Proof Since $d^{\langle m\rangle}(K_n, K) \leq d^{\langle m+1\rangle}(K_n, K)$ by Lemma 4.5 and since $d^{\langle\infty\rangle}(K_n, K)$ is defined as the increasing limit of $d^{\langle m\rangle}(K_n, K)$ as $m \to \infty$, it is clear that $d^{\langle\infty\rangle}(K_n, K) \to 0$ implies $d^{\langle 2\rangle}(K_n, K) \to 0$. To prove the opposite implication, it suffices to show that $d^{\langle 2\rangle}(K_n, K) \to 0$ implies

$$\sup_{m\geq 1} d^{\langle m\rangle}(K_n, K) \underset{n\to\infty}{\longrightarrow} 0. \tag{4.30}$$

By Lemma 4.5, $d^{\langle 2\rangle}(K_n, K) \to 0$ implies $\varepsilon_n := d^{\langle 1\rangle}(K_n, K) \to 0$. Lemmas 4.10 and 4.11 now imply that

$$\sup_{m\geq 1} d^{\langle m\rangle}(K_n, K) \leq m_{\varepsilon_n}(K_n, K) + \varepsilon_n \underset{n\to\infty}{\longrightarrow} 0. \tag{4.31}$$

∎

Proof of Theorem 2.10 By the definition of $d_{\rm part}$ and Proposition 4.8 we have $d_{\rm part} = d^{\langle 2\rangle}$ and $d_{\rm tot} = d^{\langle\infty\rangle}$. By Lemma 4.12 both metrics generate the same topology on $\mathcal{K}_{\rm tot}(\mathcal{X})$. If d

and d' generate the same topology on $\mathcal{X}$ and $d_{\rm part}$ and $d'_{\rm part}$ are defined in terms of d and d' as in (2.13), then by Lemmas 2.7 and 2.8, $d_{\rm part}$ and $d'_{\rm part}$ generate the same topology on $\mathcal{K}_{\rm tot}(\mathcal{X})$. The inequalities (2.16) follow from the fact that $d^{\langle 1\rangle}(K_1,K_2)\leq d^{\langle 2\rangle}(K_1,K_2)\leq d^{\langle\infty\rangle}(K_1,K_2)$ by Lemma 4.5. On the other hand, Lemma 4.6 shows that if $\mathcal{X}=[0,1]$, then for each $\varepsilon>0$ we can find $K_1,K_2\in\mathcal{K}_{\rm tot}(\mathcal{X})$ such that

$$d^{\langle 2\rangle}(K_1,K_2)\leq\varepsilon\quad\text{while}\quad 1/2\leq d^{\langle 3\rangle}(K_1,K_2)\leq d^{\langle\infty\rangle}(K_1,K_2),\tag{4.32}$$

proving the final claim of the theorem. ∎

4.5 Polishness

In this subsection, we prove Proposition 2.11. We start with the following lemma, announced in the introduction, that shows that even when $(\mathcal{X},d)$ is complete, it is in general not true that the metrics $d_{\rm part}$ and $d_{\rm tot}$ are complete on $\mathcal{K}_{\rm tot}(\mathcal{X})$.

Lemma 4.13 (Metric not complete) *Let $\mathcal{X}=[0,1]$, equipped with the usual distance. Then the metrics $d^{\langle m\rangle}$ with $2\leq m\leq\infty$ are not complete on $\mathcal{K}_{\rm tot}(\mathcal{X})$.*

Proof It suffices to construct a Cauchy sequence that does not converge. In view of Lemma 4.5, it suffices to construct a Cauchy sequence in the metric $d^{\langle\infty\rangle}$, which by Proposition 4.8 equals $d_{\rm tot}$. Let ε_n be positive constants converging to zero, and let $K_n:=\{0,1,\varepsilon_n\}$ equipped with a total order such that $0\prec 1\prec\varepsilon_n$. For each n,m, we define a monotone correspondence $R_{n,m}$ between K_n and K_m by $R_{n,m}:=\{(0,0),(1,1),(\varepsilon_n,\varepsilon_m)\}$. Then

$$d_{\rm tot}(K_n,K_m)\leq\sup_{(x_1,x_2)\in R_{n,m}}|x_1-x_2|=|\varepsilon_n-\varepsilon_m|,\tag{4.33}$$

so K_n is clearly a Cauchy sequence in $d_{\rm tot}$. However, the sequence K_n does not converge in the ordered Hausdorff topology. If it had a limit K, then (in view of Lemma 4.4) this would have to be the set $K=\{0,1\}$ equipped with a total order such that $0\preceq 1$ and $1\preceq 0$, but such a totally ordered set does not exist. ∎

The proof of Proposition 2.11 needs some preparations. For each $L\in\mathcal{K}_+(\mathcal{X}^2)$ and $\varepsilon\geq 0$, we set

$$m^{\langle 2\rangle}_\varepsilon(L):=\sup\big\{d(x_1,y_1)\vee d(x_2,y_2):(x_1,y_1),(y_2,x_2)\in L,\ d(x_1,x_2)\vee d(y_1,y_2)\leq\varepsilon\big\}.\tag{4.34}$$

In particular, this implies $m^{\langle 2\rangle}_\varepsilon(K^{\langle 2\rangle})=m_\varepsilon(K)$ $(K\in\mathcal{K}_{\rm tot}(\mathcal{X}))$.

Lemma 4.14 (Right continuity) *For any metric space $\mathcal{X}$ and $L\in\mathcal{K}_+(\mathcal{X}^2)$, the function $[0,\infty)\ni\varepsilon\to m^{\langle 2\rangle}_\varepsilon(L)$ is right-continuous.*

Proof The function $\varepsilon\mapsto m^{\langle 2\rangle}_\varepsilon(L)$ is clearly nondecreasing, so it suffices to prove that

$$m^{\langle 2\rangle}_\varepsilon(L)\geq\lim_{\eta\downarrow\varepsilon}m^{\langle 2\rangle}_\eta(L)\quad(\varepsilon\geq 0)\tag{4.35}$$

where the limit exist by monotonicity. Fix $\varepsilon_n>\varepsilon$ such that $\varepsilon_n\to\varepsilon$. Then for each $\delta>0$ and for each n, we can choose $(x^n_1,y^n_1),(y^n_2,x^n_2)\in L$ such that $d(x^n_1,x^n_2)\vee d(y^n_1,y^n_2)\leq\varepsilon_n$ and $d(x^n_1,y^n_1)\vee d(x^n_2,y^n_2)\geq m^{\langle 2\rangle}_{\eta_n}(L)-\delta$. Since L is compact, by going to a subsequence, we can assume that $(x^n_1,y^n_1)\to(x_1,y_1)$ and $(y^n_2,x^n_2)\to(y_2,x_2)$ for some $(x_1,y_1),(y_2,x_2)\in L$. Then $d(x_1,x_2)\vee d(y_1,y_2)\leq\varepsilon$ and $d(x_1,y_1)\vee d(x_2,y_2)\geq\lim_{\eta\downarrow\varepsilon}m^{\langle 2\rangle}_\eta(L)-\delta$, which proves that $m^{\langle 2\rangle}_\varepsilon(L)\geq\lim_{\eta\downarrow\varepsilon}m^{\langle 2\rangle}_\eta(L)-\delta$. Since $\delta>0$ is arbitrary, this implies (4.35). ∎

Lemma 4.15 (Upper semi-continuity) *Let $\mathcal{X}$ be a metric space and let $L_n, L \in \mathcal{K}_+(\mathcal{X}^2)$ satisfy $L_n \to L$. Then*

$$m^{\langle 2\rangle}_\varepsilon(L) \geq \limsup_{n\to\infty} m^{\langle 2\rangle}_\varepsilon(L_n) \quad (\varepsilon \geq 0). \tag{4.36}$$

Proof By the compactness of $[0,\infty]$ we can select a subsequence for which $\lim_{n\to\infty} m^{\langle 2\rangle}_\varepsilon(L_n)$ exists and is equal to the limit superior of the original sequence. Let $\delta_n > 0$ converge to zero and pick $(x^n_1, y^n_1), (y^n_2, x^n_2) \in L_n$ such that $d(x^n_1, x^n_2) \vee d(y^n_1, y^n_2) \leq \varepsilon$ and $d(x^n_1, y^n_1) \vee d(x^n_2, y^n_2) \geq m^{\langle 2\rangle}_\varepsilon(L_n) - \delta_n$. By Lemma 2.6, there exists a compact $C \subset \mathcal{X}^2$ such that $L_n \subset C$ for all n, so by going to a further subsequence we can assume that $(x^n_1, y^n_1) \to (x_1, y_1)$ and $(y^n_2, x^n_2) \to (y_2, x_2)$ for some $(x_1, y_1), (y_2, x_2) \in \mathcal{X}^2$. Then $(x_1, y_1), (y_2, x_2) \in L$ by Lemma 2.6, $d(x_1, x_2) \vee d(y_1, y_2) \leq \varepsilon$, and hence

$$m^{\langle 2\rangle}_\varepsilon(L) \geq d(x_1, y_1) \vee d(x_2, y_2) \geq \lim_{n\to\infty} \big(m^{\langle 2\rangle}_\varepsilon(L_n) - \delta_n\big). \tag{4.37}$$

Since $\delta_n \to 0$, this proves (4.36). ∎

Before we can prove Proposition 2.11 we need one more lemma. For any metric space $(\mathcal{X}, d)$, we define $\mathcal{L}(\mathcal{X}) \subset \mathcal{K}_+(\mathcal{X}^2)$ by

$$\mathcal{L}(\mathcal{X}) := \big\{K^{\langle 2\rangle} : K \in \mathcal{K}_{\rm tot}(\mathcal{X})\big\}, \tag{4.38}$$

and we let $\overline{\mathcal{L}(\mathcal{X})}$ denote the closure of $\mathcal{L}(\mathcal{X})$ in the metric space $\big(\mathcal{K}_+(\mathcal{X}^2), d^2_{\rm H}\big)$.

Lemma 4.16 (Totally ordered sets) *For any metric space $\mathcal{X}$, one has*

$$\mathcal{L}(\mathcal{X}) = \big\{L \in \overline{\mathcal{L}(\mathcal{X})} : m^{\langle 2\rangle}_0(L) = 0\big\}. \tag{4.39}$$

Proof To prove the inclusion $\subset$ in (4.39), it suffices to observe that

$$m^{\langle 2\rangle}_0(K^{\langle 2\rangle}) = \sup\big\{d(x,y) : (x,y), (y,x) \in K^{\langle 2\rangle}\big\} = 0 \tag{4.40}$$

for all $K \in \mathcal{K}_{\rm tot}(\mathcal{X})$, since $x \preceq y$ and $y \preceq x$ imply $x = y$.

We next prove the inclusion $\supset$ in (4.39). Assume that $L \in \overline{\mathcal{L}(\mathcal{X})}$ satisfies $m^{\langle 2\rangle}_0(L) = 0$. Since $L \in \overline{\mathcal{L}(\mathcal{X})}$, there exist $K_n \in \mathcal{K}_{\rm tot}(\mathcal{X})$ such that $K^{\langle 2\rangle}_n \to L$ in the topology on $\mathcal{K}_+(\mathcal{X}^2)$. Let $\pi_i(x_1, x_2) := x_i$ $(i = 1, 2)$ denote the coordinate projections $\pi_i : \mathcal{X}^2 \to \mathcal{X}$. Since $\pi_1(K^{\langle 2\rangle}_n) = K_n = \pi_2(K^{\langle 2\rangle}_n)$ for each n, using Lemma 4.3, we see that $K_n \to K$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X})$, where $K := \pi_1(L) = \pi_2(L)$. We define a relation $\preceq$ on K by setting $x \preceq y$ if and only if $(x,y) \in L$. To complete the proof, it suffices to show that $\preceq$ is a total order on K, i.e.,

(i) for each $x, y \in K$, either $x \preceq y$ or $y \preceq x$, or both,

(ii) $x \preceq y$ and $y \preceq x$ imply $x = y$,

(iii) $x \preceq y \preceq z$ imply $x \preceq z$.

To prove (i), let $x, y \in K$. Since $K_n \to K$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X})$, by Lemma 2.6, there exist $x_n, y_n \in K_n$ such that $x_n \to x$ and $y_n \to y$. Since K_n is totally ordered, either $x_n \preceq y_n$ happens for infinitely many n, or $y_n \preceq x_n$ happens for infinitely many n, or both. Since $K^{\langle 2\rangle}_n \to L$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X}^2)$, by Lemma 2.6, it folows that either $x \preceq y$ or $y \preceq x$, or both. Property (ii) follows immediately from the fact that $\sup\{d(x,y) : (x,y), (y,x) \in L\} = m^{\langle 2\rangle}_0(L) = 0$. To prove (iii), assume that $x, y, z \in K$ satisfy $x \preceq y \preceq z$. If $x = y$ or $y = z$ then trivially $x \preceq z$, so without loss of generality we may assume that $x \neq y \neq z$. Since

$K_n \to K$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X})$, by Lemma 2.6, there exist $x_n, y_n, z_n \in K_n$ such that $x_n \to x$, $y_n \to y$, and $z_n \to z$. Since K_n is totally ordered, for each n either $x_n \preceq y_n$, or $y_n \preceq x_n$, or both. But $y_n \preceq x_n$ can happen only for finitely many n since otherwise the fact that $K_n^{\langle 2\rangle} \to L$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X}^2)$ and Lemma 2.6 would imply that $y \preceq x$, which together with our assumptions $x \preceq y$ and $x \neq y$ contradicts (ii). We conclude that $x_n \preceq y_n$ for all n sufficiently large and by the same argument also $y_n \preceq z_n$ for all n sufficiently large. It follows that $x_n \preceq z_n$ for all n sufficiently large. Since $K_n^{\langle 2\rangle} \to L$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X}^2)$, Lemma 2.6 shows that $(x,z) \in L$ and hence $x \preceq z$. ∎

We are now ready to prove Proposition 2.11. We need to recall one well-known fact. A subset $A \subset \mathcal{X}$ of a topological space $\mathcal{X}$ is called a *G_δ-set* if A is a countable intersection of open sets. Our proof of Proposition 2.11 makes use of the following fact, that we cite from [Bou58, §6 No. 1, Theorem. 1] (see also [Oxt80, Thms 12.1 and 12.3]).

Lemma 4.17 (Subsets of Polish spaces) *A subset $\mathcal{Y} \subset \mathcal{X}$ of a Polish space $\mathcal{X}$ is Polish in the induced topology if and only if $\mathcal{Y}$ is a G_δ-subset of $\mathcal{X}$.*

Proof of Proposition 2.11 We first observe that if $\mathcal{X}$ is Polish, then so is $\mathcal{X}^2$, equipped with the product topology, and hence, by Lemma 2.7, also $\mathcal{K}_+(\mathcal{X}^2)$. Therefore, since $\big(\mathcal{K}_{\rm tot}(\mathcal{X}), d_{\rm part}\big)$ is isometric to $\big(\mathcal{L}(\mathcal{X}), d_{\rm H}^2\big)$ defined in (4.38), in view of Lemma 4.17, it suffices to show that $\mathcal{L}(\mathcal{X})$ is a G_δ-subset of $\mathcal{K}_+(\mathcal{X}^2)$.

It follows from Lemma 4.15 that for each $\varepsilon, \delta > 0$, the set

$$\mathcal{A}_{\delta,\varepsilon} := \big\{L \in \mathcal{K}_+(\mathcal{X}^2) : m_\varepsilon^{\langle 2\rangle}(L) \geq \delta\big\} \tag{4.41}$$

is a closed subset of $\mathcal{K}_+(\mathcal{X}^2)$ and hence its complement $\mathcal{A}^{\rm c}_{\varepsilon,\delta}$ is open. As a consequence,

$$\mathcal{G} := \bigcap_{n=1}^\infty \bigcup_{m=1}^\infty \mathcal{A}^{\rm c}_{1/n,1/m} \tag{4.42}$$

is a G_δ-set. Since each closed set is a G_δ-set, and the intersection of two G_δ-sets is a G_δ-set, using Lemmas 4.14 and 4.16, we conclude that

$$\begin{aligned} &\overline{\mathcal{L}(\mathcal{X})} \cap \mathcal{G} = \big\{L \in \overline{\mathcal{L}(\mathcal{X})} : \forall \delta > 0\ \exists \varepsilon > 0 \text{ s.t. } m_\varepsilon^{\langle 2\rangle}(L) < \delta\big\} \\ &\quad = \big\{L \in \overline{\mathcal{L}(\mathcal{X})} : \lim_{\varepsilon\to 0} m_\varepsilon^{\langle 2\rangle}(L) = 0\big\} = \big\{L \in \overline{\mathcal{L}(\mathcal{X})} : m_0^{\langle 2\rangle}(L) = 0\big\} = \mathcal{L}(\mathcal{X}) \end{aligned} \tag{4.43}$$

is a G_δ-set. ∎

4.6 Compactness criterion

In this subsection, we prove Theorem 2.12.

Proof of Theorem 2.12 Since the map $K \mapsto K^{\langle 2\rangle}$ is a homeomorphism from $\mathcal{K}_{\rm tot}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$, equipped with the induced topology from $\mathcal{K}_+(\mathcal{X}^2)$, a set $\mathcal{A} \subset K^{\langle 2\rangle}$ is precompact if and only if $\mathcal{B} := \{K^{\langle 2\rangle} : K \in \mathcal{A}\}$ is a precompact subset of $\mathcal{K}_+(\mathcal{X}^2)$ and its closure $\overline{\mathcal{B}}$ is contained in $\mathcal{L}(\mathcal{X})$. We will show that $\mathcal{B}$ is a precompact subset of $\mathcal{K}_+(\mathcal{X}^2)$ if and only if (2.18 (i) holds. Moreover, if (2.18 (i) holds, then $\overline{\mathcal{B}}$ is contained in $\mathcal{L}(\mathcal{X})$ if and only if (2.18 (ii) holds.

If (2.18 (i) holds, then C^2 is a compact subset of $\mathcal{X}^2$ and $K^{\langle 2\rangle} \subset C^2$ for all $K \in \mathcal{A}$, so $\mathcal{B}$ is a precompact subset of $\mathcal{K}_+(\mathcal{X}^2)$ by Lemma 2.8. Conversely, if $\mathcal{B}$ is a precompact subset of $\mathcal{K}_+(\mathcal{X}^2)$, then by Lemma 2.8 there exists a compact subset $D \subset \mathcal{X}^2$ such that $K^{\langle 2\rangle} \subset D$

for all $K \in \mathcal{A}$. Without loss of generality, we may assume that D is of the form $D = C^2$ for some compact subset C of $\mathcal{X}$; for example, we may take for C the union of the two coordinate projections of D. Then $K \subset C$ for all $K \in \mathcal{A}$, proving that (2.18 (i) holds.

To complete the proof, assume that (2.18 (i) holds. We must show that $\overline{\mathcal{B}}$ is contained in $\mathcal{L}(\mathcal{X})$ if and only if (2.18 (ii) holds. Assume, first, that (2.18 (ii) does not hold. Then we can find a $\delta > 0$ and $\varepsilon_n > 0$ tending to zero, as well as $K_n \in \mathcal{A}$, such that $m^{\langle 2\rangle}_{\varepsilon_n}(K_n) \geq \delta$ for each n. By (2.18 (i), going to a subsequence if necessary, we can assume that $K_n^{\langle 2\rangle} \to L$ for some $L \in \mathcal{K}_+(\mathcal{X}^2)$. Now Lemma 4.15 implies that

$$m^{\langle 2\rangle}_{\varepsilon}(L) \geq \limsup_{n\to\infty} m^{\langle 2\rangle}_{\varepsilon}(K_n^{\langle 2\rangle}) \geq \limsup_{n\to\infty} m^{\langle 2\rangle}_{\varepsilon_n}(K_n^{\langle 2\rangle}) \geq \delta \tag{4.44}$$

for each $\varepsilon > 0$, so letting $\varepsilon \downarrow 0$, using Lemma 4.14, we conclude that $m^{\langle 2\rangle}_0(L) \geq \delta$. By Lemma 4.16, we conclude that $L \notin \mathcal{L}(X)$ and hence $\overline{\mathcal{B}}$ is not contained in $\mathcal{L}(\mathcal{X})$.

Assume now that (2.18) (ii) holds. We must show that $\overline{\mathcal{B}}$ is contained in $\mathcal{L}(\mathcal{X})$. Assume that $K_n^{\langle 2\rangle} \to L$ for some $K_n \in \mathcal{A}$. Then clearly $L \in \overline{\mathcal{L}(\mathcal{X})}$ so by Lemma 4.16 it suffices to prove that $m^{\langle 2\rangle}_0(L) = 0$. Assume that, conversely, there exist $x, y \in \mathcal{X}$ with $x \neq y$ such that $(x, y), (y, x) \in L$. Then by Lemma 2.6, there exist $x^n_1, x^n_2, y^n_1, y^n_2 \in K_n$ with $x^n_1 \preceq y^n_1$ and $y^n_2 \preceq x^n_2$ such that $x^n_i \to x$ and $y^n_i \to y$ as $n \to \infty$ $(i = 1, 2)$. Then for each $\varepsilon > 0$, we can choose n large enough such that $d(x^n_i, x) \leq \varepsilon/2$ $(i = 1, 2)$. It follows that $d(x^n_1, y^n_1) \vee d(x^n_2, y^n_2) \geq d(x, y) - \varepsilon$ and $d(x^n_1, x^n_2) \vee d(y^n_1, y^n_2) \geq \varepsilon$ so that $m_\varepsilon(K_n) \geq d(x, y) - \varepsilon$. This clearly contradicts (2.18) (ii), so we conclude that $m^{\langle 2\rangle}_0(L) = 0$ as required. ∎

4.7 Cadlag curves

In this subsection, we prove Proposition 2.13. Let $(\mathcal{X}, d)$ be a metric space. If R is any subset of $\mathcal{X}^2$, then let us call

$$\mathrm{dist}(R) := \sup_{(x_1,x_2)\in R} d_{\mathrm{sqz}}(x_1, x_2) \tag{4.45}$$

the *distortion* of R. Then (2.8) and (2.15) say that

$$d_{\mathrm{H}}(K_1, K_2) = \inf_{R\in\mathrm{Corr}(K_1,K_2)} \mathrm{dist}(R) \quad \text{and} \quad d_{\mathrm{tot}}(K_1, K_2) = \inf_{R\in\mathrm{Corr}_+(K_1,K_2)} \mathrm{dist}(R), \tag{4.46}$$

where $\mathrm{Corr}(K_1, K_2)$ and $\mathrm{Corr}_+(K_1, K_2)$ denote the sets of all (monotone) correspondences between K_1 and K_2. Let $\overline{R}$ denote the closure of a set $R \subset \mathcal{X}^2$. Then

$$\mathrm{dist}(R) = \mathrm{dist}(\overline{R}) \qquad (R \subset \mathcal{X}^2). \tag{4.47}$$

Indeed, the inequality $\leq$ is trivial, while the opposite inequality follows from the fact that for each $(x_1, x_2) \in \overline{R}$, there exist $(x^n_1, x^n_2) \in R$ such that $(x^n_1, x^n_2) \to (x_1, x_2)$ and hence $d(x^n_1, x^n_2) \to d(x_1, x_2)$.

We need one preparatory lemma.

Lemma 4.18 (Fine partition) *Let $(\mathcal{X}, d)$ be a metric space. Then for each $\gamma \in \mathcal{D}_{[0,1]}(\mathcal{X})$ and $\varepsilon > 0$, there exist $t_0 < 0 < t_1 < \cdots < t_{n-1} < 1 < t_n$ such that*

$$\sup\big\{d\big(\gamma(s), \gamma(s')\big) : 1 \leq k \leq n,\ s, t \in [0, 1] \cap [t_{k-1}, t_k)\big\} < \varepsilon. \tag{4.48}$$

Proof By Lemma 2.4, writing $\gamma(t+) := \gamma(t)$ and $\gamma(t-) := \gamma^-(t)$, where γ^- is the caglad modification of γ, we can view γ as a continuous function on the split real interval $[0-, 1+]$, that moreover satisfies $\gamma(0-) = \gamma(0+)$ and $\gamma(1-) = \gamma(1+)$. Fix $\varepsilon > 0$ and let

$$R := \big\{(s,t) \in \mathbb{R}^2 : s < t,\ s \neq 0,\ t \neq 1,\ d\big(\gamma(\sigma), \gamma(\tau)\big) < \varepsilon\ \forall \sigma, \tau \in [s+, t-] \cap [0-, 1+]\big\}. \quad (4.49)$$

Using the properties of γ, it is easy to see that

$$\bigcup_{(s,t)\in R} [s+, t-] \supset [0-, 1+]. \quad (4.50)$$

Since the intervals $[s+, t-]$ are open in the topology of the split real line and since $[0-, 1+]$ is compact by Proposition 2.3, there exists a finite subcover, i.e., there exists a finite set $S \subset R$ such that

$$\bigcup_{(s,t)\in S} [s+, t-] \supset [0-, 1+]. \quad (4.51)$$

Let $T := \{s : (s,t) \in S\} \cup \{t : (s,t) \in S\}$. Then, letting t_0 denote the largest element of $T \cap (-\infty, 0)$, ordering the elements of $T \cap (0,1)$ as $t_1 < \cdots < t_{n-1}$, and letting t_n denote the smallest element of $T \cap (1, \infty)$, we obtain times $t_0 < \cdots < t_n$ as in (4.48). ∎

Proof of Proposition 2.13 If γ_1, γ_2 are cadlag parametrisations of K_1, K_2, and $\lambda \in \Lambda$, then let us set

$$R_\lambda := \big\{\big(\gamma_1(t), \gamma_2\big(\lambda(t)\big)\big) : t \in [0,1]\big\} = \big\{\big(\gamma_1\big(\lambda^{-1}(t)\big), \gamma_2(t)\big) : t \in [0,1]\big\}, \quad (4.52)$$

and let $\overline{R}_\lambda$ denote its closure. We claim that $\overline{R}_\lambda$ is a correspondence between K_1 and K_2. To see this, let γ_i^- denote the caglad modification of γ_i ($i = 1, 2$). By the definition of a cadlag parametrisation, each element $x_1 \in K_1$ is of the form $x_1 = \gamma_1(t)$ or $= \gamma_1^-(t)$ for some $t \in [0,1]$. If $x_1 = \gamma_1(t)$, then clearly there exists an $x_2 \in K_2$ such that $(x_1, x_2) \in R_\lambda$, namely $x_2 := \gamma_2(\lambda(t))$. If $x_1 = \gamma_1^-(t)$, then we can choose $t_n \uparrow t$ and set $x_1^n := \gamma_1(t_n)$. Then $x_1^n \to x_1$ by the left continuity of γ_1^-. We have already seen that there exist $x_2^n \in K_2$ such that $(x_1^n, x_2^n) \in R_\lambda$. Since K_2 is compact, by going to a subsequence, we can assume that $x_2^n \to x_2$ for some $x_2 \in K_2$. Then $(x_1, x_2) \in \overline{R}_\lambda$. In the same way, we see that for each $x_2 \in K_2$, there exists an $x_1 \in K_1$ such that $(x_1, x_2) \in \overline{R}_\lambda$. This completes the proof that $\overline{R}_\lambda$ is a correspondence. Using the fact that the total orders on K_1 and K_2 are compatible with the topology, it is easy to see that $\overline{R}_\lambda$ is monotone if $\lambda \in \Lambda_+$.

Using these facts as well as (4.46) and (4.47), we see that

$$\begin{aligned} d_{\mathrm{H}}(K_1, K_2) &\leq \inf_{\lambda\in\Lambda} \mathrm{dist}(\overline{R}_\lambda) = \inf_{\lambda\in\Lambda} \mathrm{dist}(R_\lambda) = \inf_{\lambda\in\Lambda} \sup_{t\in[0,1]} d\big(\gamma_1(t), \gamma_2\big(\lambda(t)\big)\big), \\ d_{\mathrm{tot}}(K_1, K_2) &\leq \inf_{\lambda\in\Lambda_+} \mathrm{dist}(\overline{R}_\lambda) = \inf_{\lambda\in\Lambda_+} \mathrm{dist}(R_\lambda) = \inf_{\lambda\in\Lambda_+} \sup_{t\in[0,1]} d\big(\gamma_1(t), \gamma_2\big(\lambda(t)\big)\big). \end{aligned} \quad (4.53)$$

To complete the proof, we must show that:

(i) For each $R \in \mathrm{Corr}(K_1, K_2)$ and $\varepsilon > 0$, there exists a $\lambda \in \Lambda$ such that $\mathrm{dist}(R_\lambda) \leq \mathrm{dist}(R) + \varepsilon$.

(ii) For each $R \in \mathrm{Corr}_+(K_1, K_2)$ and $\varepsilon > 0$, there exists a $\lambda \in \Lambda_+$ such that $\mathrm{dist}(R_\lambda) \leq \mathrm{dist}(R) + \varepsilon$.

We first prove (i). Fix $R \in \mathrm{Corr}(K_1, K_2)$ and $\varepsilon > 0$. By Lemma 4.18, for $i = 1, 2$, there exist $t_0^i < 0 < t_1^i < \cdots < t_{n_i-1}^i < 1 < t_{n_i}^i$ such that

$$\sup\big\{d\big(\gamma_i(s), \gamma_i(s')\big) : 1 \leq k \leq n_i,\ s, t \in [0,1] \cap [t_{k-1}^i, t_k^i)\big\} < \varepsilon/2 \quad (i = 1, 2). \quad (4.54)$$

For $1 \le k \le n_i$, let us write $K^i_k := \{\gamma_i(t) : t \in [0,1] \cap [t^i_{k-1}, t^i_k)\}$ $(i = 1,2)$. We can choose a correspondence S between $\{1,\ldots,n_1\}$ and $\{1,\ldots,n_2\}$ such that for each $(k_1,k_2) \in S$, there exists an $(x_1,x_2) \in R$ with $x_i \in K^i_{k_i}$ $(i = 1,2)$. Then

$$\sup_{(k_1,k_2)\in S} \sup_{\substack{x_1\in K_1\\ x_2\in K_2}} d(x_1,x_2) \le \mathrm{dist}(R) + \varepsilon. \tag{4.55}$$

By refining the partitions $t^i_0,\ldots,t^i_{n_i}$, we can make sure that for each $k_1 \in \{1,\ldots,n_1\}$, there is a unique $k_2 \in \{1,\ldots,n_2\}$ such that $(k_1,k_2) \in S$, and vice versa. We can then construct a bijection $\lambda : [0,1] \to [0,1]$ such that for each $(k_1,k_2) \in S$, the restriction of λ to $[0,1] \cap [t^1_{k_1-1}, t^1_{k_1})$ is a bijection to $[0,1] \cap [t^2_{k_2-1}, t^2_{k_2})$. Then (4.55) implies that $\mathrm{dist}(R_\lambda) \le \mathrm{dist}(R) + \varepsilon$, completing the proof of (i).

To also prove (ii), we observe that if R is a monotone correspondence, then S as we initially constructed it will be a monotone correspondence between $\{1,\ldots,n_1\}$ and $\{1,\ldots,n_2\}$, and monotonicity will be preserved after we refine the partitions so that they have the same size and S is a bijection. Now λ can be chosen monotone too, completing the proof of (ii). ∎

4.8 Betweenness

In this subsection, we prove Lemmas 2.14, 2.15, 2.16, and 2.17, as well as Lemma 4.19 below that will be used in the proof of Lemma 3.1.

Proof of Lemma 2.14 Clearly, (ii) and (iii) imply (v) and (iv) implies (vi). To prove (vii), we first observe that $x \in \langle y,z\rangle$ and $y \in \langle x,z\rangle$ imply by (vi) $\langle x,z\rangle \subset \langle y,z\rangle \subset \langle x,z\rangle$ and hence $\langle x,z\rangle = \langle y,z\rangle$. Using this as well as the assumptions $y \in \langle x,z\rangle$ and $x \in \langle y,z\rangle$ we can conclude by (i) and (iii) that $\{y\} = \langle x,y\rangle \cap \langle y,z\rangle = \langle y,x\rangle \cap \langle x,z\rangle = \{x\}$ and hence $x = y$.

To prove (viii), assume that $y,y' \in \langle x,z\rangle$ and $y' \in \langle x,y\rangle$. The statement is trivial if $y = y'$ so without loss of generality we assume that $y \ne y'$. Since $y' \in \langle x,z\rangle$ we have by (iv) that $\langle x,z\rangle = \langle x,y'\rangle \cup \langle y',z\rangle$. Since also $y \in \langle x,z\rangle$ we must have either $y \in \langle x,y'\rangle$, or $y \in \langle y',z\rangle$, or both. The first possibility would by (i) and (vii) and the fact that $y' \in \langle x,y\rangle$ imply that $y = y'$, which contradicts our assumptions, so we conclude that $y \in \langle y',z\rangle$.

The first implication $\Rightarrow$ in (2.21) follow from the fact that $y \in \langle x,y\rangle$ by (i) and (ii), while the reverse implication follows from (vi). The second equivalence in (2.21) follows from (viii) and the third equivalence follows from the first one, by the symmetry (i). It is clear that (2.21) defines a partial order $\le_{x,z}$ on $\langle x,z\rangle$. By (iv), if $y,y' \in \langle x,z\rangle$, then at least one of the conditions $y' \in \langle x,y\rangle$ and $y' \in \langle y,z\rangle$ must hold, which shows that $\le_{x,z}$ is a total order. ∎

Proof of Lemma 2.15 We need to check that our definition satisfies axioms (i)–(iv) of a betweenness. Axioms (i) and (ii) are trivial. To prove (iii) and (iv), set $r := d(x,z)$ and let $\gamma : [0,r] \to \mathcal{X}$ be the unique isometry such that $\gamma(0) = x$ and $\gamma(r) = z$. Since an isometry is one-to-one, there exists a unique $p \in [0,r]$ such that $\gamma(p) = y$. Clearly, the restrictions of γ to $[0,p]$ and $[p,r]$ are isometries, so $\langle x,y\rangle = \{\gamma(t) : 0 \le t \le p\}$ and $\langle y,z\rangle = \{\gamma(t) : p \le t \le r\}$. From these observations, axioms (iii) and (iv) follow immediately. ∎

We skip the proof of Lemma 2.16 for the moment and first prove Lemma 2.17 and the already announced Lemma 4.19.

Proof of Lemma 2.17 For the trivial betweenness, $\langle x,z\rangle = \{x,z\}$ is clearly compact for each $x,z \in \mathcal{X}$, and the continuity of the map $(x,z) \mapsto \{x,z\}$ with respect to the Hausdorff topology follows immediately from Lemma 2.6.

If a betweenness is generated by an interpolation function, then $\langle x,z\rangle$, being the image of $[0,1]$ under the continuous map $p\mapsto\varphi(x,z,p)$, is clearly compact for all $x,z\in\mathbb{Z}$. Let $x_n\to x$ and $z_n\to z$. To show that $\langle x_n,z_n\rangle\to\langle x,z\rangle$ in the Hausdorff topology, we check the conditions of Lemma 2.6. Since $x_n\to x$ and $z_n\to z$, the sets $A:=\{x\}\cup\{x_n:n\in\mathbb{N}\}$ and $B:=\{z\}\cup\{z_n:n\in\mathbb{N}\}$ are compact. Let $\varphi(A\times B\times[0,1])$ denote the image of $A\times B\times[0,1]$ under φ, which is compact. Clearly $\langle x_n,z_n\rangle\subset\varphi(A\times B\times[0,1])$ for all n. To complete the argument, it suffices to show that

$$\begin{aligned}&\big\{y\in\mathcal{X}:\exists y_n\in\langle x_n,z_n\rangle\text{ s.t. }y\text{ is a cluster point of }(y_n)_{n\in\mathbb{N}}\big\}\\&\quad\subset\langle x,z\rangle\subset\big\{y\in\mathcal{X}:\exists y_n\in\langle x_n,z_n\rangle\text{ s.t. }y_n\to y\big\}.\end{aligned}\tag{4.56}$$

For the first inclusion, assume that y is a cluster point of $y_n=\varphi(x_n,z_n,p_n)$. By going to a subsequence, we can assume that $p_n\to p$ for some $p\in[0,1]$. Then $y=\varphi(x,z,p)\in\langle x,z\rangle$. For the second inclusion, assume that $y=\varphi(x,z,p)$ for some $p\in[0,1]$. Then $y_n:=\varphi(x_n,z_n,p)\in\langle x_n,z_n\rangle$ converge to y, completing the proof that each betweenness that is generated by an interpolation function is compatible with the topology.

For the final statement of the lemma, assume that $\mathcal{X}$ is a closed subset of $\mathbb{R}$. Then clearly $\langle x,z\rangle:=[x,z]\cap\mathcal{X}$ is compact for each $x,z\in\mathbb{R}$. Let $x_n\to x$ and $z_n\to z$. To show that $\langle x_n,z_n\rangle\to\langle x,z\rangle$ in the Hausdorff topology, we again check the conditions of Lemma 2.6. Clearly, $\langle x_n,z_n\rangle\subset[S,T]\cap\mathcal{X}$ for each n, where $S:=\inf_n x_n$ and $T:=\sup_n z_n$, so to complete the argument, it again suffices to check (4.56). For the first inclusion, assume that $y\in\mathcal{X}$ is a cluster point of $y_n\in\langle x_n,z_n\rangle$. Since $x_n\leq y_n\leq z_n$ for each n, taking the limit, we see that $x\leq y\leq z$ and hence $y\in\langle x,z\rangle$. For the second inclusion, assume that $y\in\langle x,z\rangle$. If $x<y<z$, then $x_n<y<z_n$ for all n large enough, so setting $y_n:=y$ for n large enough and $y_n:=x_n$ otherwise proves that y is al element of the set on the right-hand side of (4.56). If $y\in\{x,z\}$, then setting $y_n:=x_n$ or $:=z_n$ proves the same claim, so the proof is complete. ∎

The following lemma will be used in the proof of Lemma 3.1, and is also of independent interest.

Lemma 4.19 (Segments as ordered sets) *Let $\mathcal{X}$ be a metrisable space that is equipped with a betweenness that is compatible with the topology. Then for each $x,z\in\mathcal{X}$, the segment $\langle x,z\rangle$ equipped with the total order $\leq_{x,z}$ is an element of $\mathcal{K}_{\rm tot}(\mathcal{X})$, and the map $(x,z)\mapsto\langle x,z\rangle$ is continuous with respect to the product topology on $\mathcal{X}^2$ and the topology on $\mathcal{K}_{\rm tot}(\mathcal{X})$.*

Proof To show that $\langle x,z\rangle$ is an element of $\mathcal{K}_{\rm tot}(\mathcal{X})$, we must show that the total order $\leq_{x,z}$ is compatible with the induced topology on $\langle x,z\rangle$. Assume that $y_n,y'_n,y,y'\in\langle x,z\rangle$ satisfy $y_n\to y$, $y'_n\to y'$, and $y_n\leq_{x,z}y'_n$ for all n. Then $y_n\in\langle x,y'_n\rangle$ for all n. Since the betweenness is compatible with the topology, $\langle x,y'_n\rangle\to\langle x,y'\rangle$ in the Hausdorff topology, which by Lemma 2.6 implies that $y\in\langle x,y'\rangle$ and hence $y\leq_{x,z}y'$. This shows that the total order $\leq_{x,z}$ is compatible with the induced topology on $\langle x,z\rangle$.

To show that the map $(x,z)\mapsto\langle x,z\rangle$ is continuous with respect to the topology on $\mathcal{K}_{\rm tot}(\mathcal{X})$, assume that $x_n\to x$, $z_n\to z$. We will show that

$$d_{\rm part}\big(\langle x_n,z_n\rangle,\langle x,z\rangle\big)\underset{n\to\infty}{\longrightarrow}0,\tag{4.57}$$

which is equivalent to the statement that $\langle x_n,z_n\rangle^{\langle 2\rangle}$ converges to $\langle x,z\rangle^{\langle 2\rangle}$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X}^2)$. We apply Lemma 2.6. Since $\langle x_n,z_n\rangle\to\langle x,z\rangle$, there exists a compact $C\subset\mathcal{X}$ such that $\langle x_n,z_n\rangle\subset C$ for all n and hence $\langle x_n,z_n\rangle^{\langle 2\rangle}\subset C^2$ for all n. Thus, it suffices to check that (compare (4.56))

$$\begin{aligned}&\big\{(y,y')\in\mathcal{X}^2:\exists y_n,y'_n\in\langle x_n,z_n\rangle \text{ with } y_n\leq_{x_n,z_n} y'_n \text{ s.t. } (y,y') \text{ is a cluster point of } (y_n,y'_n)_{n\in\mathbb{N}}\big\}\\ &\quad\subset\langle x,z\rangle^{\langle 2\rangle}\subset\big\{(y,y')\in\mathcal{X}:\exists y_n,y'_n\in\langle x_n,z_n\rangle \text{ with } y_n\leq_{x_n,z_n} y'_n \text{ s.t. } (y_n,y'_n)\to(y,y')\big\}.\end{aligned} \tag{4.58}$$

If (y,y') is an element of the set on the left-hand side of (4.58) and (y_n,y'_n) fulfill the conditions of the definition of this set, then by going to a subsequence we may assume that $(y_n,y'_n)\to(y,y')$. Then $y,y'\in\langle x,z\rangle$ since $\langle x_n,z_n\rangle\to\langle x,z\rangle$. Moreover $y_n\leq_{x_n,z_n} y'_n$ means $y_n\in\langle x_n,y'_n\rangle$. Since $y_n\to y$ and $\langle x_n,y'_n\rangle\to\langle x,y'\rangle$, this implies $y\in\langle x,y'\rangle$ and hence $y\leq_{x,z} y'$, proving that $(y,y')\in\langle x,z\rangle^{\langle 2\rangle}$.

To prove the second inclusion in (4.58), assume that $(y,y')\in\langle x,z\rangle^{\langle 2\rangle}$. Since $\langle x_n,z_n\rangle\to\langle x,z\rangle$, there exist $y_n,y'_n\in\langle x_n,z_n\rangle$ such that $y_n\to y$ and $y'_n\to y'$. We now distinguish two cases: $y\neq y'$ and $y=y'$. If $y\neq y'$, then we claim that $y_n\leq_{x_n,z_n} y'_n$ for all n large enough. Indeed, in the opposite case, since $\leq_{x_n,z_n}$ is a total order, by going to a subsequence, we can assume that $y'_n\leq_{x_n,z_n} y_n$ for all n, which by the arguments we have already seen implies $y'\leq_{x,z} y$, so that by the fact that $(y,y')\in\langle x,z\rangle^{\langle 2\rangle}$ we must have $y=y'$, contradicting our assumption. Since $y_n\leq_{x_n,z_n} y'_n$ for all n large enough, changing the definitions of y_n,y'_n for finitely many n, we see that there exist $y_n,y'_n\in\langle x_n,z_n\rangle$ with $y_n\leq_{x_n,z_n} y'_n$ such that s.t. $(y_n,y'_n)\to(y,y')$. In the case $y=y'$ the argument is even simpler, since now $(y_n,y_n)\to(y,y')$ while obviously $y_n\leq_{x_n,z_n} y_n$ for all n. ∎

In this subsection, it only remains to prove Lemma 2.16. The statement about the linear betweenness is trivial, but before we can prove the statement about the geodesic betweenness, we first need a better understanding of metric spaces with unique geodesics, which is provided by Proposition 4.20 below. In any metric space $(\mathcal{X},d)$, for all $x,z\in\mathcal{X}$ and $\varepsilon\geq 0$, we define

$$\eta_{x,z}(\varepsilon):=\sup\big\{d(y_1,y_2):\big[d(x,y_1)\wedge d(x,y_2)\big]+\big[d(y_1,z)\wedge d(y_2,z)\big]\leq d(x,z)+\varepsilon\big\}. \tag{4.59}$$

In other words, this is the largest distance between two points $y_1,y_2\in\mathcal{X}$ for which there exist constants $r,r'\geq 0$ with $r+r'\leq d(x,z)+\varepsilon$ such that $d(x,y_i)\leq r$ and $d(y_i,z)\leq r'$ $(i=1,2)$.

Proposition 4.20 (Unique geodesics) *Let $(\mathcal{X},d)$ be a metric space. Consider the following conditions.*

- (i) *For all $x,z\in\mathcal{X}$ and $r,r'\geq 0$ with $r+r'=d(x,z)$, there exists an $y\in\mathcal{X}$ such that $d(x,y)=r$ and $d(y,z)=r'$.*
- (ii) *$\eta_{x,z}(0)=0$ for all $x,z\in\mathcal{X}$.*
- (ii)' *$\lim_{\varepsilon\to 0}\eta_{x,z}(\varepsilon)=0$ for all $x,z\in\mathcal{X}$.*

Then $(\mathcal{X},d)$ has unique geodesics if and only if (i) and (ii) hold. Moreover, (ii)' implies (ii), and if $(\mathcal{X},d)$ is a proper metric space, then (ii)' implies (ii).

Proof In any metric space $(\mathcal{X},d)$, let us introduce the notation

$$\langle x,z\rangle:=\big\{y\in\mathcal{X}:d(x,y)+d(y,z)=d(x,z)\big\}. \tag{4.60}$$

We claim that

$$y\in\langle x,z\rangle,\ y'\in\langle x,y\rangle,\ y''\in\langle y,z\rangle\quad\Rightarrow\quad y\in\langle y',y''\rangle. \tag{4.61}$$

To see this, we note that if the assumptions in (4.61) hold but the conclusion does not, then by the triangle inequality

$$\begin{aligned} d(x,z) = d(x,y) + d(y,z) &= d(x,y') + d(y',y) + d(y,y'') + d(y'',z) \\ &> d(x,y') + d(y',y'') + d(y'',z), \end{aligned} \tag{4.62}$$

which contradicts the triangle inequality. Now let $(\mathcal{X}, d)$ be a metric space with unique geodesics and let $\langle\!\langle x, z\rangle\!\rangle$ denote the unique geodesic with endpoints x, z. Clearly $\langle\!\langle x, z\rangle\!\rangle \subset \langle x, z\rangle$. We claim that

$$y \in \langle x, z\rangle \quad \Rightarrow \quad \langle\!\langle x, y\rangle\!\rangle \cup \langle\!\langle y, z\rangle\!\rangle = \langle\!\langle x, z\rangle\!\rangle. \tag{4.63}$$

To see this, let $r := d(x, y)$, $r' := d(y, z)$, and let $\gamma : [0, r] \to \mathcal{X}$ and $\gamma'' : [r, r + r'] \to \mathcal{X}$ be the unique isometries with $\gamma(0) = x$, $\gamma(r) = \gamma'(r) = y$, and $\gamma'(r + r') = z$. We claim that $\gamma'' : [0, r + r'] \to \mathcal{X}$ defined as $\gamma''(t) = \gamma(t)$ for $t \in [0, r]$ and $:= \gamma'(t)$ for $t \in [r, r + r']$ is an isometry. So see this, let $0 \le t' < t'' \le r + r'$. We need to show that $d\big(\gamma''(t'), \gamma''(t'')\big) = t'' - t'$. This is clear when $t'' \le r$ or $r \le t''$, while in the remaining case $t' < r < t''$ the claim follows from (4.61).

We now prove that if $(\mathcal{X}, d)$ is a metric space with unique geodesics, then conditions (i) and (ii) are satisfied. Condition (i) is trivial. To prove (ii), let $x, z \in \mathcal{X}$, let $r, r' \ge 0$ satisfy $r + r' := d(x, z)$, and assume that $y_1, y_2 \in \mathcal{X}$ satisfy $d(x, y_i) = r$, $d(y_i, z) = r'$ $(i = 1, 2)$. By (4.63), there exist isometries $\gamma_i : [0, r + r'] \to \mathcal{X}$ with $\gamma_i(0) = x$, $\gamma_i(r) = y_i$, and $\gamma_i(r + r') = z$, so by the assumption that $(\mathcal{X}, d)$ has unique geodesics we conclude that $y_1 = y_2$, proving (ii).

Conversely, if $(\mathcal{X}, d)$ is a metric space for which (i) and (ii) hold, then for each $x, z \in \mathcal{X}$ with $r := d(x, z)$, we can uniquely define $\gamma_{x,z} : [0, r] \to \mathcal{X}$ by

$$\gamma_{x,z}(t) := y \quad \text{with} \quad d(x, y) = t,\ d(y, z) = r - t. \tag{4.64}$$

Clearly, if $\gamma : [0, r] \to \mathcal{X}$ is an isometry with $\gamma(0) = x$ and $\gamma(r) = z$, then we must have $\gamma = \gamma_{x,z}$, so to prove that $(\mathcal{X}, d)$ has unique geodesics, it suffices to show that $\gamma_{x,z}$ is an isometry. Let $0 \le t_1 \le t_2 \le r$ and let $y_i := \gamma_{x,z}(t_i)$ $(i = 1, 2)$. Set $y'_1 := \gamma_{x,y_2}(t_1)$. Then $d(x, y'_1) = t_1$ and $d(y'_1, z) \le d(y'_1, y_2) + d(y_2, z) = r - t_1$, which by the assumption (ii) implies $y'_1 = y_1$. Since $d(y'_1, y_2) = t_2 - t_1$, this proves that $\gamma_{x,z}$ is an isometry. This completes the proof that a metric space $(\mathcal{X}, d)$ has unique geodesics if and only if (i) and (ii) hold.

Trivially, (ii)' implies (ii), so to complete the proof of the proposition, it suffices to prove that for proper metric spaces, (ii) implies (ii)'. Assume that (ii)' does not hold for some $x, z \in \mathcal{X}$. Let $0 < \varepsilon_n \le 1$ satisfy $\varepsilon_n \to 0$. Then for some $\delta > 0$, we can find $y^n_1, y^n_2 \in \mathcal{X}$ with $d(y^n_1, y^n_2) \ge \delta$, as well as $r_n, r'_n \ge 0$ with $r_n + r'_n \le d(x, z) + \varepsilon_n$, such that $d(x, y^n_i) \le r_n$ and $d(y^n_i, z) \le r'_n$ $(i = 1, 2)$. Since $d(x, y^n_i) \le d(x, z) + 1$ for all n, by the properness assumption, we can select a subsequence such that $y^n_i \to y_i$ for some $y_1, y_2 \in \mathcal{X}$. Since $r_n + r'_n \le d(x, z) + 1$, by going to a further subsequence, we can assume that $r_n \to r$ and $r'_n \to r'$ for some $r, r' \ge 0$ with $r + r' = d(x, z)$. Then $d(x, y_i) \le r$ and $d(y_i, z) \le r'$ while $d(y_1, y_2) \ge \delta$ which shows that $\eta_{x,z}(0) \ge \delta$, violating (ii). ∎

If a metric space $(\mathcal{X}, d)$ has unique geodesics, then by conditions (i) and (ii) of Proposition 4.20, we can uniquely define a function $\varphi : \mathcal{X}^2 \times [0, 1] \to \mathcal{X}$ by

$$\varphi(x, z, p) := y \quad \text{with} \quad d(x, y) = pd(x, z) \text{ and } d(y, z) = (1 - p)d(x, z). \tag{4.65}$$

Lemma 2.16 is now implied by Proposition 4.20 and the following lemma (the statement in Lemma 2.16 about normed linear spaces being trivial).

Lemma 4.21 (Geodesic interpolation function) *Let $(\mathcal{X}, d)$ be a metric space with unique geodesics and let φ be defined as in (4.65). Then for each $x, z \in \mathcal{X}$, the unique geodesic with endpoints x, z is given by $\{\varphi(x, z, p) : p \in [0, 1]\}$. If condition (ii)' of Proposition 4.20 is satisfied, then $\varphi : \mathcal{X}^2 \times [0, 1] \to \mathcal{X}$ is continuous.*

Proof Let $x, z \in \mathcal{X}$ and $r := d(x, z)$. We observe that $\{\varphi(x, z, p) : p \in [0, 1]\} = \{\gamma_{x,z}(t) : t \in [0, r]\}$ where $\gamma_{x,z}$ is defined as in (4.64). It has already been shown in the proof of Proposition 4.20 that this is the unique geodesic with endpoints x, z. Therefore, to complete the proof, it suffices to show that condition (ii)' of Proposition 4.20 implies that φ is continuous.

Assume that $x_n, x, z_n, z \in \mathcal{X}$ and $p_n, p \in [0, 1]$ satisfy $x_n \to x$, $z_n \to z$, and $p_n \to p$. Set $y_n := \varphi(x_n, z_n, p_n)$ and $y := \varphi(x, z, p)$. We have to show that $y_n \to y$. We observe that

$$\begin{aligned} d(x, y_n) + d(y_n, z) &\leq d(x_n, y_n) + d(y_n, z_n) + d(x, x_n) + d(z, z_n) \\ &= d(x_n, z_n) + d(x, x_n) + d(z, z_n) \underset{n\to\infty}{\longrightarrow} d(x, z). \end{aligned} \tag{4.66}$$

Thus, for each $\varepsilon > 0$, we can find an m such that $d(x, y_n) + d(y_n, z) \leq d(x, z) + \varepsilon$ for all $n \geq m$. Since moreover $d(x, y) + d(y, z) = d(x, z)$, it follows that $d(y_n, y) \leq \eta_{x,z}(\varepsilon)$ for all $n \geq m$. Since $\varepsilon > 0$ is arbitrary, by (ii)', this implies $d(y_n, y) \to 0$. ∎

4.9 Squeezed space

In this subsection, we prove Lemmas 2.18, 2.19, and 2.20.

Proof of Lemma 2.18 We first prove that $d_{\rm sqz}$ is a metric on $\mathcal{R}(\mathcal{X})$. For brevity, we write $d'(x, y) := d(x, y) \wedge 1$. Then d' is a metric on E. The only nontrivial statement that we have to prove is the triangle inequality, and it suffices to prove this for the function

$$\rho\big((x, s), (y, t)\big) := \big(\phi(s) \wedge \phi(t)\big) d'(x, y) + \big|\phi(s) - \phi(t)\big|.$$

We estimate

$$\rho\big((x, s), (z, u)\big) \leq \big(\phi(s) \wedge \phi(u)\big)\big(d'(x, y) + d'(y, z)\big) + \big|\phi(s) - \phi(u)\big|. \tag{4.67}$$

If $\phi(t) \geq \phi(s) \wedge \phi(u)$, then $\phi(s) \wedge \phi(u)$ is less than $\phi(s) \wedge \phi(t)$ and also less than $\phi(t) \wedge \phi(u)$, so we can simply estimate the expression in (4.67) from above by

$$\big(\phi(s) \wedge \phi(t)\big) d'(x, y) + \big(\phi(t) \wedge \phi(u)\big) d'(y, z)\big) + \big|\phi(s) - \phi(t)\big| + \big|\phi(t) - \phi(u)\big|$$

and we are done. On the other hand, if $\phi(t) < \phi(s) \wedge \phi(u)$, then

$$\big|\phi(s) - \phi(t)\big| + \big|\phi(t) - \phi(u)\big| = \big|\phi(s) - \phi(u)\big| + 2\big(\phi(s) \wedge \phi(u) - \phi(t)\big).$$

Using the fact that $d' \leq 1$, we can now estimate the right-hand side of (4.67) from above by

$$\begin{aligned} &\phi(t)\big(d'(x, y) + d'(y, z)\big) + 2\big(\phi(s) \wedge \phi(u) - \phi(t)\big) + \big|\phi(s) - \phi(u)\big| \\ &\quad = \big(\phi(s) \wedge \phi(t)\big) d'(x, y) + \big(\phi(t) \wedge \phi(u)\big) d'(y, z) \\ &\qquad + \big|\phi(s) - \phi(t)\big| + \big|\phi(t) - \phi(u)\big|, \end{aligned}$$

and again we are done. This completes the proof that $d_{\rm sqz}$ is a metric on $\mathcal{R}(\mathcal{X})$.

It remains to prove that

$$\big(\phi(t_n) \wedge \phi(t)\big)\big(d(x_n, x) \wedge 1\big) + \big|\phi(t_n) - \phi(t)\big| + d_{\overline{\mathbb{R}}}(t_n, t) \underset{n\to\infty}{\longrightarrow} 0 \tag{4.68}$$

if and only if conditions (i) and (ii) of the lemma are satisfied. Because of the third term on the left-hand side, a necessary condition for (2.26) is that $t_n \to t$, and this condition also guarantees that the second term tends to zero. If $t \in \{-\infty, +\infty\}$, then this is all one needs since the first term now tends to zero regardless of the values of x_n and x, but if $t \in \mathbb{R}$, then one needs in addition that $d(x_n, x) \to 0$. ∎

Proof of Lemma 2.19 If D is a countable dense subset of $(\mathcal{X}, d)$, then $D \times \mathbb{Q}$ is a countable dense subset of $(\mathcal{R}(\mathcal{X}), d_{\rm sqz})$, proving (a).

To prove (b), let (x_n, t_n) be a Cauchy sequence in $(\mathcal{R}(\mathcal{X}), d_{\rm sqz})$. Then by (2.26) t_n is a Cauchy sequence in $\overline{\mathbb{R}}$ and hence $t_n \to t$ for some $t \in \overline{\mathbb{R}}$. If $t \in \mathbb{R}$, then by (2.26) x_n is a Cauchy sequence in $(\mathcal{X}, d)$ so by the completeness of the latter, $x_n \to x$ for some $x \in \mathcal{X}$. By Lemma 2.18, it follows that (x_n, t_n) converges, proving the completeness of $(\mathcal{R}(\mathcal{X}), d_{\rm sqz})$. ∎

Proof of Lemma 2.20 Assume that $A \subset \mathcal{R}(\mathcal{X})$ has the property that for each $T < \infty$, there exists a compact set $K \subset \mathcal{X}$ such that $\{x \in \mathcal{X} : (x, t) \in A,\ t \in [-T, T]\} \subset K$. To show that A is precompact, we will show that each sequence $(x_n, t_n) \in A$ has a convergent subsequence. By the compactness of $\overline{\mathbb{R}}$, we can select a subsequence (x'_n, t'_n) such that $t'_n \to t$ for some $t \in \overline{\mathbb{R}}$. If $t = \pm\infty$, then by Lemma 2.18 $(x'_n, t'_n) \to (*, \pm\infty)$ and we are done. Otherwise, there exists a $T < \infty$ such that $t'_n \in [-T, T]$ for all n large enough. By assumption, there then exists a compact set $K \subset \mathcal{X}$ such that $x'_n \in K$ for all n large enough, so we can select a further subsequence such that (x''_n, t''_n) converges to a limit $(x, t) \in \mathcal{X} \times \mathbb{R}$.

Assume, on the other hand, that $A \subset \mathcal{R}(\mathcal{X})$ has the property that for some $T < \infty$, there does not exist a compact set $K \subset \mathcal{X}$ such that $\{x \in \mathcal{X} : (x, t) \in A,\ t \in [-T, T]\} \subset K$. Set

$$B := \big\{x \in \mathcal{X} : (x, t) \in A \text{ for some } t \in [-T, T]\big\}$$

The closure of B cannot be compact, since this would contradict our assumption. It follows that there exists a sequence $x_n \in B$ that does not contain a convergent subsequence, and there exist $t_n \in [-T, T]$ such that $(x_n, t_n) \in A$. But then, in view of Lemma 2.18, the sequence (x_n, t_n) cannot contain a convergent subsequence either, proving that A is not precompact. ∎

5 Proofs of the main results

5.1 Closed and filled-in graphs

In this subsection, we prove Lemmas 3.1 and 3.2, as well an analogue of Lemma 3.2 that will later be used in the proof of Theorem 3.6.

Proof of Lemma 3.1 We will show that each sequence $(x_n, t_n) \in \mathcal{G}_{\rm f}(\pi)$ has a subsequence that converges to a limit in $\mathcal{G}_{\rm f}(\pi)$. Since $I(\pi)$ is closed, we can select a subsequence such that $t_n \to t$ for some $t \in I(\pi) \cup \{-\infty, \infty\}$. If $t = \pm\infty$, then Lemma 2.18 tells us that $(x_n, t_n) \to (*, \pm\infty) \in \mathcal{G}_{\rm f}(\pi)$ so we are done, so from now on we can assume that $t \in \mathbb{R}$. By going to a further subsequence, we can assume that we are in one of the following three cases: (i) $t_n < t$ for all n, (ii) $t_n > t$ for all n, and (iii) $t_n = t$ for all n. In case (i), we use the cadlag property of π and the fact that the betweenness is compatible with the topology to see, using Lemma 2.14 (v), that

$$x_n \in \langle \pi(t_n-), \pi(t_n+) \rangle \underset{n\to\infty}{\longrightarrow} \{\pi(t-)\} \tag{5.1}$$

from which we conclude that (x_n, t_n) converges to $\big(\pi(t-), t\big) \in \mathcal{G}_{\rm f}(\pi)$. In case (ii), the same argument shows that (x_n, t_n) converges to $\big(\pi(t+), t\big) \in \mathcal{G}_{\rm f}(\pi)$. In case (iii), finally, using the compactness of $\langle \pi(t-), \pi(t+) \rangle$, we can select a further subsequence such that $(x_n, t) \to (x, t)$ for some $x \in \langle \pi(t-), \pi(t+) \rangle$. Since also in this case the limit (x, t) is an element of $\mathcal{G}_{\rm f}(\pi)$, we are done.

To see that the total order $\preceq$ on $\mathcal{G}_{\rm f}(\pi)$ is compatible with the (induced) topology on $\mathcal{G}_{\rm f}(\pi)$, it suffices to show that

$$S := \big\{ \big((x, s), (y, t) \big) \in \mathcal{G}_{\rm f}(\pi) : (x, s) \prec (y, t) \big\} \tag{5.2}$$

is an open subset of $\mathcal{G}_{\rm f}(\pi)^2$. If $\big((x, s), (y, t)\big)$ is an element of S, then either: (i) $s < t$, or: (ii) $s = t \in \mathbb{R}$ and $x \neq y$. In case (i), we can choose $s < S < T < t$. Then

$$O := \big\{ \big((x', s'), (y', t') \big) \in \mathcal{G}_{\rm f}(\pi) : s' < S,\ T < t' \big\} \tag{5.3}$$

is an open subset of $\mathcal{G}_{\rm f}(\pi)$ such that $\big((x, s), (y, t)\big) \in O \subset S$. In case (ii), we recall that by definition $(x, t) \preceq (y, t)$ if $x \leq_{\pi(t-),\pi(t+)} z$, where by Lemma 4.19 the total order $\leq_{\pi(t-),\pi(t+)}$ on $\langle \pi(t-), \pi(t+) \rangle$ is compatible with the topology. It follows that we can choose $\varepsilon > 0$ small enough such that for $z \in \langle \pi(t-), \pi(t+) \rangle$, if $d(z, x) < \varepsilon$ then $(z, t) \prec (y, t)$, while if $d(z, y) < \varepsilon$ then $(x, t) \prec (z, t)$. Next, we use the cadlag property of π to choose $\delta > 0$ small enough such that $d\big(\pi(s\pm), x\big) > \varepsilon$ for all $t < s < t + \delta$ and $d\big(\pi(s\pm), y\big) > \varepsilon$ for all $t - \delta < s < t$. Then

$$O := \big\{ \big((x', s), (y', u) \big) \in \mathcal{G}_{\rm f}(\pi) : |s - t| \vee |u - t| < \delta,\ d(x', x) \vee d(y', y) < \varepsilon \big\} \tag{5.4}$$

is an open subset of $\mathcal{G}_{\rm f}(\pi)$ such that $\big((x, s), (y, t)\big) \in O \subset S$. Together, these observations prove that S is an open subset of $\mathcal{G}_{\rm f}(\pi)^2$. ∎

The following lemma is similar to Lemma 3.2.

Lemma 5.1 (Characterisation of continuous graphs) *Let $\mathcal{X}$ be a metrisable space. Assume that $G \in \mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ and $(*, \pm\infty) \in G$. Then G is the closed graph of a path $\pi \in \Pi_{\rm c}(\mathcal{X})$ if and only if for each $t \in \mathbb{R}$, the set $\{x \in \mathcal{X} : (x, t) \in G\}$ has at most one element.*

Proof Clearly, if G is the closed graph of a path $\pi \in \Pi_{\rm c}(\mathcal{X})$, then for each $t \in \mathbb{R}$, the set $\{x \in \mathcal{X} : (x, t) \in G\}$ has at most one element. Conversely, if G has this property and $(*, \pm\infty) \in G$, then we define

$$I(\pi) := \big\{ t \in \mathbb{R} : \exists x \in \mathcal{X} \text{ s.t. } (x, t) \in G \big\}, \tag{5.5}$$

and we use the fact that G contains at each time at most one point to define $\pi : I(\pi) \to \mathcal{X}$ by

$$\big\{ \pi(t) \big\} := \{ x \in \mathcal{X} : (x, t) \in G \}. \tag{5.6}$$

We observe that $I(\pi)$ is the intersection of $\mathbb{R}$ with the image of G under the continuous map $(x, t) \mapsto t$. Since the continuous image of a compact set is compact, this proves that $I(\pi)$ is closed. To complete the proof, it suffices to show that $\pi(t_n) \to \pi(t)$ for all $t_n, t \in I(\pi)$ such that $t_n \to t$. It suffices to show that $\{\pi(t_n) : n \in \mathbb{N}\}$ is precompact and its only cluster point is $\pi(t)$. Equivalently, we may show that each subsequence of $\pi(t_n)$ contains a further subsequence that converges to $\pi(t)$. By the compactness of G, for any subsequence, we can select a further subsequence such that $\pi(t_n) \to x$ for some $x \in \mathcal{X}$ with $(x, t) \in G$. But then $x = \pi(t)$ by (5.6). ∎

Proof of Lemma 3.2 By Lemma 3.1, the filled-in graph of a path $\pi \in \Pi(\mathcal{X})$ corresponds to an element of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$. Properties (i) and (ii) now follow from the definition of the total order $\preceq$ on π and property (vi) of Lemma 2.14.

Assume that conversely, $(G, \preceq) \in \mathcal{K}_{\rm tot}(\mathcal{X})$ contains $(*, \pm\infty)$ and satisfies (i) and (ii). We claim that for each $t \in \mathbb{R}$, there exist unique $\pi(t-), \pi(t+) \in \mathcal{X}$ such that $\pi(t-) \preceq \pi(t+)$ and

$$S_t := \big\{ x \in \mathcal{X} : (x, t) \in G \big\} = \langle \pi(t-), \pi(t+) \rangle. \tag{5.7}$$

Indeed, S_t is a compact metrisable set, so we can choose a countable dense set $\{x_n : n \in \mathbb{N}\} \subset S_t$. Set $y_0 := x_0$ and define y_n as the maximum of x_n and y_{n-1} in the total order $\preceq$ $(n \geq 1)$. By the compactness of S_t, by going to a subsequence, we can assume that $y_n \to \pi(t+)$ for some $\pi(t+) \in S_t$. Then $y' \preceq y$ for all $y' \in D$ and hence also for all $y' \in S_t$ since the order is compatible with the topology. In the same way, we see that S_t has a (necessarily unique) minimal element $\pi(t-)$. By (ii), we conclude that $S_t = \langle \pi(t-), \pi(t+) \rangle$. We now define

$$I(\pi) := \big\{t \in \mathbb{R} : \exists x \in \mathcal{X} \text{ s.t. } (x,t) \in G\big\} \quad \text{and} \quad I_{\mathfrak{s}}(\pi) := \big\{t\pm : t \in I(\pi)\big\}, \tag{5.8}$$

and we use the claim we have just proved to define $\pi : I_{\mathfrak{s}}(\pi) \to \mathcal{X}$ by

$$\langle \pi(t-), \pi(t+) \rangle := \{x \in \mathcal{X} : (x,t) \in G\} \quad \text{with} \quad \pi(t-) \preceq \pi(t+) \qquad \big(t \in I(\pi)\big). \tag{5.9}$$

Since $I(\pi)$ is the intersection of $\mathbb{R}$ with the image of G under the continuous map $(x,t) \mapsto t$, which is compact, we see that $I(\pi)$ is closed.

To complete the proof, it suffices to show that $\pi : I_{\mathfrak{s}}(\pi) \to \mathcal{X}$ is continuous. By symmetry, it suffices to show that if $\tau_n \in I_{\mathfrak{s}}(\pi)$ and $t \in I(\pi)$ satisfy $\underline{\tau}_n > t$ for all n and $\underline{\tau}_n \to t$ as $n \to \infty$, then $\pi(\tau_n) \to \pi(t+)$. As in the proof of Lemma 5.1, it suffices to show that each subsequence of $\pi(\tau_n)$ contains a further subsequence that converges to $\pi(t+)$. By the compactness of G, for any subsequence, we can select a further subsequence such that $\pi(\tau_n) \to x$ for some $x \in \mathcal{X}$ such that $(x,t) \in G$. By (iii), we have $\big(\pi(t+), t\big) \preceq \big(\pi(\tau_n), \underline{\tau}_n\big)$ for all n, so using the fact that the total order is compatible with the topology, we see that $\big(\pi(t+), t\big) \preceq (x,t)$, which using the fact that $\pi(t+)$ is the maximal element of $\langle \pi(t-), \pi(t+) \rangle$ with respect to the order $\preceq$ identifies x as $\pi(t+)$. ∎

5.2 Polishness

In this subsection, we prove Propositions 3.3 and 3.4, and Lemma 3.5. Let $\mathcal{X}$ be a metrisable space that is equipped with a betweenness that is compatible with the topology. By a slight abuse of notation, for any $G \in \mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$, we set

$$\begin{aligned} m_{T,\delta}(G) &:= \sup\big\{d(x_1,x_2) : (x_i,t_i) \in G,\ -T \leq t_i \leq T\ \forall i = 1,2, \\ &\qquad\qquad (x_1,t_1) \preceq (x_2,t_2),\ t_2 - t_1 \leq \delta\big\} \\ m^{\rm S}_{T,\delta}(G) &:= \sup\big\{d\big(x_2, \langle x_1, x_3 \rangle\big) : (x_i,t_i) \in G \text{ and } -T \leq t_i \leq T\ \forall i = 1,2,3, \\ &\qquad\qquad (x_1,t_1) \preceq (x_2,t_2) \preceq (x_3,t_3),\ t_3 - t_1 \leq \delta\big\}. \end{aligned} \tag{5.10}$$

Then $m_{T,\delta}(\mathcal{G}_{\rm f}(\pi)) = m_{T,\delta}(\pi)$ for each $\pi \in \Pi_{\rm c}(\mathcal{X})$ and $m^{\rm S}_{T,\delta}(\mathcal{G}_{\rm f}(\pi)) = m^{\rm S}_{T,\delta}(\pi)$ for each $\pi \in \Pi(\mathcal{X})$. The following lemma is similar to Lemma 4.15.

Lemma 5.2 (Upper semi-continuity) *Let $\mathcal{X}$ be a metrisable space and assume that $G_n, G \in \mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ satisfy $G_n \to G$. Then, for each $T < \infty$ and $\delta > 0$,*

$$m_{T,\delta}(G) \geq \limsup_{n\to\infty} m_{T,\delta}(G_n) \quad \text{and} \quad m^{\rm S}_{T,\delta}(G) \geq \limsup_{n\to\infty} m^{\rm S}_{T,\delta}(G_n). \tag{5.11}$$

Proof We only prove the statement for the Skorohod modulus of continuity. The proof for the traditional modulus of continuity is basically the same, but a bit simpler. The proof will be very similar to the proof of Lemma 4.15. By the compactness of $[0,\infty]$ we can select a subsequence for which $\lim_{n\to\infty} m^{\rm S}_{T,\delta}(G_n)$ exists and is equal to the limit superior of the original sequence. Let $\varepsilon_n > 0$ converge to zero and pick $(x^n_i, t^n_i) \in G_n$ with $-T \leq t^n_i \leq T$ $(i = 1,2,3)$,

$(x_1^n,t_1^n) \preceq (x_2^n,t_2^n) \preceq (x_3^n,t_3^n)$, and $t_3^n - t_1^n \leq \delta$, such that $d\big(x_2^n,\{x_1^n,x_3^n\}\big) \geq m^{\rm S}_{T,\delta}(G_n) - \varepsilon_n$. Since $G_n \to G$ in the topology on $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$, by the first inequality in (2.16) they also converge in the topology on $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$, so by Lemma 2.6 there exists a compact $C \subset \mathcal{R}(\mathcal{X})$ such that $G_n \subset C$ for all n. It follows that we can select a subsequence such that $(x_i^n,t_i^n) \to (x_i,t_i)$ for some $(x_i,t_i) \in G$ $(i=1,2,3)$. Recall from Proposition 4.8 that $d_{\rm tot} = d^{\langle\infty\rangle}$ so by Lemma 4.5, convergence in $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ implies that $G_n^{\langle m\rangle} \to G^{\langle m\rangle}$ in the Hausdorff topology for any $1 \leq m \leq \infty$. In particular, $G_n^{\langle 3\rangle} \to G^{\langle 3\rangle}$, which by Lemma 2.6 implies $(x_1,t_1) \preceq (x_2,t_2) \preceq (x_3,t_3)$. Moreover $-T \leq t_i \leq T$ $(i=1,2,3)$ and $t_3 - t_1 \leq \delta$, so

$$m^{\rm S}_{T,\delta}(G) \geq d\big(x_2,\{x_1,x_3\}\big) \geq \lim_{n\to\infty}\big(m^{\rm S}_{T,\delta}(G_n) - \varepsilon_n\big) = \lim_{n\to\infty} m^{\rm S}_{T,\delta}(G_n), \tag{5.12}$$

where we have used that the betweenness is compatible with the topology. Since we have chosen our subsequence such that the right-hand side is equal to the limit superior of the original sequence, this proves the claim. ■

Proof of Proposition 3.3 We observe that if $\mathcal{X}$ is Polish, then, by Lemma 2.19, so is $\mathcal{R}(\mathcal{X})$ and hence, by Proposition 2.11, also $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$. By identifying a path with its filled-in graph, we can identity $\Pi(\mathcal{X})$ with a subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$. The Skorohod topology on $\Pi(\mathcal{X})$ is then the induced topology from $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$. Therefore, in view of Lemma 4.17, it suffices to show that $\Pi(\mathcal{X})$, viewed as a subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$, is a G_δ-subset of the latter.

We start by showing that condition (i) of Lemma 3.2 can be replaced by

(i)' $\lim_{\delta\to 0} m^{\rm S}_{T,\delta}(G) = 0 \ \forall T < \infty$.

To see this, we argue as follows. If (i) does not hold, then for some $t \in \mathbb{R}$, there exist $(x_i,t_i) \in G$ $(i=1,2,3)$ with $(x_1,t_1) \preceq (x_2,t_2) \preceq (x_3,t_3)$ and $x_2 \notin \langle x_1,x_3\rangle$, which implies that $m^{\rm S}_{T,\delta}(G) \geq d\big(x_2,\langle x_1,x_3\rangle\big) > 0$ for all $\delta > 0$ and $T < \infty$ such that $-T \leq t \leq T$, so (i)' clearly does not hold. Conversely, if (i)' does not hold, then for some $T < \infty$ and $\varepsilon > 0$ we can choose $\delta_n > 0$ tending to zero and $(x_i^n,t_i^n) \in G$ $(i=1,2,3)$ with $(x_1^n,t_1^n) \preceq (x_2^n,t_2^n) \preceq (x_3^n,t_3^n)$ such that $d\big(x_2^n,\langle x_1^n,x_3^n\rangle\big) \geq \varepsilon$. By the compactness of G, we can select a subsequence such that $(x_i^n,t_i^n) \to (x_i,t_i)$ $(i=1,2,3)$. Then clearly $t_1 = t_2 = t_3 =: t$ for some $-T \leq t \leq T$. Since the order is compatible with the topology moreover $(x_1,t_1) \preceq (x_2,t_2) \preceq (x_3,t_3)$. The fact that the betweenness is compatible with the topology allows us to conclude that $d\big(x_2,\langle x_1,x_3\rangle\big) = \lim_{n\to\infty} d\big(x_2^n,\langle x_1^n,x_3^n\rangle\big) \geq \varepsilon$. This shows that $x_2 \notin \langle x_1,x_3\rangle$ and hence (i) does not hold.

Let $\mathcal{H}$ denote the set of all elements of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ that satisfy condition (ii) of Lemma 3.2. By what we have just proved,

$$\Pi(\mathcal{X}) = \big\{G \in \mathcal{H} : \lim_{\delta\to 0} m^{\rm S}_{T,\delta}(G) = 0 \ \forall T < \infty\big\}. \tag{5.13}$$

It follows from Lemma 5.2 that for each $T < \infty$ and $\varepsilon,\delta > 0$, the set

$$\mathcal{G}_{T,\varepsilon,\delta} := \big\{G \in \mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X})) : m^{\rm S}_{T,\delta}(L) \geq \varepsilon\big\} \tag{5.14}$$

is a closed subset of $\mathcal{K}_+(\mathcal{X}^2)$ and hence its complement $\mathcal{A}^{\rm c}_{\varepsilon,\delta}$ is open. As a consequence,

$$\mathcal{G} := \bigcap_{N=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty} \mathcal{G}^{\rm c}_{N,1/n,1/m} \tag{5.15}$$

is a G_δ-set. Formula (5.13) says that $\Pi(\mathcal{X}) = \mathcal{G} \cap \mathcal{H}$. It is easy to see that $\mathcal{H}$ is a closed subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$, and hence a G_δ-set. Since the intersection of two G_δ-sets is a G_δ-set, this yields the statement we wanted to prove. ■

Proof of Proposition 3.4 By Theorem 2.10, $d^{\rm S}_{\rm part}$ and $d^{\rm S}_{\rm tot}$ generate the same topology on $\Pi(\mathcal{X})$, and by (2.16) convergence in any of these two metrics implies convergence in $d^{\rm H}$. Therefore, to show that the conditions (i)–(iii) are equivalent, it suffices to show that if $\pi_n \in \Pi(\mathcal{X})$ and $\pi \in \Pi_c(\mathcal{X})$, then (iii) implies (i). More precisely, we will show that for any betweenness on $\mathcal{X}$ that is compatible with the topology, $\mathcal{G}_f(\pi_n) \to \mathcal{G}(\pi)$ in the Hausdorff topology implies

$$\mathcal{G}_f(\pi_n)^{\langle 2\rangle} \underset{n\to\infty}{\longrightarrow} \mathcal{G}(\pi)^{\langle 2\rangle} \tag{5.16}$$

in the Hausdorff topology. By Lemma 2.6, convergence of $\mathcal{G}_f(\pi_n)$ implies the existence of a compact set $C \subset \mathcal{R}(\mathcal{X})$ such that $\mathcal{G}_f(\pi_n) \subset C$ for all n, which implies $\mathcal{G}_f(\pi_n)^{\langle 2\rangle} \subset C^2$. To complete the proof, by Lemma 2.6, we need to prove the following two statements.

(i) For every $\big((x,s),(y,t)\big) \in \mathcal{G}(\pi)^{\langle 2\rangle}$, there exist $\big((x_n,s_n),(y_n,t_n)\big) \in \mathcal{G}_f(\pi_n)^{\langle 2\rangle}$ such that $\big((x_n,s_n),(y_n,t_n)\big) \to \big((x,s),(y,t)\big)$.

(ii) If a sequence $\big((x_n,s_n),(y_n,t_n)\big) \in \mathcal{G}_f(\pi_n)^{\langle 2\rangle}$ has a cluster point $\big((x,s),(y,t)\big) \in \mathcal{R}(\mathcal{X})^2$, then $\big((x,s),(y,t)\big) \in \mathcal{G}(\pi)^{\langle 2\rangle}$.

To prove (i), we use the fact that $\mathcal{G}_f(\pi_n) \to \mathcal{G}(\pi)$ to find $(x_n,s_n),(y_n,t_n) \in \mathcal{G}_f(\pi_n)$ such that $(x_n,s_n) \to (x,s)$ and $(y_n,t_n) \to (y,t)$. If $s<t$, then $\big((x_n,s_n),(y_n,t_n)\big) \in \mathcal{G}_f(\pi_n)^{\langle 2\rangle}$ for n large enough, so (i) follows. On the other hand, if $s=t$, then $\big((x_n,s_n),(x_n,s_n)\big) \in \mathcal{G}_f(\pi_n)^{\langle 2\rangle}$ so (i) also holds in this case.

To prove (ii), we use the fact that $\mathcal{G}_f(\pi_n) \to \mathcal{G}(\pi)$ to conclude that any cluster point $\big((x,s),(y,t)\big)$ satisfies $(x,s),(y,t) \in \mathcal{G}(\pi)$ with $s \le t$, and hence by the continuity of π either $s<t$ or $(x,s)=(y,t)$, from which we conclude that $\big((x,s),(y,t)\big) \in \mathcal{G}(\pi)^{\langle 2\rangle}$.

It remains to prove that $\Pi_c(\mathcal{X})$ is Polish. This is very similar to the proof of Proposition 3.3, but simpler, so we only sketch the argument. By Lemma 5.1, we may identify $\Pi_c(\mathcal{X})$ with the subset of $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ consisting of all G that contain $(*,\pm\infty)$ and have the property that for each $t\in\mathbb{R}$, the set $\{x\in\mathcal{X} : (x,t)\in G\}$ has at most one element. In this identification, the Hausdorff metric on $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ induces the metric $d^{\rm H}$ which generates the topology on $\Pi_c(\mathcal{X})$. We claim that the condition that for each $t\in\mathbb{R}$, the set $\{x\in\mathcal{X} : (x,t)\in G\}$ has at most one element is equivalent to

$$\lim_{\delta\to 0} m_{T,\delta}(G) = 0 \ \forall T<\infty. \tag{5.17}$$

This follows by the same sort of argument as in the proof of Proposition 3.3, where it was shown that condition (i)' there is equivalent to condition (i) of Lemma 3.2. Thus, identifying a path with its closed graph, we have

$$\Pi_c(\mathcal{X}) = \big\{G\in\mathcal{K}_+(\mathcal{X}) : \lim_{\delta\to 0} m_{T,\delta}(G) = 0 \ \forall T<\infty\big\}. \tag{5.18}$$

By the same argument as in the proof of Proposition 3.3, it follows that $\Pi_c(\mathcal{X})$ is a G_δ-subset of $\mathcal{K}_+(\mathcal{X})$ and hence, by Lemmas 2.7 and 4.17, a Polish space if $\mathcal{X}$ is Polish. ∎

Our next aim is the proof of Lemma 3.5. The proof of the final statement of that lemma needs a bit of preparation. Assume that $\pi\in\Pi(\mathcal{X})$ is not the trivial path. Then we define

$$\mathcal{G}^*(\pi) := \big\{(x,t)\in\mathcal{G}(\pi) : t\in\overline{I}(\pi)\big\}, \tag{5.19}$$

where $\overline{I}(\pi)$ denotes the closure of $I(\pi)$ in $\overline{\mathbb{R}}$. This is almost the same as the closed graph $\mathcal{G}(\pi)$, except that we include the points at infinity $(*,\pm\infty)$ only if their time coordinate lies in $\overline{I}(\pi)$. Since $\mathcal{G}^*(\pi)$ is a subset of $\mathcal{G}(\pi)$, it is naturally equipped with a total order that is compatible with the topology, so we can view it as an element of the space $\mathcal{K}_{\rm tot}(\mathcal{X})$. The reason why we usually work with $\mathcal{G}(\pi)$ instead of $\mathcal{G}^*(\pi)$ is that if we would use the latter throughout, we would end up with a space of paths that contains three trivial paths, whose graphs would be $\{(*,-\infty)\}$, $\{(*,-\infty)\}$, and the union of these two. The following lemma says that as long as we restrict ourselves to nontrivial paths whose domain is an interval, it does not matter which definition of the closed graph we use.

Lemma 5.3 (Convergence of graphs) *Assume that $\pi_n, \pi \in \Pi^{|}(\mathcal{X})$ and that π is not the trivial path. Then $\mathcal{G}(\pi_n) \to \mathcal{G}(\pi)$ in the topology om $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ if and only if $\mathcal{G}^*(\pi_n) \to \mathcal{G}^*(\pi)$ in the topology om $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$.*

Proof Let $\pi_1, \pi_2 \in \Pi(\mathcal{X})$. Each correspondence between $\mathcal{G}^*(\pi_1)$ and $\mathcal{G}^*(\pi_2)$ can be extended to a correspondence between $\mathcal{G}(\pi_1)$ and $\mathcal{G}(\pi_2)$ by adding the points $\big((*,-\infty),(*,-\infty)\big)$ and $\big((*,+\infty),(*,+\infty)\big)$. Adding these extra points does not change the supremum in (2.8), so we see that

$$d_{\rm H}\big(\mathcal{G}(\pi_1),\mathcal{G}(\pi_2)\big) \le d_{\rm H}\big(\mathcal{G}^*(\pi_1),\mathcal{G}^*(\pi_2)\big) \qquad \big(\pi_1,\pi_2 \in \Pi(\mathcal{X})\big). \tag{5.20}$$

From this we see immediately that $\mathcal{G}^*(\pi_n) \to \mathcal{G}^*(\pi)$ in the Hausdorff topology implies $\mathcal{G}(\pi_n) \to \mathcal{G}(\pi)$ in the Hausdorff topology. This part of the argument holds for general $\pi_n, \pi \in \Pi(\mathcal{X})$.

To complete the proof, we must show that if $\pi_n, \pi \in \Pi^{|}(\mathcal{X})$ and π is not the trivial path, then, conversely, $\mathcal{G}(\pi_n) \to \mathcal{G}(\pi)$ in the Hausdorff topology implies $\mathcal{G}^*(\pi_n) \to \mathcal{G}^*(\pi)$ in the Hausdorff topology. By Lemma 2.6, there exists a compact $C \subset \mathcal{R}(\mathcal{X})$ such that $\mathcal{G}(\pi_n) \subset C$ for all n. Since the $\mathcal{G}^*(\pi_n)$ are subsets of $\mathcal{G}(\pi_n)$, they are contained in C too, so by Lemma 2.6 it suffices to show that:

(i) For each $(x,t) \in \mathcal{G}^*(\pi)$, there exist $(x_n,t_n) \in \mathcal{G}^*(\pi_n)$ such that $(x_n,t_n) \to (x,t)$.

(ii) If (x,t) is a cluster point of $(x_n,t_n) \in \mathcal{G}^*(\pi_n)$, then $(x,t) \in \mathcal{G}^*(\pi)$.

To prove (i), we observe that since $\mathcal{G}^*(\pi) \subset \mathcal{G}(\pi)$, for each $(x,t) \in \mathcal{G}^*(\pi)$, there exist $(x_n,t_n) \in \mathcal{G}(\pi_n)$ such that $(x_n,t_n) \to (x,t)$. If $t \in \mathbb{R}$, then $t_n \in \mathbb{R}$ for n large enough and hence $(x_n,t_n) \in \mathcal{G}^*(\pi_n)$ and we are done. If $t = \infty$, then by the fact that $\pi \in \Pi^{|}(\mathcal{X})$ and π is not the trivial path, we see that for each $T < \infty$, we must have $I(\pi_n) \cap [T,\infty) \neq \emptyset$ for all n large enough. Using this, we see that there exist $(x_n,t_n) \in \mathcal{G}(\pi_n)$ with $t_n < \infty$ such that $(x_n,t_n) \to (x,\infty)$. But then $(x_n,t_n) \in \mathcal{G}^*(\pi_n)$, as required. The proof when $t = -\infty$ is the same, so the proof of (i) is complete.

To prove (ii), we observe that since $\mathcal{G}^*(\pi_n) \subset \mathcal{G}(\pi_n) \to \mathcal{G}(\pi)$, if (x,t) is a cluster point of $(x_n,t_n) \in \mathcal{G}^*(\pi_n)$, then $(x,t) \in \mathcal{G}(\pi)$. If $t \in \mathbb{R}$, then clearly $(x,t) \in \mathcal{G}^*(\pi)$ and we are done. If $t = \infty$, then by the assumption that π is not trivial we can choose $(y,s) \in \mathcal{G}(\pi)$ with $s \in \mathbb{R}$, and by the assumption that $\mathcal{G}(\pi_n) \to \mathcal{G}(\pi)$ we can choose $(y_n,s_n) \in \mathcal{G}(\pi_n)$ such that $(y_n,s_n) \to (y,s)$. Since $\pi_n \in \Pi^{|}(\mathcal{X})$ for all n, it follows that $I(\pi_n)$ contains (s_n,t_n) for each n and hence by the assumption that $\mathcal{G}(\pi_n) \to \mathcal{G}(\pi)$, the domain $I(\pi)$ contains (s,∞), which implies that $(*,\infty) \in \mathcal{G}^*(\pi)$. The argument when $t = -\infty$ is the same so we are done. ∎

The following lemma reveals a pleasant property of $\mathcal{G}^*(\pi)$ that $\mathcal{G}(\pi)$ does not have.

Lemma 5.4 (Connected graphs) *Assume that $\pi \in \Pi(\mathcal{X})$ is not the trivial path. Then $\pi \in \Pi^{|}_{\rm c}(\mathcal{X})$ if and only if $\mathcal{G}^*(\pi)$ is connected.*

Proof If $\pi \in \Pi^{|}_{\rm c}(\mathcal{X})$, then $\mathcal{G}^*(\pi)$ is the image of the compact set $\overline{I}(\pi)$ under the continuous map from $\overline{\mathbb{R}}$ to $\mathcal{R}(\mathcal{X})$ given by $t \mapsto \big(\pi(t),t\big)$ (with $\pm\infty \mapsto (*,\pm\infty)$). Since $\overline{I}(\pi)$ is connected and the continuous image of a connected set is connected, we conclude that $\mathcal{G}^*(\pi)$ is connected.

Conversely, if $\mathcal{G}^*(\pi)$ is connected, then $\overline{I}(\pi)$ must be connected and hence $\pi \in \Pi^{|}(\mathcal{X})$. To see that π is moreover continuous, assume that conversely, $\pi(t-) \neq \pi(t+)$ for some $t \in I(\pi)$. Then we can define new paths π', π'' with domains $I(\pi') := (-\infty, t] \cap I(\pi)$ and $I(\pi'') := [0, \infty) \cap I(\pi)$, by setting $\pi'(s) := \pi(s)$ and $\pi''(s) := \pi(s)$ for $s \neq t$, and $\pi'(t\pm) := \pi(t-)$ and $\pi''(t\pm) := \pi(t+)$. By Lemma 3.1, $\mathcal{G}^*(\pi')$ and $\mathcal{G}^*(\pi'')$ are compact sets. Since $\mathcal{G}^*(\pi') \cap \mathcal{G}^*(\pi'') = \emptyset$ and $\mathcal{G}^*(\pi') \cup \mathcal{G}^*(\pi'') = \mathcal{G}^*(\pi)$, this proves that $\mathcal{G}^*(\pi)$ is not connected. ∎

Proof of Lemma 3.5 By the first inequality in (2.16), convergence $\pi_n \to \pi$ in $\Pi(\mathcal{X})$ implies convergence of $\mathcal{G}_{\rm f}(\pi_n)$ to $\mathcal{G}_{\rm f}(\pi)$ in the Hausdorff topology, which by Lemma 4.3 implies convergence of $I(\pi_n) \cup \{\pm\infty\}$ to $I(\pi) \cup \{\pm\infty\}$ in $\mathcal{K}_+(\overline{\mathbb{R}})$. Using Lemma 2.6, it is easy to see that if I_n are closed subintervals of $\mathbb{R}$ such that $I_n \cup \{\pm\infty\}$ converges in $\mathcal{K}_+(\overline{\mathbb{R}})$ to a limit, then this limit must be of the form $I \cup \{\pm\infty\}$ for some (possibly empty) closed interval $I \subset \mathbb{R}$. This shows that $\Pi^{|}(\mathcal{X})$ is closed and in the same way we also see that $\Pi^{\uparrow}(\mathcal{X})$ and $\Pi^{\downarrow}(\mathcal{X})$ are closed.

Now assume that the betweenness is the trivial betweenness. Assume that $\pi_n \in \Pi^{|}_{\rm c}$ converge to $\pi \in \Pi(\mathcal{X})$. We need to show that $\pi \in \Pi^{|}_{\rm c}$. This is certainly true if π is the trivial path, so we assume from now on that π is nontrivial. By the first inequality in (2.16), convergence of π_n to π in the topology on $\Pi(\mathcal{X})$ implies that $\mathcal{G}(\pi_n) \to \mathcal{G}(\pi)$ in the Hausdorff topology, which by Lemma 5.3 implies that also $\mathcal{G}^*(\pi_n) \to \mathcal{G}^*(\pi)$ in the Hausdorff topology. By Lemma 5.4, the graphs $\mathcal{G}^*(\pi_n)$ are connected and hence Lemma 2.9 implies that $\mathcal{G}^*(\pi)$ is connected, which by Lemma 5.4 implies that $\pi \in \Pi^{|}_{\rm c}$. ∎

5.3 Compactness criteria

In this subsection, we prove Theorems 3.6 and 3.7. Lemmas 3.2 and 5.1 will play an essential role here.

Proof of Theorem 3.6 By Lemma 5.1, we may identify $\Pi_{\rm c}(\mathcal{X})$ with the subset of $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ consisting of all G that contain $(*, \pm\infty)$ and have the property that for each $t \in \mathbb{R}$, the set $\{x \in \mathcal{X} : (x, t) \in G\}$ has at most one element. In this identification, the Hausdorff metric on $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ induces the metric $d^{\rm H}$ on $\Pi_{\rm c}(\mathcal{X})$, which by the definition above Proposition 3.4 generates the topology on $\Pi_{\rm c}(\mathcal{X})$. For any $G \in \mathcal{K}_+(\mathcal{R}(\mathcal{X}))$, we define

$$m_{T,\delta}(G) := \sup\big\{d(x_1, x_2) : (x_1, t_1), (x_2, t_2) \in G,\ -T \leq t_1 < t_2 \leq T,\ t_2 - t_1 \leq \delta\big\}. \tag{5.21}$$

In the special case that G is (the closed graph of) a path in $\Pi_{\rm c}(\mathcal{X})$, this coincides with the definition of the modulus of continuity in (3.10).

Let $\mathcal{A} \subset \Pi_{\rm c}(\mathcal{X})$. Then $\mathcal{A}$ is precompact if and only if its closure $\overline{\mathcal{A}}$ in $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ is compact and $\overline{\mathcal{A}} \subset \Pi_{\rm c}(\mathcal{X})$. By Lemmas 2.8 and 2.20, $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ if and only if $\mathcal{A}$ satisfies the compact containment condition. Therefore, to complete the proof, it suffices to show that if $\mathcal{A} \subset \Pi_{\rm c}(\mathcal{X})$ satisfies the compact containment condition, then $\overline{\mathcal{A}} \subset \Pi_{\rm c}(\mathcal{X})$ if and only if $\mathcal{A}$ is equicontinuous.

Assume that $\mathcal{A}$ satisfies the compact containment condition and is equicontinuous, and that $G_n \in \mathcal{A}$ satisfy $G_n \to G$ for some $G \in \mathcal{K}_+(\mathcal{R}(\mathcal{X}))$. We need to show that $G \in \Pi_{\rm c}(\mathcal{X})$. Clearly $(*, \pm\infty) \in G$, so it suffices to show that for each $t \in \mathbb{R}$, the set $\{x \in \mathcal{X} : (x, t) \in G\}$ has at most one element. Assume that conversely, there exist $(x_1, t), (x_2, t) \in G$ with $t \in \mathbb{R}$ and $x_1 \neq x_2$. Then by Lemma 2.6, there exist $(x^n_1, t^n_1), (x^n_2, t^n_2) \in G_n$ such that $(x^n_i, t^n_i) \to (x_i, t)$ as $n \to \infty$ $(i = 1, 2)$. Choose $T < \infty$ such that $-T < t < T$. Then for each $\delta > 0$ we have for n large enough that $-T < t^n_1, t^n_2 < T$, $|t^n_1 - t^n_2| \leq \delta$, and $d(x^n_1, x^n_2) \geq d(x_1, x_2)/2$. This proves that

$$\sup_{\pi \in \mathcal{A}} m_{T,\delta}(\pi) \geq d(x_1, x_2)/2 \qquad \forall \delta > 0, \tag{5.22}$$

contradicting the equicontinuity of $\mathcal{A}$.

Assume, on the other hand, that $\mathcal{A}$ satisfies the compact containment condition and is not equicontinuous. Let δ_n be positive constants tending to zero. Since $\mathcal{A}$ is not equicontinuous, for some $T < \infty$ and $\varepsilon > 0$ we can find $G_n \in \mathcal{A}$ such that $m_{T,\delta_n}(G_n) \geq \varepsilon$ for all n. Since $\mathcal{A}$ satisfies the compact containment condition, $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$, so by going to a subsequence we may assume that $G_n \to G$ for some $G \in \mathcal{K}_+(\mathcal{R}(\mathcal{X}))$. Since $m_{T,\delta_n}(G_n) \geq \varepsilon$ we can find $(x_1^n, t_1^n), (x_2^n, t_2^n) \in G_n$ such that $-T \leq t_1^n < t_2^n \leq T$, $t_2^n - t_1^n \leq \delta_n$, and $d(x_1^n, x_2^n) \geq \varepsilon$. By Lemma 2.6, going to a further subsequence if necessary, we can assume that $(x_i^n, t_i^n) \to (x_i, t)$ as $n \to \infty$ $(i = 1, 2)$ for some $(x_1, t), (x_2, t) \in G$. Then $-T \leq t \leq T$ and $d(x_1, x_2) \geq \varepsilon$, which shows that $\{x \in \mathcal{X} : (x, t) \in G\}$ has more than one element and hence $\overline{\mathcal{A}}$ is not contained in $\Pi_{\rm c}(\mathcal{X})$. ∎

Proof of Theorem 3.7 By Lemma 3.2, identifying a path with its filled-in graph, we may identify $\Pi(\mathcal{X})$ with the subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ consisting of all $(G, \preceq)$ that contain $(*, \pm\infty)$ and satisfy conditions (i) and (ii) of the lemma. In this identification, the metrics $d_{\rm part}$ and $d_{\rm tot}$ on $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ induce metrics $d^{\rm S}_{\rm part}$ and $d^{\rm S}_{\rm tot}$ on $\Pi(\mathcal{X})$ that both generate the Skorohod topology.

Let $\mathcal{A} \subset \Pi(\mathcal{X})$. Then $\mathcal{A}$ is precompact in the Skorohod topology if and only if its closure $\overline{\mathcal{A}}$ in $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ is compact and $\overline{\mathcal{A}} \subset \Pi(\mathcal{X})$. By Theorem 2.12 and Lemma 2.20, $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ if and only if $\mathcal{A}$ satisfies the compact containment condition and

$$\lim_{\varepsilon \to 0} \sup_{G \in \mathcal{A}} m_\varepsilon(G) = 0, \tag{5.23}$$

where $m_\varepsilon(G)$ denotes the mismatch modulus of G. To complete the proof we will prove the following three statements.

I Assume that $\mathcal{A}$ is satisfies the compact containment condition and is Skorohod-equicontinuous. Then $\mathcal{A}$ satisfies (5.23).

II Assume that $\mathcal{A}$ is Skorohod-equicontinuous. Then $\overline{\mathcal{A}} \subset \Pi(\mathcal{X})$.

III Assume that $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ and that $\overline{\mathcal{A}} \subset \Pi(\mathcal{X})$. Then $\mathcal{A}$ is Skorohod-equicontinuous.

Now if $\mathcal{A}$ satisfies the compact containment condition and is Skorohod-equicontinuous, then by our earlier remarks I implies that $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ and II implies that $\overline{\mathcal{A}} \subset \Pi(\mathcal{X})$, so $\mathcal{A}$ is precompact in the Skorohod topology. Conversely, if $\mathcal{A}$ is precompact in the Skorohod topology, then $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ and hence by our earlier remarks $\mathcal{A}$ satisfies the compact containment condition, and moreover $\overline{\mathcal{A}} \subset \Pi(\mathcal{X})$ which by III implies that $\mathcal{A}$ is Skorohod-equicontinuous.

We start by proving I. Since $\mathcal{A}$ satisfies the compact containment condition, by Lemma 2.20, we see that there exists a compact $C \subset \mathcal{R}(\mathcal{X})$ such that $G \subset C$ for all $G \in \mathcal{A}$. Since $\sup_{G \in \mathcal{A}} m_\varepsilon(G)$ is nondecreasing as a function of ε, the limit in (5.23) always exists. Let ε_n be positive constants, tending to zero. If the limit in (5.23) is positive, then there exists a $\delta > 0$ such that for each n we can find a $G \in \mathcal{A}$ and $(x_i^n, s_i^n), (y_i^n, t_i^n) \in G$ $(i = 1, 2)$ with $(x_1^n, s_1^n) \preceq (y_1^n, t_1^n)$, $(y_2^n, t_2^n) \preceq (x_2^n, s_2^n)$ such that

$$d_{\rm sqz}\big((x_1^n, s_1^n), (x_2^n, s_2^n)\big) \vee d_{\rm sqz}\big((y_1^n, t_1^n), (y_2^n, t_2^n)\big) \leq \varepsilon_n,$$
$$d_{\rm sqz}\big((x_i^n, s_i^n), (y_i^n, t_i^n)\big) \geq \delta \quad (i = 1, 2).$$

Since $G \subset C$ for all $G \in \mathcal{A}$, by going to a subsequence, we may assume that $(x_i^n, s_i^n) \to (x, s)$ and $(y_i^n, t_i^n) \to (y, t)$ $(i = 1, 2)$ for some $(x, s), (y, t) \in \mathcal{R}(\mathcal{X})$. Then $d_{\rm sqz}\big((x, s), (y, t)\big) \geq \delta$ and hence $(x, s) \neq (y, t)$. Since $(x_1^n, s_1^n) \preceq (y_1^n, t_1^n)$ and $(y_2^n, t_2^n) \preceq (x_2^n, s_2^n)$ we have $s_1^n \leq t_1^n$ and $t_2^n \leq s_2^n$ for all n which implies $s = t$ and hence $x \neq y$, since $(x, t) \neq (y, t)$. By the structure of $\mathcal{R}(\mathcal{X})$, this implies $t \in \mathbb{R}$. Let (x_-^n, s_-^n) (resp. (x_+^n, s_+^n)) be the smallest (resp. largest) of the points (x_i^n, s_i^n) $(i = 1, 2)$ with respect to the order $\preceq$, and define $(y_\pm^n, t_\pm^n)$ similarly. Since G is totally ordered, by going to a subsequence, we can assume that we are in one of the following two cases. 1. $(x_-^n, s_-^n) \preceq (y_-^n, t_-^n)$ for all n, or 2. $(y_-^n, t_-^n) \preceq (x_-^n, s_-^n)$ for all n. Let us assume that we are in case 1. Then $(x_-^n, s_-^n) \preceq (y_-^n, t_-^n) \preceq (x_+^n, s_+^n)$ for all n. Since the betweenness is compatible with the topology,

$$d\big(y_-^n, \langle x_-^n, x_+^n\rangle\big) \underset{n\to\infty}{\longrightarrow} d(y, x) > 0, \tag{5.24}$$

which contradicts the Skorohod-equicontinuity. Case 2 is completely the same, exchanging the roles of x and y.

We next prove II. Assume that $\pi_n \in \mathcal{A}$ converge in $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ to a limit G. Recall from Subsection 4.3 that $d_{\rm part} = d^{\langle 2\rangle} \leq d^{\langle m\rangle} \leq d^{\langle\infty\rangle} = d_{\rm tot}$ for all $m \geq 2$. In particular, $\pi_n \to G$ in $\mathcal{K}_{\rm tot}(\mathcal{X})$ implies that $\pi_n^{\langle m\rangle} \to G^{\langle m\rangle}$ in the Hausdorff topology for all $m \geq 2$. It suffices to check that G satisfies conditions (i) and (ii) of Lemma 3.2. Condition (ii) easily follows from the fact that $\pi_n^{\langle 2\rangle} \to G^{\langle 2\rangle}$ in the Hausdorff topology, using Lemma 2.6. It remains to prove that G satisfies condition (i) of Lemma 3.2. Assume that conversely, for some $t \in \mathbb{R}$, there exist $(x_1, t), (x_2, t), (x_3, t) \in G$ with $(x_1, t) \preceq (x_2, t) \preceq (x_3, t)$ such that $x_2 \notin \langle x_1, x_3\rangle$. Since $\pi_n^{\langle 3\rangle} \to G^{\langle 3\rangle}$ in the Hausdorff topology, by Lemma 2.6 there exist $\tau_i^n \in I_\mathfrak{s}(\pi_n) \cup \{\pm\infty\}$ $(i = 1, 2, 3)$ with $\tau_1^n \leq \tau_2^n \leq \tau_3^n$ such that $\underline{\tau}_i^n \to t$ and $\pi_n(\tau_i^n) \to x_i$ $(i = 1, 2, 3)$. Using the fact that the betweenness is compatible with the topology, we see that

$$d\big(\pi_n(\tau_2^n), \langle \pi_n(\tau_1^n), \pi_n(\tau_3^n)\rangle\big) \underset{n\to\infty}{\longrightarrow} d\big(x_2, \langle x_1, x_3\rangle\big), \tag{5.25}$$

which is easily seen to contradict Skorohod-equicontinuity, completing the proof of II.

To prove III, finallly, we will show that if $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ and $\mathcal{A}$ is not Skorohod-equicontinuous, then $\overline{\mathcal{A}}$ is not contained in $\Pi(\mathcal{X})$. Let δ_n be positive constants, tending to zero. Since $\mathcal{A}$ is not Skorohod-equicontinuous, there exists a $\varepsilon > 0$, $T < \infty$, $\pi_n \in \mathcal{A}$, and $\tau_i^n \in I_\mathfrak{s}(\pi_n)$ $(i = 1, 2, 3)$ such that $\tau_1 \leq \tau_2 \leq \tau_3$, $-T \leq \underline{\tau}_1^n$, $\underline{\tau}_3^n \leq T$, $\underline{\tau}_3^n - \underline{\tau}_1^n \leq \delta_n$, and $d\big(\pi(\tau_2^n), \langle \pi(\tau_1^n), \pi(\tau_3^n)\rangle\big) \geq \varepsilon$. Since $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$, we can select a subsequence such that $\pi_n \to G$ for some $G \in \mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$. Then $\pi_n^{\langle 3\rangle} \to G^{\langle 3\rangle}$ in the Hausdorff topology, so by Lemma 2.6 there exist $C \subset \mathcal{R}(\mathcal{X})^3$ such that $\pi_n^{\langle 3\rangle} \subset C$ for all n. It follows that by going to a further subsequence we can assume that $\big(\pi_n(\tau_i^n), \underline{\tau}_i^n\big) \to (x_i, t_i)$ as $n \to \infty$ for some $(x_i, t) \in G$ $(i = 1, 2, 3)$ with $(x_1, t_1) \preceq (x_2, t_2) \preceq (x_3, t_3)$ and $-T \leq t \leq T$. Using the fact that the betweenness is compatible with the topology, we see that $d(x_2, \langle x_1, x_3\rangle) \geq \varepsilon$, which shows that G is not the filled-in graph of a path $\pi \in \Pi(\mathcal{X})$ and hence $\overline{\mathcal{A}}$ is not contained in $\Pi(\mathcal{X})$. ∎

5.4 Paths on fixed domains

In this subsection, we prove Lemma 3.8 and Theorem 3.9.

Proof of Lemma 3.8 We start by proving the statement for $d^{\rm H}$. Let $T := \{t > 0 : f(t-) = f(t)\}$, which is dense in $[0, \infty)$ by Lemma 4.2. Let $G := \mathcal{G}_{\rm f}(f)$, $G_n := \mathcal{G}_{\rm f}(f_n)$, $G^t := \mathcal{G}_{\rm f}(f\big|_{[0,t]})$, and $G_n^t := \mathcal{G}_{\rm f}(f_n\big|_{[0,t]})$.

We first prove the implication $\Rightarrow$. Let $t \in T$. Since $G_n \to G$, by Lemma 2.6, there exists a compact set C such that $G_n \subset C$ for all n, and hence also $G_n^t \subset C$ for all n. By Lemma 2.8, it follows that $\{G_n^t : n \in \mathbb{N}\}$ is compact, so to prove that $G_n^t \to G^t$, it suffices to show that G^t is the only subsequential limit of the G_n^t. Let G^* be such a subsequential limit. Since $G_n^t \subset G_n$ it is clear from Lemma 2.6 that $G^* \subset G$. Let ψ denote the projection $\psi(x, s) := s$ and let $\psi(G^*)$ denote the image of G^* under ψ. By Lemma 4.3, $\psi(G^*) = [0, t]$. It follows that $G^* \subset G^t$. To prove the opposite inclusion, assume that $(y,s) \in G^t$. If $s < t$, then we use that by Lemma 2.6, there exist $(x_n, s_n) \in G_n$ such that $(x_n, s_n) \to (x, s)$. Since $s < t$, we have $(x_n, s_n) \in G^t_n$ for n large enough and hence $(x,s) \in G^*$. If $s = t$, then we use that $\psi(G^*) = [0,t]$ to conclude that there must be at least one $y' \in \mathcal{X}$ such that $(y',t) \in G^*$. Since $G^* \subset G^t$ we must have $y, y' \in \langle f(t-), f(t)\rangle = \{f(t)\}$, where we have used that $t \in T$, so we conclude that $y' = y$, concluding the proof that $G^* = G^t$.

We next prove the implication $\Leftarrow$. Since $G^t_n \to G^t$ for each $t \in T$, using Lemmas 2.6 and 2.20, we see that there exists a compact set C such that $G_n \subset C$ for all n, so by Lemma 2.8 it suffices to show that if G_* is a subsequential limit of the G_n, then $G_* = G$. Since $G^t_n \subset G_n$ for each n it is clear from Lemma 2.6 that $G^t \subset G_*$ for each $t \in T$. We claim that conversely, for each $(x,s) \in G_*$ and $s < t \in T$, we have $(x,s) \in G^t$. Indeed, by Lemma 2.6 for some subsequence there exist $(x_n, s_n) \in G_n$ such that $(x_n, s_n) \to (x,s)$. Since $s_n < t$ for n large enough, it follows that $(x,s) \in G^t$. These arguments show that $\{(x,t) \in G_* : t < \infty\} = \{(x,t) \in G : t < \infty\}$, which is enough to conclude $G_* = G$.

We next prove the statement for $d^{\rm S}_{\rm tot}$. Let $\pi := f$, $\pi_n := f_n$, $\pi^t := f\big|_{[0,t]}$, and $\pi^t_n := f_n\big|_{[0,t]}$, which we view as elements of the path space $\Pi(\mathcal{X})$. We first prove the implication $\Rightarrow$. By Theorem 3.7, $d^{\rm S}_{\rm tot}(\pi_n, \pi) \to 0$ implies that $\{\pi_n : n \in \mathbb{N}\}$ is Skorohod-equicontinuous and satisfies the compact containment condition, which implies the same is true for $\{\pi^t_n : n \in \mathbb{N}\}$ for any $t \in T$. Therefore, by Theorem 3.7, it suffices to show that all subsequential limits of the π^t_n are equal to π^t. Since by Theorem 2.10, convergence in $d^{\rm S}_{\rm tot}$ implies convergence in $d^{\rm H}_{\rm tot}$, we can use what we have already proved for $d^{\rm H}_{\rm tot}$ to draw the desired conclusion. The implication $\Leftarrow$ follows in the same way, where now we use that if $\{\pi^t_n : n \in \mathbb{N}\}$ is Skorohod-equicontinuous and satisfies the compact containment condition for any $t \in T$, then the same is true for $\{\pi_n : n \in \mathbb{N}\}$. ∎

Proof of Theorem 3.9 We view $\mathcal{D}_I(\mathcal{X})$ as a subset of $\Pi(\mathcal{X})$ as in (3.14). Then $\mathcal{F}$ is compact as a subset of $\mathcal{D}_I(\mathcal{X})$ if and only if its closure $\overline{\mathcal{F}}$ in the larger space $\Pi(\mathcal{X})$ is compact and satisfies $\overline{\mathcal{F}} \subset \mathcal{D}_I(\mathcal{X})$. By Theorem 3.7, $\overline{\mathcal{F}}$ is a compact subset of $\Pi(\mathcal{X})$ if an only if conditions (i) and (ii) hold. To complete the proof, we will show that, assuming (i) and (ii), one has $\overline{\mathcal{F}} \subset \mathcal{D}_I(\mathcal{X})$ if and only if (iii) holds.

We first show that (iii) implies that $\overline{\mathcal{F}} \subset \mathcal{D}_I(\mathcal{X})$. Assume that (iii) holds and let $f_n \in \mathcal{D}_I(\mathcal{X})$ and $\pi \in \Pi(\mathcal{X})$ satisfy $f_n \to \pi$ in the Skorohod topology associated with the given betweenness. Then clearly $I(\pi) = I$. To show that $\pi \in \mathcal{D}_I(\mathcal{X})$ assume that conversely $\pi(t-) \neq \pi(t+)$ for some $t \in \partial I$. Then, by the fact that $d^{\rm S}_{\rm part}(f_n, \pi) \to 0$, there exist $s_n, t_n \in I$ with $s_n < t_n$ such that $f_n(s_n) \to \pi(t-)$ and $f_n(t_n) \to \pi(t+)$, which is easily seen to contradict (iii).

Assume, on the other hand, that (iii) does not hold for some $t \in \partial I$. Let δ_n be positive constants, tending to zero. Then there exists an $\varepsilon > 0$ such that for each n, we can find $f_n \in \mathcal{F}$ and $s_n \in I$ with $|s_n - t| \leq \delta_n$ and $d\big(f_n(s_n), f_n(t)\big) \geq \varepsilon$. By (i) and (ii), $\overline{\mathcal{F}}$ is compact in $\Pi(\mathcal{X})$ so by going to a subsequence we can assume that $f_n \to \pi$ for some $\pi \in \Pi(\mathcal{X})$. By (i), we can moreover assume that $f_n(s_n) \to x$ and $f_n(t) \to y$ for some $x, y \in \mathcal{X}$. Then $d(x,y) \geq \varepsilon$ and $(x,t), (y,t) \in \mathcal{G}_{\rm f}(\pi)$, which by Lemma 2.14 (v) shows that $\pi(t-) \neq \pi(t+)$ and hence $\overline{\mathcal{F}}$ is not contained in $\mathcal{D}_I(\mathcal{X})$. ∎

Acknowledgments

We thank Jan Seidler for useful discussions and for his help in understanding [AU29].

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